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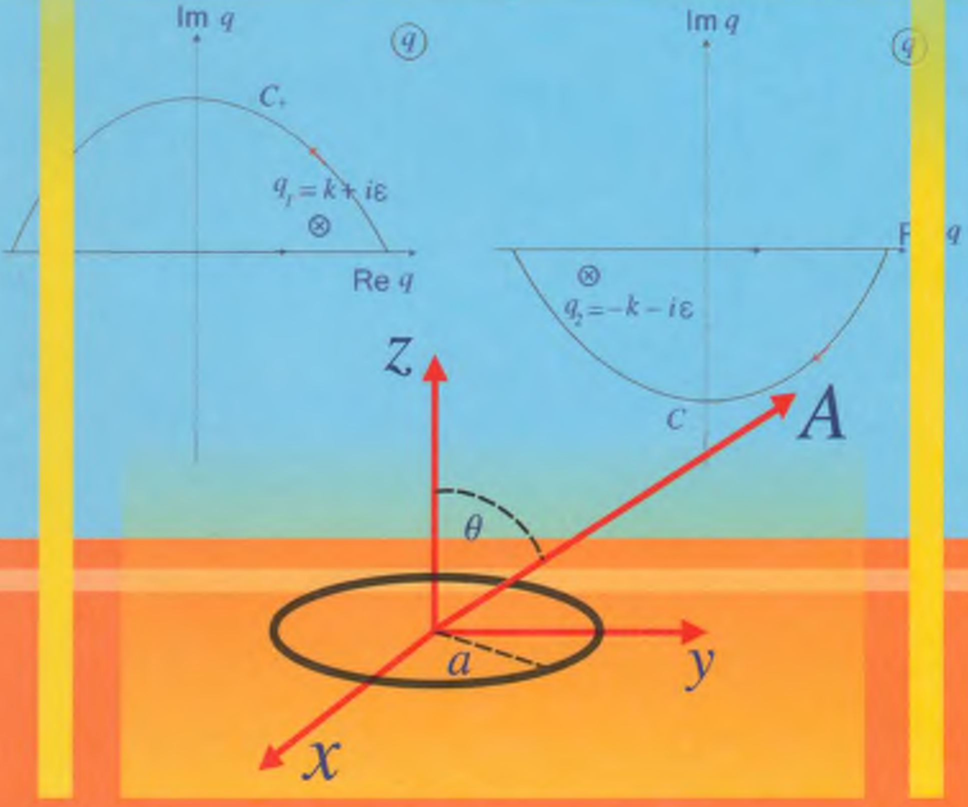
MATEMATIK FIZIKA METODLARI



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Fizika - 5140200 o'quv yo'nalishi bo'yicha ushbu darslikda fizikada eng ko'p uchraydigan maxsus funksiyalarning nazariyasi keltirilgan. Unda chiziqli xususiy hosilali ikkinchi tartibli differensial tenglamalarning klassifikatsiyasi ko'rib chiqilgan. To'lqin tarqalishi, issiqlik va massa ko'chishi kabi fizik jarayonlarni o'rganishda paydo bo'ladigan differensial tenglamalar keltirib chiqarilgan va ularni yechishning asosiy usullari berilgan.

Darslik universitetlarning fizika fakultetlari 3-kurs bakalavr-talabalariga mo'ljallangan.

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So‘z boshi

"Matematik fizika metodlari" kursi matematikaning fizikadagi beqiyos effektivligiga yaqqol misoldir. U fizik jarayonlarni va qonuniyatlarni matematik yo‘l bilan talqin qilish naqadar unumli ekanligini ko‘rsataqi. Kurs davomida talabalar fizika sohasidagi masalalarni matematik korrekt formada qo‘yish, boshlang‘ich va chegaraviy shartlarni talqin qilish va yechishni o‘rganadi. Matematik fizika tenglamalari sohasidagi tan olingan metodlarning deyarli hammasi mazkur darslikda keltirilgan. Nazariy materiallarga ularni tushuntiradigan deyarli qirqta misollar keltirilgan. Yuzdan ortiq mashqlar o‘zlarining yechimlari bilan berilgan. Bu misol va mashqlardan ko‘rinib turibdiki, matematik fizika fanining tushunchalari va metodlari to‘lqin, massa hamda issiqlik tarqalishi jarayonlarini to‘liq ravishda qamrab olgan, matematik fizika metodlari yordamida bu sohalarda yechib bo‘lmaydigan masala yo‘q.

Ushbu kitob mualliflarning O‘zbekiston Milliy universiteti fizika fakultetidagi ko‘p yillik ish tajribasi asosida yozilgan. Matematik fizika metodlari sohasida ajoyib matematik natijalar va yutuqlar juda ko‘p, ammo fizik-talabalarga o‘tiladigan kursda amaliyotga yaqin bo‘lgan masalalarni yechish metodlari va ularga misollar birinchi o‘rinda turishi kerak. Mualliflar O‘zbekiston universitetlarining fizika fakultetlari bakalavr-talabalari uchun ushbu kitobning foydasi tegadi degan umiddadir.

