O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

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TENGSIZLIKLAR-III.
MASALALAR TO'PLAMI

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Fizika –matematika fanlari doktori, professor A. A’zamov umumiy tahriri ostida.

Qo’llanmada oxirgi yillar mobaynida turli davlatlarda bo’lib o’tgan matematika olimpiadalarida taqdim qilingan tengsizliklar yechimlar bilan keltirilgan.

Qo’llanma umumiy o’rta ta’lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o’quvchilari, matematika fani o’qituvchilari hamda pedagogika oliy o’quv yurtlari talabalari uchun mo’ljallangan.

Qo’llanmadan sinfdan tashqari mashg’ulotlarda, o’quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov
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Ushbu qo’llanma Respublika ta’lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 - sonli bayyonnoma)

Qo’llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q’omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi)

Masalalar.

1. Agar a,b,c musbat sonlar va $n,k\in N$ bo'lsa, u holda quyidagi

$$\frac{a^{n+k}}{b^k}+\frac{b^{n+k}}{c^k}+\frac{c^{n+k}}{a^k}\geq a^n+b^n+c^n$$

tengsizlikni isbotlang.

2. (Rossiya -2003) Musbat *a,b,c* sonlar *a*+*b*+*c*=1 tenglikni qanoatlantirilsa, u holda

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$$

tengsizlikni isbotlang.

3. (Moldova -2005) Musbat a,b,c sonlar $a^4+b^4+c^4=3$ tenglikni qanoatlantirsa,

$$\frac{1}{4-ab}+\frac{1}{4-ac}+\frac{1}{4-bc}\leq 1$$

tengsizlikni isbotlang.

4. (Koreya -2000) Aytaylik, a,b,c, x,y,z haqiqiy sonlar quyidagi

$a\geq b\geq c>0$, $x\geq y\geq z>0$ shartlarni qanoatlantirsin, u holda

$$\frac{a^2x^2}{(by+cz)(bz+cy)}+\frac{b^2y^2}{(cz+ax)(cx+az)}+\frac{c^2z^2}{(ax+by)(ay+bx)}\geq\frac{3}{4}$$

tengsizlikni isbotlang.

5. (Yaponiya -2002) Aytaylik, $n\geq 3$ va $n\in N$ da musbat $a_1,a_2,...,a_n,b_1,b_2,....,b_n$ sonlar quyidagi $a_1+a_2+...+a_n=1$ *ва* $b_1^2+b_2^2+...+b_n^2=1$ shartlarni qanoatlantirsin. U holda

$$a_1(b_1+a_2)+a_2(b_2+a_3)+...+a_n(b_n+a_1)<1$$

tengsizlikni isbotlang.

6. Musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsa, u holda

$$\sqrt{\frac{ab}{ab+c}}+\sqrt{\frac{bc}{bc+a}}+\sqrt{\frac{ac}{ac+b}}\le\frac{3}{2}$$

tengsizlikni isbotlang.

7. (Eron -2005) Musbat a,b,c sonlar uchun

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\ge(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

tengsizlikni isbotlang.

8. (Vengriya -1996) Musbat a,b sonlarning yig'indisi birga teng bo'lsa

$\frac{a^2}{a+1}+\frac{b^2}{b+1}\ge\frac{1}{3}$ tengsizlikni isbotlang.

9. Haqiqiy musbat x,y,z sonlar $xyz\ge1$ shartni qanoatlantirsa,

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{y^5+z^2+x^2}+\frac{z^5}{z^5+x^2+y^2}\ge1$$

tengsizlikni isbotlang.

10. (XMO, Xalqaro Matematika olimpiadasi -2005) Musbat x,y,z sonlar $xyz\ge1$ shartni qanoatlantirsa,

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\ge0$$

tengsizlikni isbotlang.

11. (Bosniya -2002) Agar musbat x,y,z sonlar $xyz=x+y+z+2$ tenglikni qanoatlantirsa,

$$5(x+y+z)+18\ge8(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})$$

tengsizlik o'rinli bo'lishini isbotlang.

12. (APMO -2005) Musbat a,b,c sonlar $abc=8$ shartni qanoatlantirsa, u holda.

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}}+\frac{b^2}{\sqrt{(1+b^3)(1+c^3)}}+\frac{c^2}{\sqrt{(1+c^3)\left(1+a^3\right)}}\geq\frac{4}{3}$$

tengsizlikni isbotlang.

13. (Rossiya -1999) Musbat haqiqiy x va y sonlar $x^2+y^3\geq x^3+y^4$ tengsizlikni qanoatlantirsa, u holda $x^3+y^3\leq 2$ tengsizlikni isbotlang.

14. (APMO -2003). Agar a,b,c sonlari uchburchak tomonlarining uzunliklari bo'lib, $a+b+c=1$ shartni qanoatlantirsa, $n\in N$, $n\geq 2$ uchun

$$\sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n}<1+\frac{\sqrt[n]{2}}{2}$$

tengsizlikni isbotlang.

15. Musbat $a_1,a_2,...,a_n$ sonlar uchun quyidagi

$$n\cdot\sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}\leq n+1$$

tengsizlikni isbotlang. Bu yerda

$$g_n=\sqrt[n]{a_1a_2...a_n},\ A_n=\frac{a_1+a_2+...+a_n}{n},\ \ G_n=\sqrt[n]{A_1A_2...A_n}\ .$$

16. (Yaponiya -2005) Musbat a,b,c sonlar yig'indisi birga teng bo'lsa,

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+\sqrt[3]{1+a-b}\leq 1$$

tengsizlikni isbotlang.

17. (Kolmogorov kubogi, Rossiya -2004) Musbat haqiqiy a,b,c,d sonlar $abcd=1$ shartni qanoatlantirsa, u holda

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq 4$$

tengsizlikni isbotlang.

18. (Kolmogorov kubogi, Rossiya -2004) Musbat a,b,c sonlarning yig'indisi birga teng bo'lsa,

$$(ab+bc+ca)\left(\frac{a}{b(b+1)}+\frac{b}{c(c+1)}+\frac{c}{a(a+1)}\right)\geq\frac{3}{4}$$

tengsizlik o'rinli bo'lishini isbotlang.

19. Musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsa,

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}\geq 2$$

tengsizlikni o'rinli bo'lishini isbotlang

20. Musbat x,y,z sonlar uchun

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+x)(y+z)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1$$

tengsizlikni isbotlang.

21. (Xitoy -2004) Musbat a,b,c sonlar uchun

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\leq\frac{3\sqrt{2}}{2}$$

tengsizlikni isbotlang.

22. (Turkiya -1998) Agar $0\leq a\leq b\leq c$ shart bajarilsa,

$$(a+3b)(b+4c)(c+2a)\geq 60abc$$

tengsizlikni isbotlang.

23. (Buyuk Ipak yo'li Xalqaro olimpiadasi -2006) Musbat a,b,c sonlar uchun $abc=1$ shart bajarilsa,

$$4\left(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{c}}+\sqrt[3]{\frac{c}{a}}\right)\le 3\left(2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{2}{3}}$$

tengsizlikni isbotlang.

24. (Albaniya -2002) Musbat a,b,c sonlar uchun

$$(a+b+c)+\sqrt{a^2+b^2+c^2}\le\frac{\sqrt{3}+1}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(a^2+b^2+c^2\right)$$

tengsizlikni isbotlang.

25. (AQSh -2004) Musbat a,b,c sonlar uchun

$$\left(a^5-a^2+3\right)(b^5-b^2+3)(c^5-c^2+3)\ge\left(a+b+c\right)^3$$

tengsizlikni isbotlang.

26. Musbat a,b,c sonlar uchun

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge(ab+bc+ca)^3$$

tengsizlikni isbotlang.

27. (AQSh -2001) Agar musbat a,b,c sonlar $a^2+b^2+c^2+abc=4$ shartni qanoatlantirsa, $0<ab+bc+ca-abc\le 2$ tengsizlikni isbotlang.

28. Haqiqiy x,y sonlar $x\ne 0$, $xy(x^2-y^2)=x^2+y^2$ shartlarni qanoatlantirsa, $x^2+y^2\ge 4$ tengsizlikni isbotlang.

29. (Vyetnam -2001) Musbat a,b,c sonlar $21ab+2bc+8ac\le 12$ shartni qanoatlantirsa, $P(a,b,c)=\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ ifodaning eng kichik qiymatini toping.

30. Musbat a,b,c sonlar uchun $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}$ tengsizlikni isbotlang.

31. (Hindiston -2000) Haqiqiy $a_1,...,a_n$ conlar uchun $a_1\leq a_2\leq ...\leq a_n$ va $a_1+a_2+...+a_n=0$ shartlar bajarilsa, $na_1a_n+\sum_{i=1}^{n}a_i^2\leq 0$ tengsizlikni isbotlang.

32. (Ruminiya -1999) Musbat $x_1,x_2,...x_n$ sonlar $\prod_{i=1}^{n}x_i=1$ shartni qanoatlantirsa,

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+.....+\frac{1}{n-1+x_n}\leq 1$$

tengsizlikni isbotlang.

33. (Polsha -1991) Haqiqiy x,y,z sonlar $x^2+y^2+z^2=2$ shartni qanoatlantirsa, $x+y+z-xyz\leq 2$ tengsizlikni isbotlang.

34. (Vyetnam -2001) Musbat x,y,z sonlar $\frac{1}{\sqrt{2}}\leq z<\frac{1}{2}\min\left\{x\sqrt{2},y\sqrt{3}\right\}$, $x+z\sqrt{3}\geq\sqrt{6}$, $y\sqrt{3}+z\sqrt{10}\geq 2\sqrt{5}$ shartlarni qanoatlantirsa, $P(x,y,z)=\frac{1}{x^2}+\frac{2}{y^2}+\frac{3}{z^2}$ ifodaning eng katta qiymatini toping.

35. (AQSh -2003) Musbat a,b,c sonlar uchun

$$\frac{\left(2a+b+c\right)^2}{2a^2+(b+c)^2}+\frac{\left(2b+c+a\right)^2}{2b^2+(a+c)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

tengsizlikni isbotlang.

36. (Albaniya -2004) Musbat a,b,c sonlarning ko'paytmasi birga teng bo'lsa,

$$\frac{1}{\sqrt{a+\frac{1}{b}+0,64}}+\frac{1}{\sqrt{b+\frac{1}{c}+0,64}}+\frac{1}{\sqrt{c+\frac{1}{a}+0,64}}\geq 1,2$$

tengsizlikni isbotlang.

37. Musbat x, y, z sonlar $xyz=1$ shartni qanoatlantirsa,

$$x+y+z-\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\geq\frac{1}{x(1+z)}+\frac{1}{y(1+x)}+\frac{1}{z(1+y)}$$

tengsizlikni isbotlang.

38. Musbat x, y, z sonlar uchun

$$3(x^2-x+1)(y^2-y+1)(z^2-z+1)\geq(xyz)^2+xyz+1$$

tengsizlikni isbotlang.

39. (Belorussiya -1997) $n\in N,\ n>1$ uchun

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}<n-n^{\frac{n-1}{n}}$$

tengsizlikni isbotlang.

40. Birdan katta haqiqiy x, y, z sonlar $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ tenglikni kanoatlantirsa, $\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$ tengsizlikni isbotlang.

41. (Hindiston -2002) Musbat a,b,c sonlar $a^2+b^2+c^2=3abc$ tenglikni qanoatlantirsa, $\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq\frac{9}{a+b+c}$ tengsizlikni isbotlang.

42. (Ukraina -2002) Musbat x, y, z sonlar uchun

$$\frac{1}{(x+y+z)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}$$ t

tengsizlikni isbotlang.

43. (Ukraina -2001) $a_1,a_2,...,a_n$ haqiqiy sonlar

$a_1+a_2+...+a_n\geq n^2$, $a_1^2+a_2^2+...+a_n^2\leq n^2+1$ shartlarni qanoatlantirsa,

$n-1\leq a_k\leq n+1$ tengsizlikni isbotlang.

44. (Ukraina -2001) Musbat a,b,c *ва* α,β,γ $\alpha+\beta+\gamma=1$ sonlar uchun

$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)}\leq a+b+c$ tengsizlikni isbotlang.

45. (Ukraina -2000) Musbat a,b sonlar uchun

$$\frac{1}{a^3}+\frac{1}{3ab^2}+\frac{1}{3a^2b}+\frac{1}{b^3}\geq\frac{64}{3(a+b)^3}$$

tengsizlikni isbotlang.

46. (Ukraina -1999) Haqiqiy $x_1,x_2,...,x_6\in[0;1]$ sonlar uchun

$$\frac{x_1^3}{x_2^5+x_3^5+x_4^5+x_5^5+x_6^5+5}+\frac{x_2^3}{x_1^5+x_3^5+x_4^5+x_5^5+x_6^5+5}+...+\frac{x_6^3}{x_1^5+x_2^5+x_3^5+x_4^5+x_5^5+5}\leq\frac{3}{5}$$

tengsizlikni isbotlang.

47. (Bosniya -2002) Musbat a,b,c sonlari $a^2+b^2+c^2=1$ shartni qanoatlantir-sa,

$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ac}+\frac{c^2}{1+2ab}\geq\frac{3}{5}$ tengsizlikni isbotlang.

48. (Yugoslaviya -2002) Musbat $x_1x_2.....x_{2001}$ sonlar uchun

$$x_i^2\geq x_1^2+\frac{x_2^2}{2^3}+\frac{x_3^2}{3^3}+...+\frac{x_{i-1}^2}{(i-1)^3},\quad 2\leq i\leq 2001$$

shart bajarilsa, $\sum_{i=2}^{2001} \frac{x_i}{x_1 + x_2 + \ldots + x_{i-1}} > 1{,}999$ tengsizlikni isbotlang.

49. (Hindiston -2002) Musbat a,b,c sonlar uchun

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

ekanligini ko'rsating.

50. (XMO -1998) Agar $y_i \geq 1\ (i = 1,2,...n)$ bo'lsa,

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + ... + \frac{1}{1+y_n} \geq \frac{n}{1+\sqrt[n]{y_1 y_2 ... y_n}}$$

tengsizlikni isbotlang.

51. (Hindiston -2004) Ixtiyoriy $x_i \in (0;\frac{1}{2}]$ sonlar uchun (i=1, 2,…, n)

$$\frac{\prod_{i=1}^{n} x_i}{\left(\sum_{i=1}^{n} x_i\right)^n} \leq \frac{\prod_{i=1}^{n}(1-x_i)}{\left(\sum_{i=1}^{n}(1-x_i)\right)^n}$$

tengsizlikni isbotlang.

52. (XMO -2001) $x_1, x_2, ...x_n$ haqiqiy sonlar uchun

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + ... + \frac{x_n}{1+x_1^2+x_2^2+...+x_n^2} < \sqrt{n}$$

tengsizlikni isbotlang.

53. (XMO -2004) Musbat a,b,c sonlar $ab + bc + ca = 1$ tenglikni qanoatlantirsa,

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

tengsizlikni isbotlang.

54. (XMO -2004) Musbat x, y, z, a, b, c sonlar $ab+bc+ca=1$ shartni qanoatlantirsa

$$3abc(x+y+z) \le \frac{2}{3} + ax^3 + by^3 + cz^3$$

tengsizlikni isbotlang.

55. (Moldova -2001) $a_1, a_2, ..., a_n$ haqiqiy musbat sonlar uchun

$$\frac{1}{\sum_{i=1}^{n} \frac{1}{1+a_i}} - \frac{1}{\sum_{i=1}^{n} \frac{1}{a_i}} \ge \frac{1}{n}$$

tengsizlikni isbotlang.

56. (AQSh -1992) $a_0, a_1, a_2, ..., a_n$ musbat haqiqiy sonlari $a_{i-1} \cdot a_{i+1} \le a_i^2$ $(i = 1, 2, ..., n-1)$ shartni qanoatlantirsa

$$\frac{a_0 + a_1 + ... + a_n}{n+1} \cdot \frac{a_1 + a_2 + ... + a_{n-1}}{n-1} \ge \frac{a_0 + a_1 + ... + a_{n-1}}{n} \cdot \frac{a_0 + a_1 + ... + a_n}{n}$$

tengsizlikni isbotlang.

57. (Polsha -1996) a, b, c , x, y, z nomanfiy sonlar bo'lib,

$x + y + z = a + b + c = 1, \quad 0 \le x, y, z \le \frac{1}{2}$ shartlarni qanoatlantirsa,

$$ax + by + cz \ge 8abc$$

tengsizlikni isbotlang.

58. (XMO -1998) Yig'indisi birdan kichik bo'lgan $x_1, x_2, ..., x_n$ musbat sonlar uchun

$n^{n+1} x_1 x_2 .... x_n (1 - x_1 - x_2 - ... - x_n) \le \left(x_1 + x_2 + ... + x_n\right)(1 - x_1)(1 - x_2)...(1 - x_n)$

tengsizlikni isbotlang.

59. $0 \le a, b, c \le 1$ sonlar uchun $\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1} \le 2$ tengsizlikni isbotlang.

60. a, b, c, d $\in [1;2]$ sonlar uchun $\frac{a+b}{b+c} + \frac{c+d}{d+a} \le \frac{4(a+c)}{b+d}$ tengsizlikni isbotlang.

61. (Vyetnam -2002) *a, b, c* uchburchak tomonlari va $0 \le t \le 1$ uchun

$\sqrt{\frac{a}{b+c-ta}} + \sqrt{\frac{b}{a+c-tb}} + \sqrt{\frac{c}{a+b-tc}} \ge 2\sqrt{1+t}$ tengsizlikni isbotlang.

62. Haqiqiy *a, b, c* sonlar $a+b+c=0$ shartni qanoatlantirsa,

$a^2b^2 + b^2c^2 + c^2a^2 + 3 \ge 6abc$ tengsizlikni isbotlang.

63. (O'zbekiston -2001) $x_1, x_2, \ldots, x_{2002}$ musbat sonlar $\sum_{i=1}^{2002} \frac{1}{1+x_i^2} = 1$ shartni

qanoatlantirsa, $x_1x_2 \ldots x_{2002} \ge 2001^{1001}$ tengsizlikni isbotlang.

64. (Belorussiya -1999) Agar musbat *a, b, c* sonlar $a^2+b^2+c^2=3$ tenglikni

qanoatlantirsa, $\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ac} \ge \frac{3}{2}$ tengsizlik o'rinli bo'lishini isbotlang.

65. Agar $a \ge b \ge c > 0$ va $x, y, z \in R$ bo'lsa,

$$(ax+by+cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) \le \frac{(a+c)^2}{4ac}(x+y+z)^2$$

tengsizlikni isbotlang.

66. (Singapur -2004) Agar *a, b, c* > 0 bo'lsa, u holda

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ac}{a+c+2b} \le \frac{1}{4}(a+b+c)$$

tengsizlikni isbotlang.

67. (Polsha -2006) Agar musbat *a, b, c* sonlar *ab*+*bc*+*ca*=*abc* tenglikni qanoatlantirsa, u holda $\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(a^3+c^3)}\geq 1$ tengsizlikni isbotlang.

68. (Sankt-Peterburg -2006) Agar *a, b, c, d* musbat sonlar $a^2+b^2+c^2+d^2=1$ tenglikni qanoatlantirsa, u holda $a+b+c+d+\frac{1}{abcd}\geq 18$ tengsizlikni isbotlang.

69. (APMO , Osiyo va Tinch Okean qirg'og'i davlatlari olimpiadasi -2004) Agar *x, y, z* musbat sonlar bo'lsa,

$$(x^2+2)(y^2+2)(z^2+2)\geq 9(xy+yz+zx)$$

tengsizlik o'rinli bo'lishini ko'rsating.

70. (Xitoy -2004) Agar natural $a_1\leq a_2\leq ...\leq a_n$ sonlar ketma-ketligi $\sum_{i=1}^{n}\frac{1}{a_i}<1$ shartni qanoatlantirsa $\left(\sum_{i=1}^{n}\frac{1}{a_i^2+x^2}\right)^2\leq\frac{1}{2a_1^2-2a_1+2x^2}$ tengsizlikni isbotlang. ($x\in R$).

71. (Rossiya -2002) Musbat x,y,z sonlar $x+y+z=3$ shartni qanoatlantirsa, $\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx$ tengsizlikni isbotlang.

72. (Belorussiya -2002) Natural a,b sonlar uchun $\left|a\sqrt{2}-b\right|>\frac{1}{2(a+b)}$ tengsizikni isbotlang.

73. (Belorussiya -2002) Musbat haqiqiy a,b,c,d sonlar uchun

$$\sqrt{(a+c)^2+(b+d)^2} \le \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \le \sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$$

tengsizlikni isbotlang.

74. (APMO -2002) Musbat a,b,c sonlar $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ shartlarni qanoatlantirsa,

$\sqrt{a+bc}+\sqrt{b+ac}+\sqrt{c+ab} \ge \sqrt{abc}+\sqrt{a}+\sqrt{b}+\sqrt{c}$ ttengsizlikni isbotlang.

75. (APMO -1996) a,b,c uchburchak tomonlari bo'lsa,

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$$

tengsizlikni isbotlang.

76. Musbat a,b,c sonlar $a+b+c=3$ shartni qanoatlantirsa,

$2(a^3b+b^3c+c^3a) \ge 3(a^2b+b^2c+c^2a-1)$ tengsizlikni isbotlang.

77. (APMO -1998) a,b,c musbat sonlar uchun

$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$ tengsizlikni isbotlang.

78. (Singapur -2001) $n \in N$ va $a_1,a_2,...,a_n$ sonlar $\sum_{i=1}^{n} a_i = 1$ shartni qanoatlantirsa,

$\frac{a_1^4}{a_1^2+a_2^2}+\frac{a_2^4}{a_2^2+a_3^2}+...+\frac{a_n^4}{a_n^2+a_1^2} \ge \frac{1}{2n}$ tengsizlikni isbotlang.

79. (XMO -2000) Musbat a,b,c sonlar $abc=1$ shartni qanoatlantirsa,

$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1$ tengsizlikni isbotlang.

80. (Qozog'iston -2000) Yig'indisi birga teng bo'lgan *a, b, c* sonlar uchun

$\frac{a^7+b^7}{a^5+b^5}+\frac{b^7+c^7}{b^5+c^5}+\frac{c^7+a^7}{c^5+a^5}\geq\frac{1}{3}$ tengsizlikni isbotlang.

81. (Yaponiya -2002) Musbat x, y sonlar uchun $x+y+\frac{2}{x+y}+\frac{1}{2xy}\geq\frac{7}{2}$ tengsizlikni isbotlang.

82. (Pol'sha -1996) Musbat a,b,c sonlarning yig'indisi birga teng bo'lsa,

$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}$ tengsizlik o'rinli bo'lishini isbotlang.

83. (Gonkong -2005) Musbat a,b,c,d sonlarning yig'indisi birga teng bo'lsa,

$6\left(a^3+b^3+c^3+d^3\right)\geq\left(a^2+b^2+c^2+d^2\right)+\frac{1}{8}$ tengsizlikni isbotlang.

84. (Kanada -1998) Istalgan natural n soni $(n\geq 2)$ uchun

$\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2n}\right)$ tengsizlikni isbotlang.

85. (Bosniya -2002) Agar $a>0$ va $0<b<1$ bo'lsa, $\sqrt{1+a^2}+\sqrt{1-b^2}\leq\frac{a}{b}+\frac{b}{a}$ tengsizlikni isbotlang.

86. (Polsha -1995) Agar $x_1=\frac{1}{2}$ va $x_n=\frac{2n-3}{2n}x_{n-1},\ n>1$ bo'lsa, $\sum_{i=1}^{n}x_i<1$ tengsizlik o'rinli bo'lishini isbotlang.

87. (Bosniya -2002) Agar $x_i \in \left(0;\frac{\pi}{2}\right)$ $(i=1,2,...,n)$ sonlari $\sum_{i=1}^{n} tgx_i \le n$ shartni qanoatlantirsa, $\sin x_1 \cdot \sin x_2 \cdot ... \cdot \sin x_n \le 2^{-\frac{n}{2}}$ tengsizlikni isbotlang.

88. (Belorussiya -2000) Musbat $a,b,c;x,y,z$ sonlar uchun

$\frac{a^6}{x}+\frac{b^6}{y}+\frac{c^6}{z} \ge \frac{\left(a^2+b^2+c^2\right)}{3\left(x+y+z\right)}$ tengsizlikni isbotlang.

89. (AQSh -1997) Istalgan musbat a,b,c sonlar uchun

$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \le \frac{1}{abc}$ tengsizlikni isbotlang.

90. (Pol'sha -2000) Aytaylik $x_i > 0$ $(i=1,2,...,n)$ va $n>2$ bo'lsin.

$x_1+2x_2+3x_3+...+nx_n \le \frac{n(n-1)}{2}+x_1+x_2^2+x_3^3+...+x_n^n$ tengsizlikni isbotlang.

91. (Gretsiya -2002) Agar a,b,c musbat sonlar $a^2+b^2+c^2=1$ shartni qanoatlantirsa, $\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1} \ge \frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$ tengsizlikni isbotlang.

92. (Ukraina -2002) $a_i \ge 1$ $(i=1,2,...,n)$, $n \ge 1$, $A=1+a_1+a_2+...+a_n$

$x_k=\frac{1}{1+a_k x_{k-1}}$, $1 \le k \le n$ deb belgilash kiritsak, u holda $x_1+x_2+...+x_n > \frac{n^2 A}{n^2+A^2}$ tengsizlikni isbotlang.

93. (Sankt Peterburg -2004) Musbat a,b,c sonlar uchun

$$\frac{ab}{3a+b}+\frac{bc}{b+2c}+\frac{ac}{c+2a} \le \frac{2a+20b+27c}{49}$$

tengsizlikni isbotlang.

94. (Irlandiya -1998) $x \neq 0$ son uchun

$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0$$

tengsizlikni isbotlang.

95. (Eron -1998) Birdan katta x, y, z sonlar $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ shartni qanoatlantirsa,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

tengsizlikni isbotlang.

96. (Vyetnam-1998) $x_1, x_2, ..., x_n$ $(n \geq 2)$ musbat sonlar

$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + ... + \frac{1}{x_n + 1998} = \frac{1}{1998}$ tenglikni qanoatlantirsa,

$\frac{\sqrt[n]{x_1 x_2 ... x_n}}{n-1} \geq 1998$ tengsizlikni isbotlang.

Yechimlar.

1. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan munosabatga ko'ra,

$$\underbrace{\frac{a^{n+k}}{b^k}+....+\frac{a^{n+k}}{b^k}}_{n}+\underbrace{b^n+b^n+....+b^n}_{k}\geq (n+k)\sqrt[n+k]{\frac{\left(a^{n+k}\right)^n b^{nk}}{b^{nk}}}=(n+k)a^n$$

yoki

$$n\cdot\frac{a^{n+k}}{b^k}+k\cdot b^h\geq (n+k)a^n.$$

Xuddi shunday,

$$n\cdot\frac{b^{n+k}}{c^k}+kc^n\geq (n+k)b^n,$$

$$n\cdot\frac{c^{n+k}}{a^k}+k\cdot a^n\geq \left(n+k\right)c^n$$

tengsizliklarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shib,

$$\frac{a^{n+k}}{b^k}+\frac{b^{n+k}}{c^k}+\frac{c^{n+k}}{a^k}\geq a^n+b^n+c^n$$

ni hosil qilamiz.

2. Ushbu $\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$ tengsizlikdan foydalanamiz:

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\frac{1}{2}\left(\frac{1}{b+c}+\frac{1}{a+b}\right)+\frac{1}{2}\left(\frac{1}{b+c}+\frac{1}{a+c}\right)+\frac{1}{2}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\geq$$

$$\frac{1}{2}\left(\frac{4}{2b+a+c}+\frac{4}{b+a+2c}+\frac{4}{b+2a+c}\right)=\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}.$$

3. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanib,

$$\frac{2}{4-ab}=\frac{4}{8-2ab}\leq\frac{4}{8-a^2-b^2}\leq\frac{1}{4-a^2}+\frac{1}{4-b^2}$$

tengsizlikni hosil qilamiz. Xuddi shunday,

$$\frac{2}{4-bc} \le \frac{1}{4-b^2} + \frac{1}{4-c^2}, \quad \frac{2}{4-ac} \le \frac{1}{4-a^2} + \frac{1}{4-c^2} \quad .$$

Bu tengsizliklarni hadma-had qo'shib,

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ac} \le \frac{1}{4-a^2} + \frac{1}{4-b^2} + \frac{1}{4-c^2}$$

tengsizlikni hosil qilamiz. Berilgan $a^4 + b^4 + c^4 = 3$ shartdan $a^2 < 2$ ekanligi kelib chiqadi.Bundan quyidagi

$$\left(a^2 - 1\right)^2 (2 - a^2) \ge 0 \Leftrightarrow \frac{1}{4-a^2} \le \frac{a^4+5}{18}$$

tengsizlik o'rinli. Xuddi shunday,

$$\frac{1}{4-b^2} \le \frac{b^4+5}{18}, \quad \frac{1}{4-c^2} \le \frac{c^4+5}{18}$$

tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib,

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ca} \le \frac{1}{4-a^2} + \frac{1}{4-b^2} + \frac{1}{4-c^2} \le \frac{a^4+5}{18} + \frac{b^4+5}{18} + \frac{c^4+5}{18} = 1$$

ekanligini hosil qilamiz.