Mualliflar

I BOB. MAXSUS FUNKSIYALAR

§1. Silindrik funksiyalar (Bessel funksiyalari)

Quyidagi ko'rinishdagi tenglama

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0 \tag{1}$$

silindrik (yoki ***Bessel***) tenglamasi deyiladi. Keyin ko'ramizki, ushbu tipdagi tenglamalar matematik fizika tenglamalarini silindrik sistemada ochganimizda paydo bo'ladi. Tenglamaning yechimini

$$y(x) = x^s \sum_{n=0}^{\infty} c_n x^n = x^s(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)$$

ko'rinishda qidiramiz. Tenglamaning yechimini bunday ko'rinishda qidirish ***Frobenius***[1] ***metodi*** deyiladi. Hosilalarni topaylik:

$$y' = \sum_{n=0}^{\infty} c_n(n+s)x^{n+s-1} = sc_0x^{s-1} + (s+1)c_1x^s + (s+2)c_2x^{s+1} + \cdots$$

$$y'' = \sum_{n=0}^{\infty} c_n(n+s)(n+s-1)x^{n+s-2} =$$

$$= s(s-1)c_0x^{s-2} + s(s+1)c_1x^{s-1} + (s+2)(s+1)c_2x^s + \cdots$$

Oxirgi uchta tengliklarni (1)-ga olib borib qo'yamiz va x-ning har bir darajasi oldidagi koeffisientlarni yig'ib nolga tenglashtiramiz. Umumiy ko'rinishda

$$\sum_{n=0}^{\infty} \Big[c_n(n+s)(n+s-1)x^{n+s} + c_n(n+s)x^{n+s} + (x^2 - \nu^2)c_nx^{n+s}\Big] = 0. \tag{2}$$

Bu cheksiz qatorning birinchi bir necha hadlarini ochib yozib olaylik:

$$c_0s(s-1)x^s + c_1s(s+1)x^{s+1} + \cdots + c_0sx^s + c_1(s+1)x^{s+1} + \cdots$$

$$+(x^2 - \nu^2)(c_0x^s + c_1x^{s+1} + \cdots) = 0.$$

[1] Ferdinand Georg Frobenius (1840-1917) - nemis matematigi

x–ning darajasi eng past bo'lgan had x^s, uning oldidagi koeffisientlarni yig'amiz:

$$c_0(s^2 - \nu^2) = 0. \tag{3}$$

x^{s+1}–monomning oldidagi koeffisientlarni yig'aylik:

$$c_1[(s+1)^2 - \nu^2] = 0. \tag{4}$$

Umumiy ko'rinishda (2)-ning yechimi quyidagicha:

$$c_n = -\frac{1}{(s+n)^2 - \nu^2} c_{n-2}. \tag{5}$$

(3)-dan quyidagi xulosaga kelamiz:

$$c_0 = 0 \quad \text{yoki} \quad s = \pm\nu. \tag{6}$$

(4)-dan esa

$$c_1 = 0 \quad \text{yoki} \quad s = \pm\nu - 1.$$

Bizning maqsadimizga

$$s = \nu \quad \text{va} \quad c_1 = 0 \tag{7}$$

deb qabul qilish mos keladi. Ko'rilayotgan differensial tenglama - ikkinchi tartibli, $s = -\nu$ hol ikkinchi yechimni berishi kerak, ammo bunday tanlangan ikkinchi yechim $\nu = n$ butun son bo'lgan hollarda mustaqil yechim bo'lmaydi (buni keyin (11)-formuladan ko'ramiz). Shuning uchun ikkinchi yechimni boshqacha yo'l bilan keyin ta'riflaymiz. Demak, (5)-formula quyidagi ko'rinishni oladi:

$$c_n = -\frac{1}{n^2 + 2\nu n} c_{n-2}. \tag{8}$$

Bu formulaning nomi - ***rekurrent munosabat***, uni (7)-formula bilan solishtirsak faqat c_0, c_2, c_4, $c_6, \ldots$ largina noldan farqli ekanligini ko'ramiz, va $c_1 = c_3 = c_5 = \cdots = 0$ bo'ladi. Ya'ni, faqatgina juft indeksli c_n lar noldan farqli. Shu sababdan qulaylik uchun

$$n = 2k,\ k = 0,\ 1,\ 2,\ 3, \ldots$$

deb olamiz. Bu bizni

$$c_{2k} = -\frac{1}{2k \cdot 2(k+\nu)} c_{2(k-1)} \tag{9}$$

formulaga olib keladi. Ushbu rekurrent munosabatni yechish qiyin emas:

$$c_{2k} = -\frac{1}{2k \cdot 2(k+\nu)} c_{2(k-1)} = (-1)^2 \frac{c_{2(k-2)}}{2^2 k(k-1) \cdot 2^2(k+\nu)(k+\nu-1)} =$$

$$= \cdots = (-1)^k \frac{\nu!}{2^{2k} k!(k+\nu)!} c_0.$$

Demak, quyidagi yechimni topdik:

$$y(x) = c_0 \sum_{n=0}^{\infty} (-1)^k \frac{\nu!}{2^{2k} k!(k+\nu)!} x^{2k+\nu}.$$