4. Berilgan tengsizlikni chap tomonida turgan qo'shiluvchilarni mos ravishda *A,B,C* deb belgilaymiz.

$$\left(by + cz\right)(bz + cy) = (b^2 + c^2)yz + bc(y^2 + z^2) \le \left(\frac{y^2 + z^2}{2}\right)(b^2 + c^2) + bc(y^2 + z^2) =$$

$$= \frac{1}{2}\left(y^2 + z^2\right)\left(b + c\right)^2 \Rightarrow A \ge 2\left(\frac{a}{b+c}\right)^2 \frac{x^2}{y^2 + z^2}$$

Xuddi shunday , $B \ge 2\left(\frac{b}{a+c}\right)^2 \frac{y^2}{t^2 + x^2}, C \ge 2\left(\frac{c}{a+b}\right)^2 \frac{z^2}{x^2 + y^2}$ tengsizliklarni hosil qilamiz.Berilgan shartlarga ko'ra

$$\frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b}, \quad \frac{x^2}{y^2+z^2} \ge \frac{y^2}{z^2+x^2} \ge \frac{z^2}{x^2+y^2}$$

munosabatlar o'rinli. Chebishev tengsizligini qo'llasak,

$$A+B+C \ge 2\cdot\frac{1}{3}\left\{\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{a+c}\right)^2+\left(\frac{c}{a+b}\right)^2\right\}\left\{\frac{x^2}{y^2+z^2}+\frac{y^2}{x^2+z^2}+\frac{z^2}{x^2+y^2}\right\} \ge$$

$$\ge 2\cdot\frac{1}{3}\cdot\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\left(\frac{x^2}{y^2+z^2}+\frac{y^2}{z^2+x^2}+\frac{z^2}{x^2+y^2}\right).$$

Musbat α,β,γ sonlar uchun $\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}\ge\frac{3}{2}$ tengsizlikni isbotlaymiz.

$\alpha+\beta=\tau,\quad \beta+\gamma=s,\ \gamma+\alpha=t$ belgilash kiritib,

$$\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}=\frac{\tau+t-s}{2s}+\frac{\tau+s-t}{2t}+\frac{s+t-\tau}{2\tau}=$$

$$=\frac{1}{2}\left(\frac{\tau}{s}+\frac{t}{s}+\frac{\tau}{t}+\frac{s}{t}+\frac{s}{\tau}+\frac{t}{\tau}-3\right)\ge\frac{1}{2}(2+2+2-3)=\frac{3}{2}$$

ni hosil qilamiz. Bundan $A+B+C\ge 2\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\left(\frac{3}{2}\right)^2\cdot\frac{3}{2}=\frac{3}{4}$

Tenglik $a=b=c$ va $x=y=z$ bo’lganda bajariladi.

5. $t=a_1^2+a_2^2+...+a_n^2$ deb belgilab,

$$\begin{aligned}&a_1a_2+a_2a_3+...+a_na_{n-1}+a_na_1\le\\ &\le a_1a_2+a_1a_3+a_1a_4+...+a_1a_n+\\ &+a_2a_3+a_2a_4+...+a_2a_n+\\ &+a_3a_4+...+a_3a_n+\\ &+.....+\\ &+a_{n-1}a_n=\\ &=\frac{1}{2}\left\{(a_1+a_2+...a_n)^2-(a_1^2+a_2^2+...a_n^2)\right\}=\frac{1}{2}(1-t)\end{aligned}$$

munosabatlarni hosil qilamiz. Bu yerdan $t<1$ kelib chiqadi.

Koshi-Bunyakovskiy-Shvarts tengsizligini qo’llab,

$$(a_1b_1+a_2b_2+...+a_nb_n)^2\le(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_2^2)=t$$

yoki $a_1b_1+a_2b_2+...+a_nb_n\le\sqrt{t}$ tengsizlikni topamiz. Bundan

$$a_1(b_1+a_2)+a_2(b_2+a_3)+...+a_n(b_n+a_1)=(a_1b_1+a_2b_2+...+a_nb_n)+$$
$$+(a_1a_2+a_2a_3+....+a_na_1)\le\sqrt{t}+\frac{1}{2}(1-t)=-\frac{1}{2}(\sqrt{t}-1)^2+1<1$$

6. Berilgan tengsizlikni chap tomonini T bilan belgilab, o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanamiz, ya'ni:

$$T=\sqrt{\frac{ab}{ab+1-a-b}}+\sqrt{\frac{bc}{bc+1-b-c}}+\sqrt{\frac{ac}{ac+1-a-c}}=$$
$$=\sqrt{\frac{ab}{(1-a)(1-b)}}+\sqrt{\frac{bc}{(1-b)(1-c)}}+\sqrt{\frac{ac}{(1-a)(1-c)}}\le$$
$$\le\frac{1}{2}\left(\frac{a}{1-b}+\frac{b}{1-a}\right)+\frac{1}{2}\left(\frac{b}{1-c}+\frac{c}{1-b}\right)+\frac{1}{2}\left(\frac{a}{1-c}+\frac{c}{1-a}\right)\le$$
$$\le\frac{1}{2}\left(\frac{a}{a+c}+\frac{b}{b+c}+\frac{b}{b+a}+\frac{c}{c+a}+\frac{a}{b+a}+\frac{c}{b+c}\right)=\frac{3}{2}$$

7. Tengsizlikni ikkala qismidagi qavslarni ochib ixchamlasak

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ \ (*)$$

ifoda hosil bo'ladi.Endi $\frac{a}{b}=x,\ \frac{b}{c}=y,\ \frac{c}{a}=z$ deb belgilash kiritsak, u holda (*) tengsizlik

$$x^2+y^2+z^2+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge x+y+z+3\ \ (**)$$

ko'rinishga keladi. $xyz=1$ ekanligidan quyidagi

$$x^2+y^2+z^2\ge\frac{(x+y+z)^2}{3}\ge x+y+z\ \ (1)$$

va $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge 3$ (2) tengsizliklar o'rinli. (1) va (2) larni hadma-had qo'shib (**) ni hosil qilamiz. Bundan (*) isbotlandi.

8. $a+b=1$ ekanligidan foydalanib yuqoridagi tengsizlikni quyidagi shakldda yozamiz:

$$\frac{1}{3}\le\frac{a^2}{(a+b)\big(a+(a+b)\big)}+\frac{b^2}{(a+b)\big(b+(a+b)\big)}$$

Bundan $a^2b+ab^2\le a^3+b^3$ yoki $\left(a^3+b^3\right)-\left(a^2b+ab^2\right)=\left(a-b\right)^2\left(a+b\right)\ge 0$

9. $yz(y^2+z^2)=y^3z+yz^3\le y^4+z^4$ tengsizlik o'rinli, chunki

$$y^4-y^3z-yz^3+z^4=\left(y^3-z^3\right)(y-z)\ge 0\ \Rightarrow x(y^4+z^4)\ge xyz\left(y^2+z^2\right)\ge y^2+z^2$$

yoki

$$\frac{x^5}{x^5+y^2+z^2}\ge\frac{x^5}{x^5+x(y^4+z^4)}=\frac{x^4}{x^4+y^4+z^4}\ .$$

Xuddi shunday,

$$\frac{y^5}{y^5+x^2+z^2}\ge\frac{y^4}{x^4+y^4+z^4},\ \frac{z^5}{z^5+x^2+y^2}\ge\frac{z^4}{x^4+y^4+z^4}\ .$$

Bu tengsizliklarni hadma-had ko'shib, isboti talab qilingan tengsizlikni hosil qilamiz.

10. Yuqoridagi tengsizlikni quyidagicha yozib olamiz:

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{y^2+x^2+z^2}{x^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le 3$$

va Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llab,

$$(x^5+y^2+z^2)(yz+y^2+z^2)\ge(x^{\frac{5}{2}}(yz)^{\frac{1}{2}}+y^2+z^2)^{2}\ge(x^2+y^2+z^2)^2$$

yoki $\dfrac{x^2+y^2+z^2}{x^5+y^2+z^2}\le\dfrac{yz+y^2+z^2}{x^2+y^2+z^2}$. Xuddi shunday,

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}\le\frac{xz+x^2+z^2}{x^2+y^2+z^2},\quad \frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le\frac{xy+x^2+y^2}{x^2+y^2+z^2}$$

munosabatlarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shsak,

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+x^2+z^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le 2+\frac{xy+yz+zx}{x^2+y^2+z^2}\le 3$$

11. α,β,γ uchburchak burchaklari uchun

$1=\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$ tenglikdan foydalanib,

$$\frac{\cos\alpha}{\cos\beta\cos\gamma}\cdot\frac{\cos\beta}{\cos\gamma\cos\alpha}\cdot\frac{\cos\gamma}{\cos\alpha\cos\beta}=\frac{\cos\alpha}{\cos\beta\cos\gamma}+\frac{\cos\beta}{\cos\gamma\cos\alpha}+\frac{\cos\gamma}{\cos\alpha\cos\beta}+2$$

ifodani hosil qilamiz .

$$x=\frac{\cos\alpha}{\cos\beta\cos\gamma},\ y=\frac{\cos\beta}{\cos\gamma\cos\alpha},\ z=\frac{\cos\gamma}{\cos\alpha\cos\beta}$$

deb belgilash kiritib, $\cos\alpha+\cos\beta+\cos\gamma\le\frac{3}{2}$ tengsizlikdan foydalansak,

$$\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}+}\frac{1}{\sqrt{zx}}\le\frac{3}{2}\Leftrightarrow\ 2(\sqrt{x}+\sqrt{y}+\sqrt{z})\le3\sqrt{xyz}\Leftrightarrow$$

$$4(x+y+z+2(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}))\le9xyz\Leftrightarrow$$

$$8(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})\le9(x+y+z+2)-4(x+y+z)=5(x+y+z)+18$$

12. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanib,

$$\sqrt{1+a^3}=\sqrt{(1+a)(1+a^2-a)}\le\frac{2+a^2}{2},\ \sqrt{1+b^3}=\sqrt{(1+b)(1-b+b^2)}\le\frac{2+b^2}{2},$$

$$\sqrt{1+c^3}=\sqrt{(1+c)(1-c+c^3)}\le\frac{2+c^2}{2}$$

munosabatlarni topamiz. Endi quyidagi tengsizlikni isbotlasak yetarli:

$$\frac{4a^2}{(2+a^2)(2+b^2)}+\frac{4b^2}{(2+b^2)(2+c^2)}+\frac{4c^2}{(2+c^2)(2+a^2)}\ge\frac{4}{3}\Leftrightarrow$$

$$3\left(a^2(2+c^2)+b^2(2+a^2)+c^2(2+b^2)\right)\ge(2+a^2)(2+b^2)(2+c^2)\Leftrightarrow$$

$$a^2b^2+b^2c^2+c^2a^2+2(a^2+b^2+c^2)\ge a^2b^2c^2+8=72$$

Bu tengsizlik quyidagi tengsizliklarni hadma-had qo'shishdan hosil qilinadi:

$$a^2b^2+b^2c^2+c^2a^2\ge3\sqrt[3]{(abc)^4}=48,\quad 2(a^2+b^2+c^2)\ge6\sqrt[3]{a^2b^2c^2}=24$$

Bulardan isboti talab qilingan tengsizlikni hosil qilamiz.

13. Birinchi navbatda $x+y^2 \geq x^2+y^3$ tengsizlikni isbotlaymiz. Faraz qilaylik, $x+y^2 < x^2+y^3$ bo'lsin. $x^3+y^4 \leq x^2+y^3$ tengsizlikdan foydalansak, farazimizga zid bo'lgan $2\left(x^2+y^3\right) \geq \left(x+x^3\right)+\left(y^2+y^4\right) \geq 2x^2+2y^3$ tengsizlik hosil bo'ladi. Shuning uchun

$$x+y^2 \geq x^2+y^3 \geq x^3+y^4 \Rightarrow 2\left(x+y^2\right) \geq x^2+y^3+x^3+y^4 \geq$$
$$\geq x^3+y^3+2x-1+2y^2-1 \Rightarrow x^3+y^3 \leq 2.$$

14. Umumiylikni chegaralamasdan $a \geq b \geq c$ deb olib, uchburchak tengsiz-ligini qo'llasak, $1=a+b+c>2a \Rightarrow b \leq a < \frac{1}{2}$ va bundan

$$a^n+b^n < \frac{1}{2^n}+\frac{1}{2^n}=\frac{2}{2^n} \Rightarrow \left(a^n+b^n\right)^{\frac{1}{n}} < \frac{2^{\frac{1}{n}}}{2} \quad (*) \ .$$

Endi qo'yidagini qaraymiz:

$$\left(b+\frac{c}{2}\right)^n = b^n+\frac{n}{2}c\,b^{n-1}+....+\frac{c^n}{2^n} > b^n+c^n \ \text{(chunki } \frac{n}{2}cb^{n-1}>c^n\text{)}.$$

Xuddi shunday,

$$\left(a+\frac{c}{2}\right)^n > a^n+c^n \ .$$

Demak,

$$\left(b^n+c^n\right)^{\frac{1}{n}}+\left(a^n+b^n\right)^{\frac{1}{n}} < b+\frac{c}{2}+a+\frac{c}{2}=1. \quad (**)$$

(*) va (**) larni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

15. $\frac{A_{k-1}}{A_k}=x_k$ va $x_1=1$ deb belgilash kiritsak,

$$n\sqrt[n]{\frac{G_n}{A_n}}=n\cdot\sqrt[n^2]{\frac{A_1A_2...A_n}{A_n^n}}=n\cdot\sqrt[n^2]{\frac{A_1}{A_2}\left(\frac{A_2}{A_3}\right)^2\left(\frac{A_3}{A_4}\right)^3...\left(\frac{A_{n-1}}{A_n}\right)^{n-1}}=$$

$$= n\sqrt[n^2]{x_2 x_3^2 x_4^3 \ldots x_n^{n-1}} = n\sqrt[n^2]{x_1^{\frac{n(n+1)}{2}} \cdot x_2\, x_3^2\, x_4^3 \ldots . x_n^{n-1}};$$

$$\frac{a_k}{A_k} = \frac{kA_k - (k-1)A_{k-1}}{A_k} = k - (k-1)\frac{A_{k-1}}{A_k} = k - (k-1)x_k;$$

$$\frac{g_n}{G_n} = \sqrt[n]{\frac{a_1}{A_1} \cdot \frac{a_2}{A_2} \ldots \frac{a_n}{A_n}} = \sqrt[n]{1.(2 - x_2)(3 - 2x_3)\ldots(n - (n-1)x_k)}\,.$$

Umumlashgan Koshi tengsizligidan foydalansak

$$(a_i > 0,\, \alpha_i > 0,\, i = 1,2,\ldots,n \qquad \sqrt[\alpha_1+\alpha_2+\alpha_n]{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}} \le \frac{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n}{\alpha_1 + \alpha_2 + \ldots + \alpha_n}),$$

$$n\sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} = n\sqrt[n^2]{x_1^{\frac{n(n+1)}{2}} x_2 x_3^2 \ldots x_n^{n-1}} + \sqrt[n]{1 \cdot (2 - x_2)(3 - 2x_3)\ldots(n - (n-1)x_n)} \le$$

$$\le \frac{1}{n}\left(\frac{n(n+1)}{2} + x_2 + 2x_3 + \ldots + (n-1)x_n\right) + \frac{1}{n}(1 + (2 - x_2) + (3 - 2x_3) + \ldots + (n-1)x_n)$$

$$\frac{n+1}{2} + \frac{1}{n}(x_2 + 2x_3 + \ldots + (n-1)x_n) + \frac{(n+1)}{2} - \frac{1}{n}(x_2 + 2x_3 + \ldots + (n-1)x_n) = n + 1\,.$$

16. Bu tengsizlikning chap tomonini S deb belgilab, quyidagi usulda o’rta arifmetik va o’rta geometrik miqdorlar o’rtasidagi munosabatni qo’llaymiz:

$$S \le a\left(\frac{1 + 1 + (1 + b - c)}{3}\right) + b\left(\frac{1 + 1 + (1 + c - a)}{3}\right) + c\left(\frac{1 + 1 + (1 + a - b)}{3}\right) =$$

$$\frac{3a + 3b + 3c + ab + ac + bc - ba - ca - cb}{3} = 1$$

17.

$$\frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + cd}{1 + c} + \frac{1 + da}{1 + d} = \left(\frac{1 + ab}{1 + a} + \frac{1 + cd}{1 + c}\right) + \left(\frac{1 + bc}{1 + b} + \frac{1 + da}{1 + d}\right) =$$

$$= \left(\frac{1 + ab}{1 + a} + \frac{1 + ab}{ab(1 + c)}\right) + \left(\frac{1 + bc}{1 + b} + \frac{1 + bc}{bc(1 + d)}\right) =$$

$$= (1 + ab)\left(\frac{1}{1 + a} + \frac{1}{ab(1 + c)}\right) + (1 + bc)\left(\frac{1}{1 + b} + \frac{1}{bc(1 + d)}\right) \ge$$

$$\ge \frac{4(1 + ab)}{1 + a + ab(1 + c)} + \frac{4(1 + bc)}{1 + b + bc(1 + d)} = 4$$

18. Bu tengsizlikni chap tomonini T bilan belgilab, umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagicha qo'llaymiz:

$$T\cdot\Big(a\big(b+1\big)+b(c+1)+c(a+1)\Big)\geq(a+b+c)^3=1$$

yoki $T\geq\dfrac{1}{ab+bc+ca+1}$ tengsizlikni va undan

$$T\geq\frac{1}{ab+bc+ca+1}\geq\frac{1}{\frac{(a+b+c)^2}{3}+1}=\frac{3}{4}$$

munosabatni hosil qilamiz.

-19.

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}=\frac{a(1-b-c)+b}{b+c}+\frac{b(1-a-c)+c}{c+a}+\frac{c(1-a-b)+a}{a+b}=$$

$$=\frac{a+b}{b+c}-a+\frac{b+c}{c+a}-b+\frac{c+a}{c+b}-c=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}-1\geq$$

$$\geq3\sqrt[3]{\frac{a+b}{b+c}\cdot\frac{b+c}{c+a}\cdot\frac{c+a}{a+b}}-1=2$$

20. Ixtiyoriy $x,y,z>0$ uchun $\left(x-\sqrt{yz}\right)^2\geq0\Leftrightarrow x^2+yz\geq2\sqrt{x^2yz}\Leftrightarrow$

$$x^2+xy+xz+yz\geq xy+2\sqrt{x^2yz}+xz\Leftrightarrow(x+y)(x+z)\geq\left(\sqrt{xy}+\sqrt{xz}\right)^2\Leftrightarrow$$

$$\sqrt{(x+y)(x+z)}\geq\sqrt{xy}+\sqrt{xz}$$

munosabatni topamiz. Bundan

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(x+y)(z+y)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq$$

$$\leq\frac{x}{x+\sqrt{xy}+\sqrt{xz}}+\frac{y}{y+\sqrt{yx}+\sqrt{yz}}+\frac{z}{z+\sqrt{zx}+\sqrt{zy}}=$$

$$=\frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}+\frac{\sqrt{y}}{\sqrt{y}+\sqrt{x}+\sqrt{z}}+\frac{\sqrt{z}}{\sqrt{z}+\sqrt{x}+\sqrt{y}}=1$$

21. Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llab,

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\le\sqrt{\left(\frac{2a(a+b+c)}{(a+b)(a+c)}+\frac{2b(a+b+c)}{(b+c)(b+a)}+\frac{2c(a+b+c)}{(c+a)(c+b)}\right)}\times$$

$$\times\sqrt{\left(\frac{a+c}{2(a+b+c)}+\frac{b+a}{2(a+b+c)}+\frac{c+b}{2(a+b+c)}\right)}=$$

$$=\sqrt{2\cdot(a+b+c)\left(\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\right)}$$

munosabatni hosil qilamiz. Endi

$$(a+b+c)\left(\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\right)=$$

$$=\frac{2(a+b+c)(ab+ac+bc)}{(a+b)(b+c)(c+a)}\le\frac{9}{4}$$

yoki

$$8\big(a+b+c\big)(ab+bc+ca)\le 9(a+b)(b+c)(c+a)\Leftrightarrow$$

$$6abc\le ab(a+b)+bc(b+c)+ac(a+c)$$

tengsizlikni isbotlash yetarli. Bu tengsizlik esa o’rta arifmetik va o’rta geometrik miqdorlar o’rtasidagi munosabatga ko’ra o’rinli. Bulardan yuqoridagi isboti talab etilgan tengsizlik isbotlandi.

22. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligidan quyidagicha foydalanamiz:

$$(a+3b)(b+4c)(c+2a)=(a+b+b+b)(b+c+c+c+c)(c+a+a)\ge$$

$$\ge 4\sqrt[4]{ab^3}\cdot 5\sqrt[5]{b\cdot c^4}\cdot 3\sqrt[3]{a^2c}=60\ a^{\frac{11}{12}}\cdot b^{\frac{19}{20}}\cdot c^{\frac{17}{15}}=$$

$$=60abc\frac{c^{\frac{2}{15}}}{a^{\frac{1}{12}}\cdot b^{\frac{1}{20}}}=60\ abc\frac{c^{\frac{1}{12}}\cdot c^{\frac{1}{20}}}{a^{\frac{1}{12}}b^{\frac{1}{20}}}=60abc\left(\frac{c}{a}\right)^{\frac{1}{12}}\cdot\left(\frac{c}{b}\right)^{\frac{1}{20}}\ge 60abc$$

23. $a=\frac{x}{y},\ b=\frac{y}{t},\ c=\frac{t}{x}$ deb belgilash kiritsak, u holda

$$4\left(\sqrt[3]{\frac{xt}{y^2}}+\sqrt[3]{\frac{yx}{t^2}}+\sqrt[3]{\frac{yt}{x^2}}\right)\le 3\left(2+\frac{x}{y}+\frac{y}{t}+\frac{t}{x}+\frac{y}{x}+\frac{t}{y}+\frac{x}{t}\right)^{\frac{2}{3}}$$

bundan

$$4\sqrt[3]{xyt}\left(\frac{1}{y}+\frac{1}{x}+\frac{1}{t}\right)\le 3\left(2+\frac{x}{y}+\frac{y}{t}+\frac{t}{x}+\frac{y}{t}+\frac{t}{y}+\frac{x}{t}\right)^{\frac{2}{3}},$$

$$\frac{4}{\sqrt[3]{(xyt)^2}}(xy+yt+tx)\le 3\left(2+\frac{x+t}{y}+\frac{t+y}{x}+\frac{t+x}{t}\right)^{\frac{2}{3}}.$$

Bundan $64(xy+yt+tx)^3\le 27((xy+yt+tx)(x+y+t)-xyt)^2$ tengsizlikni isbotlasak yetarli.

$$27((x+y+t)(xy+yt+tx)-xyt)^2\ge$$

$$\ge 27\left((x+y+t)(xy+yt+tx)-\frac{(x+y+t)(xy+yt+tx)}{9}\right)^2=$$

$$=27\left(\frac{8}{9}(x+y+t)(xy+yt+tx)\right)^2=64(xy+yt+tx)^2\frac{(x+y+t)^2}{3}\ge 64(xy+yt+tx)^3$$

24. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligiga ko’ra

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right)^{\frac{1}{3}},\qquad a+b+c\ge 3\sqrt[3]{abc}$$

tengsizliklar o’rinli. Bulardan $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\ge 9$ ekanligini topamiz. Bu tengsizlik va $a+b+c\le\sqrt{3(a^2+b^2+c^2)}$ tengsizliklarni hadma-had ko’paytirib,

$3\sqrt{3}\le\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\sqrt{a^2+b^2+c^2}\right)$ va bundan

$$\frac{\sqrt{3}+1}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a^2+b^2+c^2)\ge\sqrt{3(a^2+b^2+c^2)}+\sqrt{a^2+b^2+c^2}\ge$$

$$\ge(a+b+c)+\sqrt{a^2+b^2+c^2}.$$

25. Musbat x son uchun x^2-1 va x^3-1 ifodalar bir xil ishoraladir, ya’ni

$0 \le (x^2-1)(x^3-1) = x^5 - x^3 - x^2 + 1$ yoki $x^5 - x^2 + 3 \ge x^3 + 2$. Bu tengsizlikdan foydalansak, u holda $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a^3+2)(b^3+2)(c^3+2)$. Bundan $\left(a^3+2\right)(b^3+2)(c^3+2) \ge (a+b+c)^3$ tengsizlikni isbotlash yetarli. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$$(a^3+2)(b^3+2)(c^3+2) = (a^3+1+1)(1+b^3+1)(1+1+c^3) \ge (a+b+c)^3$$

26. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$$\begin{gathered}(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) = \\ = (a^2+ab+b^2)(c^2+b^2+bc)(ac+a^2+c^2) \ge (ac+ab+bc)^3\end{gathered}$$

27. Agar $a,b,c>1$ bo'lsa, $a^2+b^2+c^2+abc>4$ bo'ladi. Agar $a \le 1$ bo'lsa, u holda $ab+bc+ca-abc \ge bc-abc = bc(1-a) \ge 0$. Endi $ab+bc+ca-abc \le 2$ tengsizlikni isbotlaymiz. $a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C$ va $A,\ B,\ C \in \left[0, \frac{\pi}{2}\right]$ deb belgilash kiritsak, shartga ko'ra $A+B+C=\pi$ ekanligini topamiz va

$\cos A\cos B + \cos B\cos C + \cos A\cos C - 2\cos A\cos B\cos C \le \frac{1}{2}$

tengsizlikni isbotlasak yetarli bo'ladi. Faraz etaylik, $A \ge \frac{\pi}{3}$ yoki $1 - 2\cos A \ge 0$

.Bundan

$$\begin{gathered}\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C = \\ = \cos A(\cos B + \cos C) + \cos B\cos C(1 - 2\cos A)\end{gathered}$$

Quyidagi $\cos B + \cos C \le \frac{3}{2} - \cos A$ va

$2\cos B\cos C = \cos(B-C) + \cos(B+C) \le 1 - \cos A$ tengsizliklardan foydalansak,

$\cos A(\cos B + \cos C) + \cos B\cos C(1-2\cos A) \le \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A)$

28. $x \neq 0$ bo'lgani uchun $x^2+y^2>0$ bo'ladi. Bundan $\dfrac{(2xy)^2}{(x^2+y^2)^2}+\dfrac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)^2}=1$

ekanligini topamiz. Oxirgi tenglikda har bir qo'shiluvchi [-1;1] oraliqqa tegishli ekanligidan

$$\frac{(2xy)^2}{\left(x^2+y^2\right)^2}=\sin^2\alpha \ \text{ va } \ \frac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)^2}=\cos^2\alpha$$

deb belgilash kiritish mumkin. Bundan

$$\cos^2\alpha\sin^2\alpha=\frac{4\left(xy\left(x^2-y^2\right)\right)^2}{\left(x^2+y^2\right)^4}=\frac{4}{\left(x^2+y^2\right)^2},\ (x^2+y^2)^2=\frac{16}{\sin^2 2\alpha}\geq 16,$$

$$x^2+y^2\geq 4.$$

29. $a=\dfrac{x}{3}, b=\dfrac{4y}{5}, c=\dfrac{3z}{2}$ deb belgilash kiritsak, u holda masalaning sharti

quyidagicha ko'rinishga ega bo'ladi: $7xy+3yz+5xz\leq 15$

O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini yuqoridagi tengsizlikka qo'llab $15\geq 7xy+3yz+5xz\geq 15\sqrt[15]{x^{12}y^{10}z^8}$ yoki

$x^6y^5z^4\leq 1$ (*) tengsizlikni topamiz.

Endi (*) dan foydalansak:

$$P(a,b,c)=\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=\frac{3}{x}+\frac{5}{2y}+\frac{2}{z}=\underbrace{\frac{1}{2x}+\ldots+\frac{1}{2x}}_{6\ ma}+\underbrace{\frac{1}{2y}+\ldots+\frac{1}{2y}}_{5\ ma}+$$

$$+\underbrace{\frac{1}{2z}+\ldots+\frac{1}{2z}}_{4\ ma}\geq\frac{15}{2}\sqrt[15]{\frac{1}{x^6y^5z^4}}\geq\frac{15}{2}$$

Tenglik $x=y=z=1$ yoki $a=\dfrac{1}{3}, b=\dfrac{4}{5},\ c=\dfrac{2}{3}$ bo'lganda bajariladi.

30. Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llasak,

$$\frac{a^2}{b}-b+\frac{b^2}{c}-c+\frac{c^2}{a}-a=\frac{\left(a-b\right)^2}{a}+\frac{(b-c)^2}{b}+\frac{\left(a-c\right)^2}{c}=$$

$$=\frac{1}{a+b+c}\left(\frac{\left(a-b\right)^2}{a}+\frac{(b-c)^2}{b}+\frac{(a-c)^2}{c}\right)(a+b+c)\geq$$

$$\frac{1}{a+b+c}\left(|a-b|+|b-d|+|c-d|\right)^2=$$

$$=\frac{1}{a+b+c}\left(2\max\{a,b,c\}-2\min\{a,b,c\}\right)^2\geq\frac{4(a-b)^2}{a+b+c}$$

31. Umumiylikni chegaralamasdan $a_1\leq a_2\leq\ldots\leq a_{i-1}\leq 0\leq a_i\leq a_{i+1}\leq\ldots.\leq a_n$ deb olamiz va $a_k=-b_k$ $(k=1,2,\ldots,i-1)$, $b_k>0$ deb belgilaymiz, u holda $a_i\leq a_{i+1}\leq\ldots\leq a_n$ va $b_1\geq b_2\geq\ldots\geq b_{i-1}$, $b_1+b_2+..+b_{i-1}=a_i+a_{i+1}+..+a_n$ bo'ladi.