(1)-tenglama chiziqli bo'lgani uchun c_0 koeffisientni tanlab olish o'zimizning qo'limizda. Odatda uni

$$c_0 = \frac{1}{2^\nu \nu!}$$

ko'rinishda tanlab olish qabul qilingan. Hosil bo'lgan funksiya ***silindrik***, yoki ***Bessel funksiyasi*** [2] deyiladi va quyidagicha belgilanadi:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{10}$$

1.1-mashq.

$$J_\nu(-x) = (-1)^\nu J_\nu(x)$$

ekanligiga ishonch hosil qiling.

1.2-mashq. Agar $\nu = n$ butun son bo'lsa

$$J_n(x) = (-1)^n J_{-n}(x) \tag{11}$$

ekanligini ko'rsating.

Bessel tenglamasi ikkinchi tartibli tenglama, demak, uning ikkita chiziqli mustaqil yechimi mavjud bo'lishi kerak. Ikkinchi yechimni (6)-ga qarab $s = -\nu$ ga mos keladigan qilib tanlab olishimiz mumkin deb o'ylashimiz mumkin, ammo (11)-dan ko'rinib turibdiki, $\nu = n$ butun son bo'lgan holda bu yechimlar mustaqil bo'lmaydi. Shu sababdan ikkinchi yechim boshqacharoq ko'rinishda olinadi. Uning ta'rifi:

$$N_\nu(x) = \frac{\cos \nu\pi \cdot J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}. \tag{12}$$

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[2]Silindrik tenglama va silindrik funksiyalar shveytsar matematigi Daniel Bernoulli (1700 - 1782) tomonidan ochilgan, ammo nemis matematigi va astronomi Friedrich Wilhelm Bessel (1784 - 1846) bu tenglamaning yechimlarini birinchi bo'lib klassifikatsiya qilib chiqqan

Bunday tanlab olingan funksiyalar ***Neumann***[3] ***funksiyalari*** deyiladi. Ko'rinib turibdiki, $\nu = n$ holda bu munosabatning surati va maxraji nolga teng, uni l'Hôpital [4] qoidasi bo'yicha ochish kerak.

1.3-mashq. $\nu = n$ butun son bo'lgan holda

$$N_n(x) = \left.\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^\nu \frac{\partial J_{-\nu}(x)}{\partial \nu}\right|_{\nu=n}$$

ekanligini ko'rsating.

Chiziqli tenglama yechimlarining ixtiyoriy chiziqli kombinatsiyasi yana shu tenglamaning yechimi bo'ladi. Masalan,

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \tag{13}$$

funksiyalar (ularning nomi - birinchi va ikkinchi tur ***Hankel*** [5] ***funksiyalari***) ham Bessel tenglamasi (1)-ning yechimlaridir. Bundan keyin Bessel funksiyalari uchun keltirib chiqariladigan rekurrent munosabatlar mana shu to'rta funksiya uchun o'rinlidir.

§1.1. Bessel funksiyalari uchun hosil qiluvchi funksiyasi

Quyidagi munosabatni isbot qilaylik:

$$g(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{14}$$

Bu tenglikning chap tomonidagi $g(x,t)$ funksiya Bessel funksiyalarining hosil qiluvchi funksiyasi deyiladi, qator esa shu funksiyaning Laurent qatoridir. Isbot qiyin emas:

$$g(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}} = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{xt}{2}\right)^l \cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x}{2t}\right)^k =$$

$$= \sum_{l,k=0}^{\infty} \frac{(-1)^k}{l!k!}\left(\frac{x}{2}\right)^{l+k} t^{l-k}.$$

Quyidagi almashtirish kiritaylik: $l - k = n$, unda $l = n + k$ bo'ladi va n soni $-\infty$ dan ∞ gacha o'zgaradi:

$$g(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right)t^n = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

[3] Karl Gottfried Neumann (1832-1925) - nemis matematigi
[4] Guillaume Fransois Antoine de l'Hôpital (1661-1704) - fransuz matematigi, rus tilida - Лопиталь.
[5] Hermann Hankel (1839-1873) - nemis matematigi

§1.2. Bessel funksiyalari uchun rekurrent munosabatlar

Hosil qiluvchi funksiyadan foydalanib rekurrent munosabatlarni keltirib chiqaraylik. Buning uchun (14)-tenglikdan bir marta t bo'yicha, bir marta x bo'yicha hosila olamiz. t bo'yicha hosila olaylik:

$$\frac{\partial}{\partial t}g(x,t) = \frac{x}{2}\left(1+\frac{1}{t^2}\right)e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}.$$

Bu tenglikning chap tomonini ochib yozaylik:

$$\frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n-2} = \sum_{n=-\infty}^{\infty} nJ_n(x)t^{n-1}.$$

Tenglikning chap va o'ng tomonlaridagi t^n darajalari oldidagi hadlar bir-biriga teng bo'lishi kerak:

$$\frac{x}{2}J_n + \frac{x}{2}J_{n+2} = (n+1)J_{n+1},$$

yoki,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x). \qquad (15)$$

Demak, bizga $(n-1)-$ indeksli va $(n)-$indeksli Bessel funksiyalari berilgan bo'lsa biz $(n+1)-$ indeksli Bessel funksiyasini ular orqali ifodalab olishimiz mumkin ekan. Bunday munosabatlar ***rekurrent*** munosabatlar deyiladi. Hosilalarni o'z ichiga olgan rekurrent munosabatlar ham bor. Buning uchun hosil qiluvchi funksiyadan x bo'yicha hosila olamiz:

$$\frac{\partial}{\partial x}g(x,t) = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J'_n(x)t^n.$$

Yana (14)-ta'rifni ishlatamiz, ya'ni, olingan tenglikning chap tomonini u yordamida ochamiz:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_n(x)t^{n-1} = \sum_{n=-\infty}^{\infty} J'_n(x)t^n.$$

Chap va o'ng tomonlardagi t ning bir xil tartibli darajalarini solishtirsak,

$$J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \qquad (16)$$

ko'rinishga ega bo'lgan rekurrent munosabatga kelamiz.

1.1-misol.

$$J_0'(x) = \frac{1}{2}\left(J_{-1}(x) - J_1(x)\right) = \frac{1}{2}(-J_1(x) - J_1(x)) = -J_1(x).$$

(15)- va (16)-larni keltirib chiqarishda biz faqat butun indeksli Bessel funksiyalari J_n lardan foydalandik, ammo ular

- ixtiyoriy butun bo'lmagan ν indeksli silindrik funksiyalar uchun o'rinlidir;
- hamma silindrik funksiyalar uchun - J_ν, N_ν, $H_\nu^{(1,2)}$ - o'rinlidir.

Rekurrent munosabatlarning yana bir qulay formasi bor. Ularni olish uchun (15)- va (16)-larni bir marta qo'shamiz va bir marta ayiramiz. Natijada

$$J_{n-1} = J_n' + \frac{n}{x}J_n \quad \text{va} \quad J_{n+1} = \frac{n}{x}J_n - J_n'$$

ko'rinishdagi munosabatlarni olamiz. Ularning birinchisini x^n ga va ikkinchisini x^{-n} ga ko'paytirsak quyidagi tez uchrab turadigan munosabatlarga kelamiz:

$$\frac{d}{dx}\Big[x^n J_n(x)\Big] = x^n J_{n-1}(x) \qquad \text{va} \qquad \frac{d}{dx}\Big[x^{-n} J_n(x)\Big] = -x^{-n} J_{n+1}(x). \quad (17)$$

Bu munosabatlarni eslab qolish yanada oson bo'lgan ko'rinishga keltirib olishimiz qiyin emas:

$$\frac{d}{xdx}\left[x^n J_n(x)\right] = x^{n-1} J_{n-1}(x) \qquad \text{va} \qquad \frac{d}{xdx}\left[\frac{J_n(x)}{x^n}\right] = -\frac{J_{n+1}(x)}{x^{n+1}}. \quad (18)$$

1.4-mashq. Quyidagilarni isbot qiling:

$$\left(\frac{d}{xdx}\right)^m \Big[x^\nu J_\nu(x)\Big] = x^{\nu-m} J_{\nu-m}; \quad (19)$$

$$\left(\frac{d}{xdx}\right)^m \left[\frac{J_\nu(x)}{x^\nu}\right] = (-1)^m \frac{J_{\nu+m}(x)}{x^{\nu+m}}. \quad (20)$$

§1.3. Bessel funksiyasi uchun integral tasavvur

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

formula chap tomondagi funksiyaning Laurent qatoridir. Kompleks o'zgaruvchilar nazariyasidan ma'lumki, qator koeffisienti (bizning holda bu J_n) uchun quyidagi formulaga egamiz:

$$J_n(x) = \frac{1}{2\pi i}\oint_C \frac{e^{\frac{x}{2}(z-\frac{1}{z})}}{z^{n+1}} dz \quad (21)$$

n butun son bo'lganda C kontur koordinat boshini o'z ichiga olgan yopiq konturdir, masalan, birlik radiusli aylana.