$na_1a_n=-nb_1a_n$ ekanligidan $\sum_{i=1}^{n}a_i^2\leq nb_1a_n$ ko'rsatish yetarli.

$$nb_1a_n=\left(\underbrace{b_1a_n+b_1a_n+\ldots+b_1a_n}_{(i-1)-ma}\right)+\left(\underbrace{b_1a_n+b_1a_n+\ldots+b_1a_n}_{(n+1-i)-ma}\right)=$$

$$=b_1\left(a_n+a_n+\ldots+a_n\right)+a_n\left(b_1+b_1+\ldots+b_1\right)\geq b_1\left(a_i+a_{i+1}+\ldots+a_n\right)+$$
$$+a_n\left(b_1+b_2+\ldots+b_{i-1}\right)=b_i\left(b_1+b_2+\ldots+b_{i-1}\right)+a_n\left(a_i+a_{i+1}+\ldots+a_n\right)\geq$$
$$\geq b_1^2+b_2^2+\ldots+b_{i-1}^2+a_i^2+a_{i+1}^2+\ldots+a_n^2=\sum_{i=1}^{n}a_i^2.$$

32. Tengsizlikni

$$\frac{n-1}{n-1+x_1}+\frac{n-1}{n-1+x_2}+\ldots+\frac{n-1}{n-1+x_n}\leq n-1$$

shaklda yozib, undan

$$\frac{x_1}{n-1+x_1}+\frac{x_2}{n-1+x_1}+\ldots+\frac{x_n}{n-1+x_1}\geq 1$$

tengsizlikni xosil kilamiz. $y_i=\dfrac{x_i}{n-1+x_i}(*)$ belgilash kiritsak, u holda

$S = y_1 + y_2 + ... + y_n \geq 1$ tengsizlikni isbotlash yetarli. O'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatga ko'ra,

$$S - y_1 \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_1}},$$

$$S - y_2 \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_2}},$$

…………………

$$S - y_n \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_n}}$$

tengsizliklar o'rinli. Bu tengsizliklarning mos kismlarini kupaytirib

$$(S - y_1)(S - y_2)...(S - y_n) \geq (n-1)^n y_1 y_2 ... y_n;$$

tengsizlikni va (*) ko'ra $x_i = \dfrac{(n-1)y_i}{1 - y_i}$ yoki

$(n-1)^n y_1 y_2 ... y_n = (1 - y_1)(1 - y_2)...(1 - y_n)$ bulardan

$(S - y_1)(S - y_2)...(S - y_n) \geq (1 - y_1)(1 - y_2)...(1 - y_n)$ (**) munosabatni xosil kilamiz. Agar $0 < S < 1$ bo'lsa, $S - y_i < 1 - y_i$ yoki $\prod_{i=1}^{n}(S - y_i) \leq \prod_{i=1}^{n}(1 - y_i)$. Bu (**) ga ziddir. Demak $S \geq 1$ ekan.

33. Masalaning shartidan $|x| = \sqrt{2 - y^2 - z^2}$ va $yz \leq 1$ ekanligini ko'rish mumkin. Koshi-Bunyakovskiy-Shvarts tengsizliginidan quyidagi usulda foydalanib,

$$x + y + z - xyz = x(1 - yz) + y + z \leq |x| \cdot |1 - yz| + |y + z| =$$

$$= \sqrt{2 - y^2 - z^2}\,|1 - yz| + |y + z| \leq \sqrt{\left(\left(2 - y^2 - z^2\right) + \left(y + z\right)^2\right)\left(\left(1 - yz\right)^2 + 1\right)} =$$

$$= \sqrt{\left(2 + 2yz\right)\left(2 - 2yz + y^2 z^2\right)}$$

munosabatni hosil qilamiz. Endi $\left(1 + yz\right)(2 - 2yz + y^2 z^2) \leq 2$ ekanligini ko'rsatish yetarli. Bu tengsizlikning chap tomonidagi qavslarni ochib ixchamlash natijasida $y^3 z^3 \leq y^2 z^2$ yoki $yz \leq 1$ ni xosil qilamiz, bundan esa $x + y + z - xyz \leq 2$ tengsizlik isbotlandi.

34. $\frac{1}{x\sqrt{2}}=a,\ \frac{1}{y\sqrt{3}}=b,\ \frac{1}{2z}=c$ deb belgilash kiritsak, u holda

$Q(a,b,c)=2a^2+6b^2+12c^2$ ifodaning eng katta qiymatini topsak masala yechiladi. a,b,c musbat sonlar quyidagi shartlarni qanoatlantiradi:

$$\max\{a,b,\}<c\le\frac{1}{2} \qquad (1)$$

$$c\sqrt{2}+a\sqrt{3}\ge 2\sqrt{6}ac \qquad (2)$$

$$c\sqrt{2}+b\sqrt{5}\ge 2\sqrt{10}bc \qquad (3)$$

(2) dan

$$\frac{\sqrt{2}}{a}+\frac{\sqrt{3}}{c}\ge 2\sqrt{6}\Rightarrow\frac{2}{a^2}+\frac{3}{c^2}\ge 12\Rightarrow\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)\ge 2a^2,$$

Bundan

$$a^2+c^2=2a^2+c^2-a^2\le\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+c^2\left(1-\frac{a^2}{c^2}\right)\le\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+\frac{1}{2}\left(1-\frac{a^2}{c^2}\right)=\frac{5}{6}$$

Xuddi shunday (1) va (3) dan $b^2+c^2\le\frac{7}{10}$ ekanligini topamiz.

Shunday qilib, $Q(a,b,c,)=2(a^2+c^2)+6(b^2+c^2)+4c^2\le\frac{118}{15}$

Tenglik $Q(a,b,c)=Q\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{2}}\right)=\frac{118}{15}$ bajariladi va $a=\frac{1}{\sqrt{3}},\ b=\frac{1}{\sqrt{5}},\ c=\frac{1}{\sqrt{2}}$

qiymatlar (1)-(2)-(3) shartlarni qanoatlantiradi.

Bundan $\max P(x,y,z)=\max Q(a,b,c)=\frac{118}{15}$.

35. $a+b=x,\ \ b+c=y,\ \ c+a=t$ belgilash kiritsak yuqoridagi tengsizlik quyidagi ko'rinishga keladi:

$$\frac{2(x+t)^2}{(x+t-y)^2+2y^2}+\frac{2(y+t)^2}{(y+t-x)^2+2x^2}+\frac{2(x+y)^2}{(x+y-t)^2+2t^2}\le 8 \quad (*)$$

Ushbu $2(t^2+p^2)\ge(t+p)^2$ tengsizlikdan foydalansak,

$$\frac{4(x+t)^2}{(2(x+t-y)^2+2y^2)+2y^2}+\frac{4\left(y+t\right)^2}{(2(y+t-x)^2+2x^2)+2x^2}+\frac{4\left(x+y\right)^2}{(2(x+y-t)^2+2t^2)+2t^2}\le$$

$$\le\frac{4\left(x+t\right)^2}{(x+t)^2+2y^2}+\frac{4\left(y+t\right)^2}{(y+t)^2+2x^2}+\frac{4\left(x+y\right)^2}{(x+y)^2+2t^2}=$$

$$=\frac{4}{1+\frac{2y^2}{(x+t)^2}}+\frac{4}{1+2\frac{2x^2}{(y+t)^2}}+\frac{4}{1+2\frac{t^2}{(x+y)^2}}\le\frac{4}{1+\frac{y^2}{x^2+t^2}}+\frac{4}{1+\frac{x^2}{y^2+t^2}}+\frac{4}{1+\frac{t^2}{x^2+y^2}}=$$

$$=\frac{4\left(x^2+t^2\right)}{x^2+y^2+t^2}+\frac{4\left(y^2+t^2\right)}{x^2+y^2+t^2}+\frac{4\left(x^2+y^2\right)}{x^2+y^2+t^2}=8.$$

Bundan (*) isbotlandi.

36. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo’llaymiz:

$$\frac{0,6}{\sqrt{0,36(a+\frac{1}{b}+0,64)}}+\frac{0,6}{\sqrt{0,36(b+\frac{1}{c}+0,64)}}+\frac{0,6}{\sqrt{0,36(c+\frac{1}{a}+0,64)}}\ge$$

$$\ge1,2\left(\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1}\right)=1,2$$

Chunki

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1}=\frac{1}{ab+b+1}+\frac{1}{bc+c+1}+\frac{1}{ac+a+1}=$$

$$=\frac{ac}{ac(ab+b+1)}+\frac{a}{a(bc+c+1)}+\frac{1}{ac+a+1}=\frac{ac}{a+ac+1}+\frac{a}{1+ac+a}+\frac{1}{ac+a+1}=1$$

37. Bu tengsizlikni shakl almashtirish natijasida

$$\left(z-\frac{y+1}{(x+1)y}\right)+\left(x-\frac{z+1}{(y+1)z}\right)+\left(y-\frac{x+1}{(z+1)x}\right)\geq 0 \quad \text{yoki}$$

$\frac{x+z}{x+1}+\frac{x+y}{y+1}+\frac{y+z}{z+1}\geq 3$ (*)ga teng kuchli tengsizlikka olib kelamiz.

Koshi-Bunyakovskiy-Shvarts tengsizligiga ko'ra $x+1\leq\sqrt{(x+xy)(x+z)}$

munosabat o'rinli. Bundan $x+z\geq\frac{(x+1)^2}{x(1+y)}$ yoki $\frac{x+z}{x+1}\geq\frac{x+1}{x(1+y)}$ munosabatni

hosil qilamiz. Demak

$$\frac{x+z}{x+1}+\frac{x+y}{y+1}+\frac{z+y}{z+1}\geq\frac{x+1}{x(1+y)}+\frac{1+y}{y(1+z)}+\frac{1+z}{z(1+x)}\geq$$
$$\geq 3\sqrt[3]{\frac{x+1}{x(1+y)}\cdot\frac{1+y}{y(1+z)}\cdot\frac{1+z}{z(1+x)}}=3$$

38. Avval $\forall t>0$ uchun $3(t^2-t+1)^3\geq t^6+t^3+1$ (*) tengsizlik o'rinli ekanligini ko'rsatamiz. (*) ni shakl almashtirib, quyidagi munosabatni hosil qilamiz: $(t-1)^4(2t^2-t+2)\geq 0$ bu tengsizlik $\forall t>0$ uchun o'rinli. Bundan

$3(x^2-x+1)^3 3(y^2-y+1)^3 3(z^2-z+1)^3\geq(x^6+x^3+1)(y^6+y^3+1)(z^6+z^3+1)$ (**)

munosabatni hosil qilamiz. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llasak,

$(x^6+x^3+1)(y^6+y^3+1)(z^6+z^3+1)\geq(x^2y^2z^2+xyz+1)^3$ (***)

(**) va (***) larni hadma-had ko'paytirib, isboti talab etilgan tengsizlikni hosil qilamiz.

39. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llaymiz:

$$1+\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+...+\left(1-\frac{1}{n}\right)=1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{n-1}{n}\geq n\sqrt[n]{1\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}...\frac{n-1}{n}}=$$

$$=n\sqrt[n]{\frac{1}{n}}=n^{\frac{n-1}{n}}.$$

40. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\leq\sqrt{x+y+z}\cdot\sqrt{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}=\sqrt{x+y+z}$$

munosabatni hosil qilamiz.

41. Berilgan tengsizlikdan $(a^3+b^3+c^3)(a+b+c)\geq 9a^2b^2c^2=(a^2+b^2+c^2)^2$ tengsizlikni hosil qilamiz. Bu tengsizlik esa Koshi - Bunyakovskiy tengsizligiga ko'ra o'rinlidir.

42. O'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatni quyidagi usulda qo'llaymiz:

$$\frac{1}{(x+y+t)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{9}\left(\frac{9}{(x+y+z)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{8}{9}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geq$$

$$\geq\frac{4}{9}\sqrt[4]{\frac{9}{(x+y+z)^2x^2y^2z^2}}+\frac{8}{3}\sqrt[3]{\frac{1}{x^2y^2z^2}}\geq\frac{4}{9}\sqrt{\frac{3}{(x+y+z)xyz}}+\frac{8}{3}\sqrt{\frac{3}{xyz(x+y+z)}}=$$

$$=\frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}$$

43. $d_k=|n-a_k|$ belgilash kiritsak, u holda

$$\sum_{k=1}^{n}d_k^2=\sum_{k=1}^{n}\left(n-a_k\right)^2=n\cdot n^2-2n\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}a_k^2\leq n^3-2n\cdot n^2+n^3+1=1$$

munosabatni hosil bo'ladi. Bundan $d_k\leq 1$ yoki $n-1\leq a_k\leq n+1$.

44. Tengsizlikni ikkala qismini $a+b+c$ ga bo'lib va

$x=\frac{a}{a+b+c},\ y=\frac{b}{a+b+c},\ z=\frac{c}{a+b+c}$ belgilash kiritsak

$$\alpha x+\beta y+\gamma z+2\sqrt{\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)\left(xy+yz+zx\right)}\le\frac{\alpha^2}{2}+\frac{x^2}{2}+\frac{\beta^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+\frac{\gamma^2}{2}+$$

$$+\gamma\alpha+\alpha\beta+\beta\gamma+xy+yz+zx=\frac{1}{2}\left(\alpha+\beta+\gamma\right)^2+\frac{1}{2}\left(x+y+z\right)^2=\frac{1}{2}+\frac{1}{2}=1.$$

45. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$\frac{1}{a^3}+\frac{1}{3ab^2}+\frac{1}{3a^2b}+\frac{1}{b^3}=(\frac{1}{a^3}+\frac{1}{b^3})+\frac{1}{3ab}\left(\frac{1}{a}+\frac{1}{b}\right)\ge2\sqrt{\frac{1}{a^3b^3}}+\frac{2}{3ab}\sqrt{\frac{1}{ab}}=\frac{8}{3\sqrt{(ab)^3}}\ge$$

$$\ge\frac{64}{3(a+b)^3}$$

46. $x_1,x_2....x_6\in\left[0;1\right]$ ekanligidan tengsizlikning chap qismi quyidagi ifodadan kichik yoki teng

$$\frac{x_1^3}{x_1^5+x_2^5+.....+x_6^5+4}+\frac{x_2^3}{x_1^5+x_2^5+......+x_6^5+4}+...+\frac{x_6^3}{x_1^5+x_2^5+.....+x_6^5+4}=$$

$$=\frac{x_1^3+x_2^3+...+x_6^3}{x_1^5+x_2^5+...+x_6^5+4}\le\frac{3}{5}.(*)$$

Ixtiyoriy $t\ge0$ uchun $3t^5+2\ge5t^3\Leftrightarrow\left(t-1\right)^2\left(3t^3+6t^2+4t+2\right)\ge0$ munosabat o'rinli ekanligidan foydalansak (*) kelib chiqadi.

47. Tengsizlikni chap qismini S bilan belgilab, Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$S\cdot\left(a^2\left(1+2bc\right)+b^2(1+2ac)+c^2(1+2ab)\right)\ge(a^2+b^2+c^2)^2$ yoki

$$S\ge\frac{1}{1+2abc(a+b+c)}$$

munosabatni hosil qilamiz. $a^2+b^2+c^2 \geq \sqrt{3abc(a+b+c)}$ tengsizlikka ko'ra

$$S \geq \frac{1}{1+\frac{2}{3}3abc(a+b+c)} \geq \frac{1}{1+\frac{2}{3}\left(a^2+b^2+c^2\right)^2} = \frac{3}{5}$$

ekanligini hosil qilamiz.

48. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\left(\frac{i(i-1)}{2}\right)^2 \cdot x_i^2 \geq \left(x_1^2+\frac{x_2^2}{2^3}+\frac{x_3^2}{3^3}+\ldots+\frac{x_{i-1}^2}{(i-1)^3}\right)\left(1+2^3+3^3+\ldots+(i-1)^3\right) \geq$$

$$\geq \left(x_1+x_2+x_3+\ldots+x_{i-1}\right)^2 \text{ ёки } \frac{x_i}{x_1+x_2+\ldots+x_{i-1}} \geq \frac{2}{i(i-1)}, \quad 2 \leq i \leq 2001$$

ekanligini topamiz. Bundan

$$\sum_{i=2}^{2001} \frac{x_i}{x_1+x_2+\ldots+x_{i-1}} \geq 2\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{2000\cdot 2001}\right) = 2\left(1-\frac{1}{2001}\right) > 1{,}999.$$

49. Tengsizlikni $\left(\frac{a}{b}-\frac{c+a}{c+b}+1\right)+\left(\frac{b}{c}-\frac{a+b}{a+c}+1\right)+\left(\frac{c}{a}-\frac{b+c}{b+a}+1\right) \geq 3$ yoki

$\frac{b^2+ac}{b(c+b)}+\frac{c^2+ab}{c(a+c)}+\frac{a^2+bc}{a(b+a)} \geq 3$ kurinishda yozamiz.Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llasak,

$$\frac{b^2+ac}{b(c+b)}+\frac{c^2+ab}{c(a+c)}+\frac{a^2+bc}{a(a+b)} \geq \frac{(b+c)a}{b(a+c)}+\frac{b(a+c)}{c(a+b)}+\frac{c(a+b)}{a(b+c)} \geq$$

$$\geq 3\sqrt[3]{\frac{(b+c)a}{b(a+c)}\cdot\frac{b(a+c)}{c(a+b)}\cdot\frac{c(a+b)}{a(b+c)}} = 3.$$

50. Avval tengsizlikni induktsiya metodi yordamida $n=2^k, k \in N$ sonlar uchun isbotlaymiz. $n=2$ da

$$\frac{1}{y_1+1}+\frac{1}{y_2+1}-\frac{2}{\sqrt{y_1y_2}+1} = \frac{\left(\sqrt{y_1y_2}-1\right)\left(\sqrt{y_1}-\sqrt{y_2}\right)^2}{(y_1+1)(y_2+1)(\sqrt{y_1y_2}+1)} \geq 0$$

$n=2^k$ da tengsizlik o'rinli bo'lsin deb faraz qilsak, u holda $n=2^{k+1}$ da

$$\left(\frac{1}{1+y_1}+\frac{1}{1+y_2}+...+\frac{1}{1+y_{2^k}}\right)+\left(\frac{1}{1+y_{2^k+1}}+\frac{1}{1+y_{2^k+2}}+....+\frac{1}{1+y_{2^{k+1}}}\right)\geq$$

$$\geq\frac{2^k}{1+\sqrt[2^k]{y_1y_2...y_{2^k}}}+\frac{2^k}{1+\sqrt[2^k]{y_{2^k+1}y_{2^k+2}...y_{2^{k+1}}}}\geq\frac{2^{k+1}}{1+\sqrt[2^{k+1}]{y_1\cdot y_2....y_{2^{k+1}}}}$$

munosabat o'rinli va tengsizlik $n=2^k,\ k\in N,$ uchun isbotlandi. Endi tengsizlikni $n\in N$ uchun isbotlaymiz. Buning uchun $m=2^k>n, k\in N,$ uchun tengsizlik o'rinli deb faraz etsak va $y_{n+1}=y_{n+2}=....=y_m=\sqrt[n]{y_1y_2..y_n}$ deb olsak, u holda

$$\frac{1}{y_1+1}+\frac{1}{y_2+1}+...+\frac{1}{y_n+1}+\frac{m-n}{1+\sqrt[n]{y_1y_2...y_n}}\geq\frac{m}{\sqrt[n]{y_1y_2...y_n}+1}$$

bo'ladi. Bundan yuqoridagi tengsizlik isbotlanadi.

51. $x_i=\cos^2\alpha_i$, $\frac{\pi}{4}\leq\alpha_i<\frac{\pi}{2}$ belgilash kiritsak, u holda tengsizlik

$$\frac{\prod_{i=1}^{n}\cos^2\alpha_i}{\prod_{i=1}^{n}(1-\cos^2\alpha_i)}\leq\left(\frac{\sum_{i=1}^{n}\cos^2\alpha_i}{\sum_{i=1}^{n}(1-\cos^2\alpha_i)}\right)^n$$

ko'rinishga, yoki

$$\frac{1}{1+tg^2\alpha_1}+\frac{1}{1+tg^2\alpha_2}+...+\frac{1}{1+tg^2\alpha_n}\geq\frac{n}{1+\sqrt[n]{tg^2\alpha_1tg^2\alpha_2...tg^2\alpha_n}}$$

ko'rinishga keladi. Agar $tg^2\alpha_i=y_i$ deb belgilash kiritsak, $y_i\geq1$ bo'ladi va $\frac{1}{1+y_1}+\frac{1}{1+y_2}+...\frac{1}{1+y_n}\geq\frac{n}{1+\sqrt[n]{y_1y_2...y_n}}$ ni isbotlash kerak bo'ladi.Bu tengsizlik esa 50- masalada isbotlangan.

52. Tengsizlikni musbat $x_1,x_2,...,x_n$ sonlar uchun isbotlash yetarli. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\frac{x_1}{1+x_1^2}+\frac{x_2}{1+x_1^2+x_2^2}+....+\frac{x_n}{1+x_1^2+x_2^2+...+x_n^2}<$$

$$<\sqrt{n\left(\left(\frac{x_1}{1+x_1^2}\right)^2+\left(\frac{x_2}{1+x_1^2+x_2^2}\right)^2+...+\left(\frac{x_n}{1+x_1^2+...+x_n^2}\right)^2\right)}$$

munosabatni hosil qilamiz. Bundan

$$\frac{x_1^2}{(1+x_1^2)^2}+\frac{x_2^2}{(1+x_1^2+x_2^2)^2}+..+\frac{x_n^2}{(1+x_1^2+x_2^2+...+x_n^2)^2}<1$$

ekanligini ko’rsatsak, yuqoridagi tengsizlik isbotlanadi.

$$\frac{{x_k}^2}{\left(1+x_1^2+x_2^2+...+x_k^2\right)}\le\frac{x_k^2}{(1+x_1^2+x_2^2+...+x_{k-1}^2)(1+x_1^2+...+x_k^2)}=$$

$$=\frac{1}{1+x_1^2+...+x_{k-1}^2}-\frac{1}{1+x_1^2+...+x_k^2}$$

bo’lgani uchun

$$\sum_{k=1}^{n}\left(\frac{x_k}{1+x_1^2+...+x_k^2}\right)^2<1-\frac{1}{1+x_1^2+...+x_n^2}<1.$$

53. $9(a^3+b^3+c^3)\ge(a+b+c)^3$ tengsizlikdan foydalanib,

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\le\sqrt[3]{9\left(\frac{1}{a}+6b+\frac{1}{b}+6c+\frac{1}{c}+6a\right)}=$$

$$=\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{abc}+3\frac{1-ab}{c}+3\frac{1-bc}{a}+3\frac{1-ac}{b}}=\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{4-3\left((ab)^2+(bc)^2+(ca)^2\right)}{abc}}\le$$

$$\le\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{4-3\frac{(ab+bc+ca)^2}{3}}{abc}}=\frac{3}{\sqrt[3]{abc}}$$

munosabatni hosil qilamiz.Endi $\frac{3}{\sqrt[3]{abc}}\le\frac{1}{abc}$ yoki $a^2b^2c^2\le\frac{1}{27}$ ekanligi

ko’rsatamiz: $(abc)^2=(ab)(bc)(ca)\le\left(\frac{ab+bc+ca}{3}\right)^3=\frac{1}{27}$.

54. $1-bc=a(b+c)$ ekanligidan

$$3abcx=3a\sqrt[3]{b^2c\cdot c^2b\cdot x^3}\le a(b^2c+c^2b+x^3)=ax^3+bc(1-bc)=$$

$$=ax^3+bc\left(\frac{2}{3}-bc\right)+\frac{1}{3}bc\le ax^3+\frac{1}{3}bc+\left(\frac{bc+(\frac{2}{3}-bc)}{2}\right)^2=ax^3+\frac{1}{3}bc+\frac{1}{9}$$

munosabatni hosil qilamiz. Xuddi shunday

$$3abcy\le by^3+\frac{1}{3}ac+\frac{1}{9},\ 3abcz\le cz^3+\frac{1}{3}ab+\frac{1}{9}$$

tengsizliklar o'rinli.bu tengsizliklarni hadma-had qo'shib isboti talab etilgan tengsizlikni hosil qilamiz.

55. Tengsizlikni ikkala qismiga umumiy maxraj tanlab,

$$\sum_{i=1}^{n}\frac{1}{a_i}-\sum_{i=1}^{n}\frac{1}{1+a_i}\ge\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\cdot\sum_{i=1}^{n}\frac{1}{1+a_i}\quad \text{yoki}\quad n\sum_{i=1}^{n}\frac{1}{a_i(a_i+1)}\ge\sum_{i=1}^{n}\frac{1}{1+a_i}\sum_{i=1}^{n}\frac{1}{a_i}$$

munosabatni hosil qilamiz. Umumiylikni chegaralamasdan $a_1\ge a_2\ge...\ge a_n$ deb olsak, oxirgi tengsizlik Chebishev tengsizligia ko'ra o'rinlidir.

56. $a_1+a_2+....+a_{n-1}=k$ deb olamiz. U holda isboti talab etilgan tengsizlikka teng kuchli bo'lgan

$$\frac{a_0+k+a_n}{n+1}\cdot\frac{k}{n-1}\ge\frac{a_0+k}{n}\cdot\frac{a_n+k}{n}\Leftrightarrow$$
$$n^2k(a_0+a_n+k)\ge(n^2-1)(a_0+k)(a_n+k)\Leftrightarrow k(k+a_0+a_n)\ge a_0a_n(n^2-1).$$

tengsizlikni hosil qilamiz. $a_0+a_n\ge2\sqrt{a_0a_n}$ ekanligidan $k\ge\left(n-1\right)\sqrt{a_0a_n}$ tengsizlikni isbotlasak yuqoridagi tengsizlik isbotlandi.

$a_{i-1}\cdot a_{i+1}\le a_i^2\quad(i=1,2,....,n-1)$ tengsizlikka ko'ra $\frac{a_0}{a_1}\le\frac{a_1}{a_2}\le\frac{a_2}{a_3}\le...\le\frac{a_{n-1}}{a_n}$ yoki $a_0a_n\le a_1a_{n-1}\le a_2a_{n-2}\le......$ munosabatni hosil qilamiz. Bundan,

$$K = a_1 + a_2 + ... + a_{n-1} = (a_1 + a_{n-1}) + (a_2 + a_{n-2}) + ... \geq 2\sqrt{a_1 a_{n-1}} +$$

$$+2\sqrt{a_2 a_{n-2}} + ... \geq 2\sqrt{a_0 a_n} + 2\sqrt{a_0 a_n} + ... = 2 \cdot \frac{n-1}{2}\sqrt{a_0 a_n} = (n-1)\sqrt{a_0 a_n}$$

ekanligi kelib chiqadi.

57. Umumiylikni chegaralamasdan $a \leq b \leq c$ deb olamiz. U holda o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llab,

$$8abc = 8ab(1-a-b) \leq 2(a+b)^2(1-a-b) = 2(a+b)\left[(a+b)(1-a-b)\right] \leq \frac{a+b}{2}$$

tengsizlikni topamiz. $(a-c)(x-\frac{1}{2}) + (b-c)(y-\frac{1}{2}) \geq 0$ munosabat o'rinli ekanligidan $ax + by + cz \geq \frac{a+b}{2} \geq 8abc$ tengsizlikni to'g'riligini topamiz.

58. $x_{n+1} = 1 - x_1 - x_2 - ... - x_n$ bo'lsin. U holda $x_{n+1} > 0$ va o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llasak

$$1 - x_i = x_1 + x_2 + ... + x_{n+1} - x_i \geq n\sqrt[n]{\frac{x_1 x_2 ... x_{n+1}}{x_i}} \quad (i = 1, 2, ..., n+1)$$ munosabatni olamiz.

Bu tengsizliklarni hadma-had ko'paytirib,

$$\prod_{i=1}^{n+1}(1-x_i) \geq \prod_{i=1}^{n+1} n\sqrt[n]{\frac{x_1 x_2 ... x_{n+1}}{x_i}} = n^{n+1} x_1 x_2 ... x_n x_{n+1} = n^{n+1} x_1 x_2 ... x_n (1 - x_1 - x_2 - ... - x_n)$$

yoki $(1-x_1)(1-x_2)...(1-x_n)(x_1 + x_2 + ... + x_n) \geq n^{n+1} x_1 x_2 ... x_n (1 - x_1 - x_2 ... - x_n)$ tengsizlikni hosil qilamiz.

59. Umumiylikni chegaralamasdan $0 \leq a \leq b \leq c \leq 1$ deb olamiz. U holda $(1-a)(1-b) \geq 0$ yoki $a + b + c \leq a + b + 1 \leq 2 + ab$<2(1+ab) ekanligini topamiz.

Bundan,

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1} \leq \frac{a}{ab+1} + \frac{b}{ab+1} + \frac{c}{ab+1} = \frac{a+b+c}{ab+1} < \frac{2(ab+1)}{ab+1} = 2$$

munosabatni hosil qilamiz.

60. $b(a-1)+a(c-1)\geq 0 \Leftrightarrow ab+ac\geq b+a \Leftrightarrow \dfrac{a+b}{b+c}\leq a$ va

$d(c-1)+c(a-1)\geq 0 \Leftrightarrow dc+ca\geq d+c \Leftrightarrow \dfrac{d+c}{d+a}\leq c$ ekanligi rashan. Bundan

$$\frac{4(a+c)}{(b+d)}\geq \frac{4(a+c)}{4}=a+c\geq \frac{a+b}{b+c}+\frac{d+c}{d+a}$$

munosabat o'rinli ekanligi kelib chiqadi.

61. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llab $\sqrt{\dfrac{a}{b+c-ta}}=\dfrac{a\sqrt{1+t}}{\sqrt{(a+at)(b+c-ta)}}\geq \dfrac{2a\sqrt{t+1}}{a+b+c}$ tengsizlikni topamiz. Xuddi shunday $\sqrt{\dfrac{b}{a+c-tb}}\geq \dfrac{2b\sqrt{1+t}}{a+b+c}$, $\sqrt{\dfrac{c}{a+b-tc}}\geq \dfrac{2c\sqrt{1+t}}{a+b+c}$ tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

62. $(ab+1-c)^2+(bc+1-a)^2+(ac+1-b)^2\geq 0$ ekanligidan, yuqoridagi tengsizlikni o'rinli bo'lishi kelib chiqadi.