§1.4. Yarim butun indeksli Bessel funksiyalari

(10)-qatorda $\nu = 1/2$ deb olaylik:

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!(k+1/2)!} \left(\frac{x}{2}\right)^{2k+1/2}.$$

Legendrening ikkilash formulasi deyiladigan

$$k!\left(k+\frac{1}{2}\right)! = \sqrt{\pi}\, 2^{-2k-1}(2k+1)! \qquad (22)$$

formuladan foydalansak ([9], 19-bet) yuqoridagi qator quyidagi ko'rinishga keladi:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}} \sin x.$$

Xuddi shunday yo'l bilan $\nu = -1/2$ holni ham soddalashtirishimiz mumkin:

$$J_{-1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k-1/2)!} \left(\frac{x}{2}\right)^{2k-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k-1/2} 2^{-2k+1/2}}{2^{-2k}\sqrt{\pi}(2k)!} =$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \sqrt{\frac{2}{\pi x}} \cos x.$$

Ana endi (20)-rekurrent munosabatni ishlataylik. Undan kelib chiqadiki,

$$J_{m+1/2}(x) = (-1)^m x^{m+1/2} \left(\frac{d}{xdx}\right)^m \left[\frac{J_{1/2}(x)}{\sqrt{x}}\right] =$$

$$= (-1)^m \sqrt{\frac{2}{\pi}}\, x^{m+1/2} \left(\frac{d}{xdx}\right)^m \left(\frac{\sin x}{x}\right).$$

Xuddi shu yo'sinda (19)-ni ishlatsak quyidagini olamiz:

$$J_{-m-1/2}(x) = \sqrt{\frac{2}{\pi}}\, x^{m+1/2} \left(\frac{d}{xdx}\right)^m \left(\frac{\cos x}{x}\right).$$

Yarim butun indeksli Bessel funksiyalari Helmholtz tenglamasini sferik sistemada yechganda ham paydo bo'ladi (6-bobning ohiridagi shar uchun issiqlik tarqalishi masalasining yechilishida paydo bo'lgan (75)-tenglamaning analiziga qarang).

§1.5. Mavhum argumentli Bessel funksiyalari

Agar (1)-silindrik tenglamada $x \to ix$ almashtirish bajarsak,

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0 \qquad (23)$$

tenglamani olamiz. Albatta, $J_\nu(ix)$ funksiya bu tenglamaning yechimi, ammo bu holdagi yechim uchun quyidagi belgilash qabul qilingan:

$$I_\nu(x) = i^{-\nu} J_\nu(ix).$$

Keltirib chiqarish qiyin emaski,

$$I_\nu(x) = i^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{ix}{2}\right)^{2k+\nu} = \sum_{n=0}^{\infty}\frac{1}{k!(k+\nu)!}\left(\frac{x}{2}\right)^{2k+\nu}.$$

Ikkinchi yechim odatda

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

ko'rinishda tanlab olinadi. Bu funksiyaning nomi Macdonald funksiyasi (ba'zi-bir kitoblarda - Kelvin funksiyasi). Xususiy hollar:

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\mathrm{sh}\,x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\mathrm{ch}\,x.$$

$$K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,e^{-x}.$$

§1.6. Bessel funksiyalarining nollari. Ortogonallik munosabatlari

(1)-tenglamada $x = kr$ almashtirish bajaraylik:

$$r^2\frac{d^2J_\nu(kr)}{dr^2} + r\frac{dJ_\nu(kr)}{dr} + (k^2r^2 - \nu^2)J_\nu(kr) = 0.$$

Bu tenglamani

$$\frac{d}{dr}\left(r\frac{d}{dr}J_\nu(kr)\right) + \left(k^2r - \frac{\nu^2}{r}\right)J_\nu(kr) = 0 \qquad (24)$$

ko'rinishga keltirib olaylik. Shu tenglamani bir gal k_1 parametr bilan, bir gal k_2 parametr bilan yozib olib, k_1 li tenglamani $J_\nu(k_2r)$ ga, k_2 li tenglamani $J_\nu(k_1r)$ ga ko'paytiramiz va birini ikkinchisidan ayiramiz. Natijada

$$J_\nu(k_2r)\left(rJ_\nu'(k_1r)\right)' - J_\nu(k_1r)\left(rJ_\nu'(k_2r)\right)' = (k_2^2 - k_1^2)rJ_\nu(k_1r)J_\nu(k_2r)$$

formulani olamiz (har bir shtrih - r bo'yicha hosila). Tenglamaning chap tomonini bizning maqsadimiz uchun qulayroq ko'rinishga keltiraylik:

$$J_\nu(k_2r)\,(rJ'_\nu(k_1r))' - J_\nu(k_1r)\,(rJ'_\nu(k_2r))' =$$

$$= \frac{d}{dr}\left[r\left(J_\nu(k_2r)\frac{d}{dr}J_\nu(k_1r) - J_\nu(k_1r)\frac{d}{dr}J_\nu(k_2r)\right)\right].$$

Demak,

$$\int_0^1 J_\nu(k_1r)J_\nu(k_2r)r\,dr = \frac{1}{k_2^2-k_1^2}\left(rJ_\nu(k_2r)\frac{d}{dr}J_\nu(k_1r) - rJ_\nu(k_1r)\frac{d}{dr}J_\nu(k_2r)\right)\Bigg|_0^1. \tag{25}$$