63. $\dfrac{1}{1+x_i^2}=y_i$ $(\mathrm{i}=1,2,\ \ldots,2002)$ deb belgilash kiritsak, u holda $y_1+y_2+\ldots+y_{2002}=1$ bo'ladi. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llasak, $1-y_i=y_1+y_2+\ldots+y_{2002}-y_i\geq 2001\sqrt[2001]{\dfrac{y_1y_2\ldots y_{2002}}{y_i}}$

$(i=1,2,\ \ldots\ 2002)$

ekanligini topamiz va bu tengsizliklarni hadma-had ko'paytirib

$$\prod_{i=1}^{2002}(1-y_i)\geq \prod_{i=1}^{2002}2001\sqrt[2001]{\frac{y_1y_2\ldots y_{2002}}{y_i}}=2001^{2002}y_1y_2\ldots y_{2002},\quad \prod_{i=1}^{2002}\frac{1-y_i}{y_i}\geq 2001^{2002}$$

yoki $\prod_{i=1}^{2002} x_i \geq 2001^{1001}$ tengsizlikni hosil qilamiz.

64. Musbat x, y, z va v sonlar uchun $\frac{x^2}{y}+\frac{z^2}{v}\geq\frac{(x+z)^2}{y+v}$ tengsizlik o'rinli ekanligidan foydalansak

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\geq\frac{(1+1)^2}{1+ab+1+bc}+\frac{1}{1+ac}\geq\frac{(1+1+1)^2}{3+ab+bc+ca}\geq .$$
$$\geq\frac{9}{3+a^2+b^2+c^2}=\frac{3}{2}$$

65. Tengsizlikni chap tomonini T bilan belgilab, o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llasak, u holda

$$T=\left(\frac{ax}{\sqrt{ac}}+\frac{by}{\sqrt{ac}}+\frac{cz}{\sqrt{ac}}\right)\left(\frac{x\sqrt{ac}}{a}+\frac{y\sqrt{ac}}{b}+\frac{z\sqrt{ac}}{c}\right)\leq$$
$$\leq\frac{1}{4}\left(\left(\frac{ax}{\sqrt{ac}}+\frac{x\sqrt{ac}}{a}\right)+\left(\frac{by}{\sqrt{ac}}+\frac{y\sqrt{ac}}{b}\right)+\left(\frac{cz}{\sqrt{ac}}+\frac{z\sqrt{ac}}{c}\right)\right)^2$$

tengsizlik hosil bo'ladi. Ushbu tengsizliklar o'rinli ekanligidan

$$x\left(\frac{a}{\sqrt{ac}}+\frac{\sqrt{ac}}{a}\right)\leq x\frac{a+c}{\sqrt{ac}},\ y\left(\frac{b}{\sqrt{ac}}+\frac{\sqrt{ac}}{b}\right)\leq y\frac{a+c}{\sqrt{ac}},\ z\left(\frac{c}{\sqrt{ac}}+\frac{\sqrt{ac}}{c}\right)\leq z\frac{a+c}{\sqrt{ac}}.$$

bu tengsizliklarni hadma-had qo'shib

$$S\leq\frac{1}{4}\left(\frac{a+c}{\sqrt{ac}}(x+y+z)\right)^2=\frac{(a+c)^2}{4ac}(x+y+z)^2$$

munosabatni hosil qilamiz.

66. x>0, y>0 $\frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$ tengsizlik o’rinli va bu tengsizlikni qo’llab,

$$\frac{ab}{a+b+2c}+\frac{bc}{b+c+2a}+\frac{ac}{a+c+2b} \le \frac{ab}{4}\left(\frac{1}{a+c}+\frac{1}{b+c}\right)+\frac{bc}{4}\left(\frac{1}{a+c}+\frac{1}{b+a}\right)+$$
$$+\frac{ac}{4}\left(\frac{1}{a+b}+\frac{1}{b+c}\right)=\frac{1}{4(a+c)}(ab+bc)+\frac{1}{4(b+c)}(ab+ac)+\frac{1}{4(a+b)}(ac+bc)=$$
$$=\frac{1}{4}(a+b+c)$$

munosabatni hosil qilamiz.

67. Musbat x, y sonlar uchun $\frac{x^4+y^4}{x^3+y^3} \ge \frac{x+y}{2}$ (*) tengsizlik o’rinli. Chunki

$$2(x^4+y^4) \ge (x+y)(x^3+y^3) \Leftrightarrow x^4+y^4 \ge x^3y+y^3x \Leftrightarrow (x-y)^2(x^2+xy+y^2) \ge 0.$$

Endi (*)dan foydalansak,

$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(a^3+c^3)} \ge \frac{a+b}{2ab}+\frac{b+c}{2bc}+\frac{a+c}{2ac}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$$

munosabat hosil bo’ladi.

68. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini

qo’llab, $a+b+c+d+\underbrace{\frac{1}{32abcd}+...+\frac{1}{32abcd}}_{32ma} \ge 36\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}}$

munosabatni topamiz.

Endi $36\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}} \ge 18 \Leftrightarrow 2\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}} \ge 1 \Leftrightarrow$

$\Leftrightarrow 2^{36}abcd\left(\frac{1}{32abcd}\right)^{32}=\frac{2^{36}}{2^{160}(abcd)^{31}} \ge 1 \Leftrightarrow (abcd) \le \frac{1}{2^4}$ tengsizlikni isbotlasak

masala yechiladi. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini berilgan tenglikka qo’llab,

$1=a^2+b^2+c^2+d^2\geq 4\sqrt[4]{a^2b^2c^2d^2}=4\sqrt{abcd}\Leftrightarrow\ abcd\leq\dfrac{1}{2^4}$ ekanligini topamiz.

69. Berilgan tengsizlikni chap qismidagi qavslarni ochib o’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini qo’llaymiz. U holda

$$(x^2+2)(y^2+2)(z^2+2)=x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8=$$
$$=2(x^2y^2+1)+2(y^2z^2+1)+2(z^2x^2+1)+3(x^2+y^2+z^2)+x^2y^2z^2+2+x^2+y^2+z^2\geq$$
$$\geq 4(xy+yz+zx)+3(xy+yz+zx)+(x^2+y^2+z^2)+2+x^2y^2z^2=$$
$$=x^2y^2z^2+x^2+y^2+z^2+2+7(xy+yz+zx)$$

ekanligini topamiz. Bundan $x^2y^2z^2+x^2+y^2+z^2+2\geq 2xy+2yz+2zx$ (*) tengsizlikni isbotlasak masala yechiladi.

Lemma: Istalgan *a, b, c* musbat sonlar uchun quyidagi

$(a+b-c)(b+c-a)(a+c-b)\leq abc$

tengsizlik o’rinli.

Isboti: Aytaylik *a+b-c=m, b+c-a=n, a+c-b=k* bo’lsin, bundan $a=\dfrac{m+n}{2}$, $b=\dfrac{m+n}{2}$, $c=\dfrac{k+n}{2}$ tengliklarni topamiz. U holda $8mnk\leq(m+n)(n+k)(m+k)$ ekanligini ko’rsatish yetarli. Bu tengsizlikni ushbu: $m+n\geq 2\sqrt{mn}$, $n+k\geq 2\sqrt{nk}$, $m+k\geq 2\sqrt{mk}$ tengsizliklarni hadma-had ko’paytirish natijasida hosil qilamiz.

$$(a+b-c)(b+c-a)(a+c-b)\leq abc\Leftrightarrow 3abc+a^3+b^3+c^3\geq$$
$$\geq ab(a+b)+bc(b+c)+ca(c+a)\geq 2(ab)^{\frac{3}{2}}+2(bc)^{\frac{3}{2}}+2(ca)^{\frac{3}{2}}.$$

Oxirgi tengsizlik istalgan a, b, c musbat sonlar uchun o’rinli ekanligidan, quyidagicha $a=x^{\frac{2}{3}}$, $b=y^{\frac{2}{3}}$, $c=z^{\frac{2}{3}}$ belgilash olamiz. Bundan

$3(xyz)^{\frac{2}{3}}+x^2+y^2+z^2\geq 2xy+2yz+2zx$ tengsizlikni hosil qilamiz. Bu yerdan quyidagi $2xy+2yz+2zx\leq x^2+y^2+z^2+3(xyz)^{\frac{2}{3}}\leq x^2+y^2+z^2+(xyz)+2$ munosabatni, ya’ni (*) to’g’ri ekanligini topamiz.

70. Berilgan tengsizlikni chap qismini S bilan belgilab, quyidagi usulda Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llaymiz:

$$S=\left(\sum_{i=1}^{n}\left(\frac{\sqrt{a_i}}{a_i^2+x^2}\right)\left(\frac{1}{\sqrt{a_i}}\right)\right)^2\le\left(\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}\right)\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)\le\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}.$$

Bizga ma'lumki $\left(a_i^2+x^2\right)^2>\left(a_i^2+x^2\right)^2-a_i^2>0$ (i=1, 2, …, n). Bundan

$\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}<\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{1}{a_i^2+x^2-a_i}\right)-\left(\frac{1}{a_i^2+x^2+a_i}\right)\right)$. Berilgan shartdan

$a_{i+1}\ge a_i+1$ yekanligini topamiz. Bundan $a_{i+1}^2-a_{i+1}+x^2\ge a_i^2+a_i+x^2$ va

$$\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2-a_i\right)}\le\frac{1}{\left(a_1^2+x^2-a_1\right)}+\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2+a_i\right)}-\frac{1}{a_n^2+x^2+a_n}\le\frac{1}{a_1^2+x^2-a_1}+$$

$$+\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2+a_i\right)}\Leftrightarrow\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{1}{a_i^2+x^2-a_i}\right)-\left(\frac{1}{a_i^2+x^2+a_i}\right)\right)\le\frac{1}{2a_1^2+2x^2-2a_1}$$

yoki $S\le\frac{1}{2a_1^2-2a_1+2x^2}$ munosabatni hosil qilamiz.

71. Quyidagi tenglikni qaraymiz $\left(x+y+z\right)^2=9$ yoki

$xy+yz+zx=\frac{9-x^2-y^2-z^2}{2}$. Bundan o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llab,

$$xy+yz+zx=\frac{3x-x^2}{2}+\frac{3y-y^2}{2}+\frac{3z-z^2}{2}\le\sqrt{x}+\sqrt{y}+\sqrt{z}$$

munosabatni hosil qilamiz.

72. *a* va *b* sonlar musbat va butun ekanligidan, $a\sqrt{2}\ne b$ yoki $2a^2\ne b^2$ bulardan

$\left|2a^2-b^2\right|\ge1\Leftrightarrow\left|\left(a\sqrt{2}-b\right)\left(a\sqrt{2}+b\right)\right|\ge1\Leftrightarrow\left|a\sqrt{2}-b\right|\cdot\left|a\sqrt{2}+b\right|\ge1$ munosabatni hosil qilamiz. $0 < a\sqrt{2} + b < 2a + 2b = 2(a+b)$ ekanligidan

$\left|a\sqrt{2} - b\right| \geq \frac{1}{a\sqrt{2}+b} > \frac{1}{2(a+b)}$ munosabat kelib chiqadi.

73. Avvaliga $\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$ (*) tengsizlikni isbotlaymiz: tengsizlikning ikkala qismini kvadratga oshirib, $\sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} \geq ac + bd$ yoki $(ad - bc)^2 \geq 0$ munosabatni hosil qilamiz. Endi

$$\sqrt{a^2+b^2} + \sqrt{c^2+d^2} \leq \sqrt{(a+c)^2 + (b+d)^2} + \frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}} \quad (**)$$

tengsizlikni isbotlaymiz: (**)ning ikkala qismiga umumiy mahraj tanlab va (*)dan foydalanib, $\left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2}\right)^2 \leq (a+c)^2 + (b+d)^2 + 2|ad-bc|$ (***) munosabatni hosil qilamiz. Bundan

$$\sqrt{\left(a^2+b^2\right)\left(c^2+d^2\right)} \leq ac + bd + |ad-bc| \Leftrightarrow 2(ac+bd)|ad-bc| \geq 0$$

munosabatni hosil qilamiz. Demak, (***) isbotlandi. Bulardan esa isboti talab etilgan tengsizliklar kelib chiqadi.

74. Tengsizlikni ikkala qismini $\sqrt{abc}$ ga bo'lib,

$\sqrt{\frac{1}{bc}+\frac{1}{a}} + \sqrt{\frac{1}{ac}+\frac{1}{b}} + \sqrt{\frac{1}{ab}+\frac{1}{c}} \geq 1 + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ac}} + \frac{1}{\sqrt{ab}}$ tengsizlikni hosil qilamiz.

Endi, $\sqrt{\frac{1}{bc}+\frac{1}{a}} \geq \frac{1}{a} + \frac{1}{\sqrt{bc}}$ (*) ekanligini ko'rsatamiz: (*)ni ikkala qismini kvadratga oshirib,

$$\frac{1}{bc} + \frac{1}{a} \geq \frac{1}{a^2} + \frac{2}{a\sqrt{bc}} + \frac{1}{bc} \Leftrightarrow 1 \geq \frac{1}{a} + \frac{2}{\sqrt{bc}} \Leftrightarrow \frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}} \Leftrightarrow \left(\sqrt{b} - \sqrt{c}\right)^{-2} \geq 0$$

munosabatni hosil qilamiz. Xuddi shunday:

$$\sqrt{\frac{1}{ac}+\frac{1}{b}} \geq \frac{1}{\sqrt{ac}}+\frac{1}{b}, \sqrt{\frac{1}{ab}+\frac{1}{c}} \geq \frac{1}{\sqrt{ab}}+\frac{1}{c}$$

munosabatlar o'rinli. Bu tengsizliklarni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

75. $a+b-c=x,\ b+c-a=y,\ c+a-b=z$ deb belgilash kiritsak, u holda $a=\frac{x+z}{2},\ b=\frac{x+y}{2},\ c=\frac{y+z}{2}$ tengliklarni topamiz va bu tengliklarni yuqoridagi tengsizlikka qo'yib, $\sqrt{\frac{x+z}{2}}+\sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}} \geq \sqrt{x}+\sqrt{y}+\sqrt{z}$ munosabatni hosil qilamiz. $\forall a,b>0$ sonlar uchun $\frac{\sqrt{a}+\sqrt{b}}{2} \leq \sqrt{\frac{a+b}{2}}$ (*) tengsizlik o'rinli. (*) dan foydalansak

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\frac{\sqrt{x}+\sqrt{y}}{2}+\frac{\sqrt{y}+\sqrt{z}}{2}+\frac{\sqrt{x}+\sqrt{z}}{2} \leq \sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}}+\sqrt{\frac{z+x}{2}}\ .$$

76. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$2(a^3b+b^3c+c^3a)+3 \geq (a^3b+a^3b+b)+(b^3c+b^3c+c)+(c^3a+c^3a+a) \geq$$
$$\geq 3a^2b+3b^2c+3c^2a$$

Bundan yuqoridagi tengsizlik isbotlandi.

77. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llab,

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)=\left(\frac{a}{b}+\frac{a}{c}+\frac{a}{a}\right)+\left(\frac{b}{a}+\frac{b}{c}+\frac{b}{b}\right)+\left(\frac{c}{a}+\frac{c}{b}+\frac{c}{c}\right)-1 \geq$$
$$\geq 3\frac{b}{\sqrt[3]{abc}}+3\frac{a}{\sqrt[3]{abc}}+3\frac{c}{\sqrt[3]{abc}}-1=3\frac{a+b+c}{\sqrt[3]{abc}}-1 \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

tengsizlikni hosil qilamiz.