Faraz qilaylik, k_1 va k_2 sonlar quyidagi tenglamaning yechimlaridan bo'lsin:

$$\alpha J_\nu(k) + \beta k J'_\nu(k) = 0, \quad \alpha+\beta > 0, \quad \alpha \ge 0,\ \beta \ge 0. \tag{26}$$

Unda (25)-ning o'ng tomoni $k_1 \neq k_2$ holda nolga teng bo'ladi va biz olamiz:

$$\int_0^1 J_\nu(k_1r)J_\nu(k_2r)r\,dr = 0, \qquad k_1 \neq k_2. \tag{27}$$

$k_1 = k_2$ holni quyidagicha ko'ramiz. (25)-ning o'ng tomonida $k_2 = k_1 + \delta$ deymiz va $\delta \to 0$ limitga o'tamiz:

$$\frac{1}{2k_1\delta}\left[k_1J_\nu(k_1+\delta)J'_\nu(k_1) - (k_1+\delta)J_\nu(k_1)J'_\nu(k_1+\delta)\right] \to$$

$$\to \frac{1}{2}\left[J'_\nu(k_1)\right]^2 - \frac{1}{2k_1}\left(J_\nu(k_1)J'_\nu(k_1) + k_1J_\nu(k_1)J''_\nu(k_1)\right).$$

Bessel tenglamasidan

$$k_1^2J''_\nu(k_1) + k_1J'(k_1) = (\nu^2 - k_1^2)J_\nu(k_1)$$

kelib chiqadi, shuni ishlatib

$$\int_0^1 \left[J_\nu(kr)\right]^2 r\,dr = \frac{1}{2}\left[J'_\nu(k)\right]^2 + \frac{1}{2}\left(1-\frac{\nu^2}{k^2}\right)\left[J_\nu(k)\right]^2 \tag{28}$$

munosabatga kelamiz. (27)- va (28)-formulalar Bessel funksiyalarining o'zaro ortogonalligini va normasini ko'rsatadi.

(26)-ga qaytib kelaylik. Agar $\beta = 0$ bo'lsa k soni $J_\nu(k) = 0$ tenglamaning yechimi, ya'ni, Bessel funksiyasining noli bo'ladi. Bessel funksiyalarining nollari masalasi adabiyotda keng muhokama qilinadigan masaladir. Ma'lumki, $J_0(0) = 1$ bo'ladi va $J_0(k)$ ning birinchi noli $k_1 = 2.4844$ ga teng, qolgan nollari shu songa taxminan $n\pi$, $n = 1, 2, 3, ..$ larni qo'shib olinadi. $J_n(k)$, $n \geq 1$ holda Bessel funksiyalari koordinat boshida nolga teng bo'ladi $J_n(0) = 0$, ularning boshqa nollarini matematik ladvallardan topish mumkin.

§1.7. Helmholtz tenglamasi silindrik sistemada

Quyidagi ***Helmholtz***[6] ***tenglamasi*** deb ataladigan tenglamani

$$\Delta f + k^2 f = 0$$

silindrik sistemada ochamiz:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} + k^2 f = 0.$$

Ushbu tipdagi tenglama matematik fizikaning ko'pgina qismlarida uchraydi - elektromagnit nurlanish masalalarida, issiqlik tarqalishi masalalarida va h.k. Masalada silindrik simmetriya bor deb faraz qilamiz, boshqacha so'z bilan aytganda, z ga bog'liqlik yo'q deymiz: $f = f(r, \varphi)$. Yechimni

$$f(r, \varphi) = R(r)\Phi(\varphi)$$

ko'rinishda qidiraylik:

$$\frac{\Phi(\varphi)}{r}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} + k^2 R(r)\Phi(\varphi) = 0.$$

Bu tenglamaning quyidagi ko'rinishga kelishini tekshirib ko'rish qiyin emas:

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + k^2 r^2 = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = \lambda.$$

Tenglamaning o'ng tomonida yangi konstanta λ paydo bo'ldi. Uning kelib chiqishining sababi quyidagicha. Tenglamaning chap tomoni faqat r ning funksiyasi, o'ng tomoni esa faqat φ ning. Demak, r ni o'zgartirsak, tenglikning o'ng tomoni o'zgarmaydi, bu degani, chap tomoni ham. Xuddi shunday,

---

[6] Herman Ludwig Ferdinand von Helmholtz (1821-1894)- nemis fizigi. Ruschasi - Гельмгольц

φ ni o'zgartirsak tenglikning chap tomoni o'zgarmaydi, demak, o'ng tomoni ham. Xulosa - tenglikning ikkala tomoni ham o'zgarmas son, shu sonni λ deb belgiladik. Bu son musbat bo'lishi kerak, buni tezda tushunamiz. Natijada biz ikkita tenglamaga egamiz:

$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + (k^2r^2 - \lambda)R(r) = 0;$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + \lambda\Phi(\varphi) = 0.$$

Ikkinchi tenglamaning yechimi:

$$\Phi(\varphi) = c_1\cos(\sqrt{\lambda}\varphi) + c_2\sin(\sqrt{\lambda}\varphi).$$

φ va $\varphi + 2\pi$ burchaklar bir nuqtaga mos kelgani uchun yechimdan

$$\Phi(\varphi) = \Phi(\varphi + 2\pi)$$

bo'lishini talab qilishimiz kerak. Bu degani, $\sqrt{\lambda} = m$, $m = 0, 1, 2, ...$ bo'lishi kerak. Shuni hisobga olsak, R uchun tenglamamiz quyidagi ko'rinishni oladi:

$$r^2R''(r) + rR'(r) + (k^2r^2 - m^2)R(r) = 0. \tag{29}$$

Agarda $kr = x$ va $y = R$ deb belgilasak, tenglamamiz

$$x^2y''(x) + xy'(x) + (x^2 - m^2)y(x) = 0 \tag{30}$$

ko'rinishga keladi. Bu esa Bessel tenglamasi (1)-ning o'zidir, faqatgina u yerda ixtiyoriy bo'lgan son ν ning o'rniga butun son m turibdi. Agar Helmholtz tenglamasini sferik sistemada yechsak, yarim butun indeksli Bessel funksiyalariga kelamiz - (75)-tenglamaga qarang.

1.5-mashq. (14)-formulada $t = e^{i\theta}$ almashtirish bajarib

$$e^{ix\sin\theta} = \sum_{-\infty}^{\infty} J_n(x)e^{in\theta}$$

formulani oling.

1.6-mashq. Yuqoridagi formuladan quyidagilarni keltirib chiqaring:

$$\cos(x\sin\theta) = \sum_{-\infty}^{\infty} J_n(x)\cos(n\theta); \tag{31}$$

$$\sin(x\sin\theta) = \sum_{-\infty}^{\infty} J_n(x)\sin(n\theta). \tag{32}$$

1.7-mashq. $\theta = \pi/2$ deb olib yuqoridagi formulalardan

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) + \cdots$$

$$\sin x = 2J_1(x) - 2J_3(x) + \cdots$$

larni keltirib chiqaring.

1.8-mashq. $\theta = 0$ deb olib

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \cdots$$

formulani keltirib chiqaring.

1.9-mashq.

$$\int_0^\pi \cos(n\theta)\cos(m\theta)d\theta = \frac{\pi}{2}\delta_{nm}, \qquad \int_0^\pi \sin(n\theta)\sin(m\theta)d\theta = \frac{\pi}{2}\delta_{nm} \tag{33}$$

munosabatlardan foydaslanib

$$\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos(n\theta)d\theta = \begin{cases} J_n(x), & n \text{ - juft;} \\ 0, & n \text{ - toq.} \end{cases}$$

$$\frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin(n\theta)d\theta = \begin{cases} 0, & n \text{ - juft;} \\ J_n(x), & n \text{ - toq.} \end{cases}$$

ekanligini isbot qiling.

1.10-mashq. (14)-formulada $t = ie^{i\theta}$ almashtirish bajarib

$$e^{ix\cos\theta} = \sum_{-\infty}^{\infty} i^n J_n(x)e^{in\theta}$$

formulani oling (Jacoby-Anger formulasi).

1.11-mashq.

$$J_n(x) = (-1)^n x^n \left(\frac{d}{xdx}\right)^n J_0(x)$$

formulani keltirib chiqaring.

1.12-mashq. Schlafly integralidan

$$J_n(x) = \frac{1}{\pi}\int_0^\pi d\theta\cos(n\theta - x\sin\theta), \qquad n = 0, 1, 2, 3, \ldots \tag{34}$$

ekanligini keltirib chiqaring.

1.13-mashq. (15)-formuladan foydalanib $J_5(x)$ ni $J_0(x)$ va $J_1(x)$ orqali ifodalang.

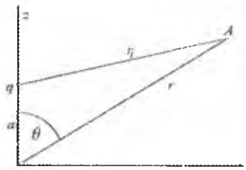
1.14-mashq. (34)-formuladan foydalanib $J_0(0) = 1$, $J_n(0) = 0$, $n \geq 1$ ekanligini isbot qiling.

1.15-mashq. (34)-formuladan foydalanib $J_0'(x) = -J_1(x)$ ekanligini isbot qiling.