78. Tengsizlikni chap qismini S bilan belgilab va Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$S\left(\left(a_1^2+a_2^2\right)+\left(a_2^2+a_3^2\right)+...+\left(a_n^2+a_1^2\right)\right)\geq\left(a_1^2+a_2^2+...+a_n^2\right)^2$ bundan

$S\geq\dfrac{a_1^2+a_2^2+...+a_n^2}{2}\geq\dfrac{\left(a_1+a_2+...+a_n\right)^2}{2n}=\dfrac{1}{2n}$ tengsizlik hosil bo'ladi.

79. $a=\dfrac{x}{y},b=\dfrac{y}{z},c=\dfrac{z}{x}$ deb belgilash kiritsak, u holda yuqoridagi tengsizlik quyidagi $(x-y+z)(y-z+x)(z-x+y)\leq xyz$ ko'rinishga keladi. Bu tengsizlik **69-**misolda **lemma** sifatida isbotlangan.

80. Avvaliga quyidagi lemmani isbotlaymiz:

Lemma. Musbat x,y sonlar uchun $\dfrac{x^7+y^7}{x^5+y^5}\geq\dfrac{x^3+y^3}{x+y}$ munosabat o'rinli.

Lemmaning isboti: Haqiqatdan ham

$$\left(x^7+y^7\right)(x+y)-\left(x^3+y^3\right)\left(x^5+y^5\right)=\left(x^7y-x^5y^3\right)+\left(y^7x-y^5x^3\right)=$$
$$=x^5y\left(x^2-y^2\right)-y^5x\left(x^2-y^2\right)=\left(x^2-y^2\right)\left(x^5y-y^5x\right)=xy\left(x^2-y^2\right)^2\left(x^2+y^2\right)\geq 0$$

tenglik $x=y$ bo'lganda bajariladi.

Lemmadan foydalansak,

$$\frac{a^7+b^7}{a^5+b^5}+\frac{b^7+c^7}{b^5+c^5}+\frac{c^7+a^7}{c^5+a^5}\geq\frac{a^3+b^3}{a+b}+\frac{b^3+c^3}{b+c}+\frac{c^3+a^3}{c+a}=$$
$$=\left(a^2-ab+b^2\right)+\left(b^2-bc+c^2\right)+\left(c^2-ac+a^2\right)=2\left(a^2+b^2+c^2\right)-(ab+bc+ca)$$

munosabat hosil bo'ladi.

Endi $2\left(a^2+b^2+c^2\right)-(ab+bc+ca)\geq\dfrac{1}{3}$ ekanligini ko'rsatamiz. Haqiqatdan ham $2\left(a^2+b^2+c^2\right)-(ab+bc+ca)\geq a^2+b^2+c^2\geq\dfrac{(a+b+c)^2}{3}=\dfrac{1}{3}$.

81. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$x+y+\frac{2}{x+y}+\frac{1}{2xy}=\frac{x+y}{4}+\frac{x+y}{4}+\frac{1}{x+y}+\frac{1}{x+y}+\frac{x}{2}+\frac{y}{2}+\frac{1}{2xy}\geq$$

$$\geq 7\sqrt[7]{\frac{x+y}{4}\cdot\frac{x+y}{4}\cdot\frac{1}{x+y}\cdot\frac{1}{x+y}\cdot\frac{x}{2}\cdot\frac{y}{2}\cdot\frac{1}{2xy}}=\frac{7}{2}$$

82. Avvaliga quyidagi $f(x)=\frac{x}{1+x^2}$, $x\in[0;1]$ funktsiyani hossalarini o'rganamiz. Ko'rinib turibdiki, ushbu funktsiya ko'rsatilgan oraliqda qavariq funktsiyadir. U holda qavariq funktsiyalar uchun ushbu $g(x)+g(y)+g(z)\leq 3g\left(\frac{x+y+z}{3}\right)$ Iensen tensizligidan foydalanib,

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}=f(a)+f(b)+f(c)\leq 3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{1}{3}\right)=\frac{9}{10}$$

munosabatni hosil qilamiz.

83. Umumiylikni chegaralamasdan $a\geq b\geq c\geq d$ va $a^2\geq b^2\geq c^2\geq d^2$ deymiz. U holda Chebishev tengsizligini qo'llab, quyidagi

$\left(a^2+b^2+c^2+d^2\right)=\left(a+b+c+d\right)\left(a^2+b^2+c^2+d^2\right)\leq 4\left(a^3+b^3+c^3+d^3\right)$ yoki

$6\left(a^3+b^3+c^3+d^3\right)\geq\frac{3}{2}\left(a^2+b^2+c^2+d^2\right)$ (*) munosabatni hosil qilamiz.

Endi Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llasak,

$\left(a^2+b^2+c^2+d^2\right)(1+1+1+1)\geq\left(a+b+c+d\right)^2$ yoki $\frac{1}{2}\left(a^2+b^2+c^2+d^2\right)\geq\frac{1}{8}$

(**) munosabat hosil bo'ladi. (*) va (**) larni hadma-had qo'shish natijasida yuqoridagi isboti talab etilgan tengsizlik kelib chiqadi.

84. Istalgan natural n uchun $\frac{1}{2n-1} > \frac{1}{2n}$ ekanligini etiborga olsak,

$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1} > \frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ munosabat o'rinlidir. Endi

$\underbrace{\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}}_{n} > \frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ yoki $\frac{1}{2} > \left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}$ ekanligidan

foydalansak, U holda

$$1+\frac{1}{3}+...+\frac{1}{2n-1} = \frac{1}{2}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n-1} > \left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}+\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)=$$
$$=\frac{n+1}{n}\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)$$

bo'ladi.

85. Berilgan tengsizlikda quyidagicha shakl almashtirish bajaramiz.

$\frac{1+a^2-(1-b^2)}{\sqrt{1+a^2}-\sqrt{1-b^2}} \le \frac{a^2+b^2}{ab}$ yoki $ab \le \sqrt{1+a^2}-\sqrt{1-b^2}$. Bu yerda

$a = tg\alpha, b = \sin\beta \quad \alpha,\beta \in \left(0;\frac{\pi}{2}\right)$ belgilash olsak, u holda

$tg\alpha\sin\beta \le \frac{1}{\cos\alpha}-\cos\beta$ yoki $\cos(\alpha-\beta) \le 1$ bo'ladi.

86. Yuqoridagi berilgan shartlarga ko'ra quyidagi tengliklarni yozamiz:

$$2\cdot 2\cdot x_2 = 2\cdot 2\cdot x_1 - 3x_1$$
$$2\cdot 3\cdot x_3 = 2\cdot 3\cdot x_2 - 3x_2$$
....
$$2\cdot n\cdot x_n = 2\cdot n\cdot x_{n-1} - 3x_{n-1}$$

va bu tengsizliklarni hadma-had qo'shib,

$$3\sum_{i-1}^{n-1}x_i = 2\cdot 2\cdot x_1 + 2\sum_{i=2}^{n-1}x_i - 2\cdot n\cdot x_n = 1 + 2\sum_{i=1}^{n-1}x_i - 2\cdot n\cdot x_n \text{ yoki } nx_n = 1-\sum_{i=1}^{n}x_i > 0$$

munosabatni hosil qilamiz.

87. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligi va $x\in\left(0;\dfrac{\pi}{2}\right)$ uchun $\sin x\left(\sin 2x-1\right)\le a$ yoki $\sin x\le\sqrt{\dfrac{tgx}{2}}$ tengsizliklarni o’rinli

$$\sin x_1\cdot\sin x_2\cdot\ldots\cdot\sin x_n\le\left(\frac{\sin x_1+\sin x_2+\ldots+\sin x_n}{n}\right)^n\le$$

ekanligini etiborga olib, $\le\left(\dfrac{\sqrt{tgx_1}+\sqrt{tgx_2}+\ldots+\sqrt{tgx_n}}{n}\right)^n\cdot 2^{-\frac{n}{2}}\le$

$$\le\left(\sqrt{\frac{tgx_1+tgx_2+\ldots+tgx_n}{n}}\right)^n\cdot 2^{-\frac{n}{2}}\le 2^{-\frac{n}{2}}$$

munosabatni hosil qilamiz.

88. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo’llab, $\left(\dfrac{a^6}{x}+\dfrac{b^6}{y}+\dfrac{c^6}{z}\right)\left(x+y+z\right)\left(1+1+1\right)\ge\left(a^2+b^2+c^2\right)^3$ munosabatni hosil qilamiz. Bundan yuqoridagi isboti talab etilgan tangsizlik kelib chiqadi.

89. Ushbu $x^3+y^3\ge xy\left(x+y\right)$ tengsizlikdan foydalanamiz:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+$$

$$+\frac{1}{ca(c+a)+abc}=\frac{1}{a+b+c}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{1}{abc}$$

90. I-usul: Ushbu $1+2+\ldots+(n-1)=\dfrac{n(n-1)}{2}$ tenglikdan va o’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanamiz:

$$\frac{n(n-1)}{2}+x_1+x_2^2+x_3^3+...+x_n^n=x_1+(1+x_2^2)+(2+x_3^3)+...+\left((n-1)+x_n^n\right)=$$
$$=(x_1)+(1+x_2^2)+(1+1+x_3^3)+...+\underbrace{(1+1+...+1+x_n^n)}_{n-1}\geq$$
$$\geq x_1+2x_2+3x_3+...+nx_n$$
$$x_1+x_2^2+...+x_n^n=x_1+\left((x_2-1)+1\right)^2+...+\left((x_n-1)+1\right)^n\geq x_1+\left(1+2x_2-1\right)$$

II-usul: Bernulli tengsizligidan quyidagi usulda foydalansak:

$$x_1+x_2^2+x_3^3+...+x_n^n+\frac{n(n-1)}{2}=x_1+\left((x_2-1)+1\right)^2+\left((x_3-1)+1\right)^3+...+\left((x_n-1)+1\right)^n+$$
$$+\frac{n(n-1)}{2}\geq x_1+\left(1+2(x_2-1)\right)+\left(1+3(x_3-1)\right)+...+\left(1+n(x_n-1)\right)+\frac{n(n-1)}{2}=$$
$$=x_1+2x_2+3x_3+...+nx_n$$

munosabat hosil bo’ladi.

91. Bu tengsizlikni chap qismini S bilan belgilab, quyidagi usulda Koshi-Bunyakovskiy-Shvarts tengsizligini qo’llaymiz:

$$S\left(a^2\left(b^2+1\right)+b^2\left(c^2+1\right)+c^2\left(a^2+1\right)\right)\geq\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

bundan,

$$S\geq\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{a^2b^2+b^2c^2+c^2a^2+1}$$

tengsizlikni va undan

$$S\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{a^2b^2+b^2c^2+c^2a^2+1}\geq\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{\frac{\left(a^2+b^2+c^2\right)^2}{3}+1}=\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

munosabatni hosil qilamiz.

92. $y_k=\dfrac{1}{x_k}$ almashtirish olsak, u holda,

$$\frac{1}{y_k}=x_k=\frac{1}{1+\frac{a_k}{y_{k-1}}}\Leftrightarrow y_k=1+\frac{a_k}{y_{k-1}}.$$

$y_{k-1}\ge 1,\ a_k\ge 1$ ekanligidan $\left(\frac{1}{y_{k-1}}-1\right)\left(a_k-1\right)\le 0\Leftrightarrow 1+a_k\frac{1}{y_{k-1}}\le a_k+\frac{1}{y_{k-1}}$ bundan

$y_k=1+\frac{a_k}{y_{k-1}}\le a_k+\frac{1}{y_{k-1}}$ munosabatni hosil qilamiz.

$$\sum_{k=1}^{n}y_k\le\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}\frac{1}{y_{k-1}}=\sum_{k=1}^{n}a_k+\frac{1}{y_0}+\sum_{k=1}^{n-1}\frac{1}{y_k}=A+\sum_{k=1}^{n-1}\frac{1}{y_k}<A+\sum_{k=1}^{n}\frac{1}{y_k}$$

$t=\sum_{k=1}^{n}\frac{1}{y_k}$ deb belgilash kiritamiz, bundan $\sum_{k=1}^{n}y_k\ge\frac{n^2t}{},\ t>0$

tengsizlikni hosil qilamiz.

$$\frac{n^2}{t}\le\sum_{k=1}^{n}y_k<A+t\Leftrightarrow t^2+At-n^2\ge 0\Leftrightarrow t>\frac{-A+\sqrt{A^2+4n^2}}{2}=\frac{2n^2}{A+\sqrt{A^2+4n^2}}=$$

$$=\frac{2n^2}{\sqrt{A\left(A+\frac{4n^2}{A}\right)}+A}\ge\frac{2n^2}{\frac{A+A+\frac{4n^2}{A}}{2}+A}+\frac{n^2A}{A^2+n^2}$$

93. $\frac{ab}{3a+b}\le\frac{12b+a}{49}$ chunki $2(a-2b)^2\ge 0,$ $\frac{bc}{b+2c}\le\frac{8b+9c}{49}$ chunki $2\left(2b-3c\right)^2\ge 0,$ $\frac{ac}{c+2a}\le\frac{18c+a}{49}$ chunki $2\left(3c-a\right)^2\ge 0$, bu tengsizliklarni hadma-had qo'shib, isbotlash kerak bo'lgan tengsizlikni hosil qilamiz.

tenglik $a=2b=3c$ bo'lganda bajariladi.

94. x^9-1, x^3-1 ifodalar ishorasi bir xil hamda $x^4>0$ bo'lgani uchun

$$x^8-x^5-\frac{1}{x}+\frac{1}{x^4}=x^5(x^3-1)-\frac{x^3-1}{x^4}=\frac{(x^9-1)(x^3-1)}{x^4}\ge 0\,.$$

95. Ravshanki, $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1$.

Koshi-Bunyakovskiy-Shvarts tengsizligiga ko'ra

$$\sqrt{x+y+z}\sqrt{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\,.$$

96.

$y_i=\frac{1998}{x_i+1998}$ almashtirish kiritamiz.

Ravshanki, $y_i\geq 0,\quad i=1,2,...,n$ va $y_1+y_2+...+y_n=1$.

Demak, $1-y_i=\sum_{j\neq i}y_j$.

Koshi tengsizligiga ko'ra $1-y_i\geq(n-1)\sqrt[n-1]{\prod_{j\neq i}y_j}$.

Bu tengsizliklarni barchasini ko'paytirsak,

$\prod_{i=1}^{n}(1-y_i)\geq(n-1)^n\prod_{i=1}^{n}y_i$ yoki $\prod_{i=1}^{n}\frac{1-y_i}{y_i}\geq(n-1)^n$ tengsizlikni hosil qilamiz.

$\frac{1-y_i}{y_i}=\frac{x_i}{1998}$ bo'lgani uchun bundan $x_1x_2...x_n\geq 1998^n(n-1)^n$ tengsizlikni hosil qilamiz.

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