§2. Legendre polinomlari. Sferik funksiyalar

Oddiy elektrostatik masaladan boshlaylik. $z = a$ nuqtada joylashgan q zaryad A nuqtada quyidagi potensial hosil qiladi:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_1}.$$



I.1-rasm: z- o'qida joylashgan zaryad

Rasmdan ko'rinib turibdiki,

$$r_1 = \sqrt{r^2 + a^2 - 2ra\cos\theta}.$$

Bu formulani masalaning geometriyasidan kelib chiqadigan vektor munosabatdan keltirib chiqarish qiyin emas:

$$\mathbf{r}_1 = \mathbf{r} - \mathbf{a} \quad \rightarrow \quad r_1^2 =$$

$$= r^2 + a^2 - 2\mathbf{r}\cdot\mathbf{a} = r^2 + a^2 - 2ra\cos\theta.$$

Demak,

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}(r^2 + a^2 - 2ra\cos\theta)^{-1/2} = \frac{q}{4\pi\varepsilon_0 r}\frac{1}{\sqrt{1 + \frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta}}.$$

ekan. Quyidagini faraz qilib: $r \gg a$, olingan ifodani a/r bo'yicha qatorga yoyaylik. Qator koeffisientlari faqat $\cos\theta$ ning funksiyasi bo'lishi mumkin:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n. \tag{35}$$

Hosil bo'lgan qatorning koeffisientlari $P_n(\cos\theta)$ ***Legendre*[7] *polinomlari*** deyiladi. Ularni quyidagi hosil qilish funksiyasi orqali ta'riflash qulaydir:

$$g(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{36}$$

§2.1. Rekurrent munosabatlar

Hosil qilish funksiyasining ta'rifidan ko'rinib turibdiki

$$P_0(x) = g(x, t = 0) = 1. \tag{37}$$

[7] Adrien-Marie Legendre (1752-1833) - fransuz matematigi. Ruschasi - Лежандр

Undan tashqari,

$$P_1(x) = \frac{\partial}{\partial t} g(x,t)\bigg|_{t=0} = x. \tag{38}$$

Albatta, bittama-bitta P_n larni bu tartibda hisoblab topish katta ishni talab qiladi. Rekurrent munosabatlardan foydalanib $P_n(x)$ larni topish bu nuqtai-nazardan katta qulaylik tug'diradi. Ularni topaylik. Buning uchun $g(x,t)$ ni bir marta t bo'yicha, bir marta x bo'yicha differensiallaymiz.

$$\frac{\partial g(x,t)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Tenglikning chap tomoni:

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2}\sum_{n=0}^{\infty} P_n(x)t^n.$$

Demak,

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

ekan. Bu tenglikdagi t ning bir xil darajalari oldidagi koeffisientlarni tenglashtirsak, quyidagi birinchi rekurrent munosabatni olamiz:

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{39}$$

1.2-misol. $n=1$ deylik:

$$3xP_1(x) = 2P_2 + P_0 \quad \to \quad P_2(x) = \frac{3x^2-1}{2}.$$

Bu yerda (37)- va (38)-formulalar ishlatildi.

1.3-misol. $n=2$ bo'lsin.

$$5xP_2 = 3P_3 + 2P_1 \quad \to \quad P_3 = \frac{5}{2}x^3 - \frac{3}{2}x.$$

1.16-mashq.
(39)-dan foydalanib $P_5(x)$ ni keltirib chiqaring.

1.17-mashq. $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ va $P_4(x)$ larning $-1 \le x \le 1$ sohadagi grafiklarini chizing.

(39)-dan ko'rinib turibdiki, $P_n(x)$ - x-ning n-darajali polinomi.

Endi hosil qiluvchi funksiyadan x bo'yicha hosila olamiz:

$$\frac{\partial g(x,t)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n,$$

yoki,

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n = t\sum_{n=0}^{\infty} P_n(x)t^n.$$

Yana chap va o'ng tomondagi t ning bir xil darajalarining oldidagi koeffisientlarni tenglashtirsak, quyidagi rekurrent munosabatni olamiz:

$$P_{n+1}'(x) + P_{n-1}'(x) = 2xP_n'(x) + P_n(x). \tag{40}$$

Agar (39)-ni differensiallasak, ikkiga ko'paytirsak va undan (40)-ni ayirsak yana bitta muhim rekurrent munosabatni olamiz:

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x).$$

Yuqoridagi uch munosabatlardan foydalanib quyidagilarni ham keltirib chiqarishimiz mumkin:

$$P_{n-1}'(x) = -nP_n(x) + xP_n'(x); \qquad P_{n+1}'(x) = xP_n'(x) + (n+1)P_n(x);$$

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x). \tag{41}$$

Oxirgi formulani olishda undan oldingisida $n \to n-1$ almashtiramiz va paydo bo'lgan P_{n-1}' ning o'rniga shu uch formulaning birinchisini ishlatamiz.

§2.2. Differensial tenglama

Legendre polinomlari bo'ysunadigan differensial tenglamani keltirib chiqaraylik. Buning uchun (41)-ning ohirgisidan bir marta hosila olaylik:

$$-2xP_n'(x) + (1-x^2)P_n''(x) = nP_{n-1}'(x) - nP_n(x) - nxP_n'(x).$$

(41)-ning birinchisidan foydalanib bu yerdagi P_{n-1}' ni yo'qotishimiz mumkin, natijada quyidagi differensial tenglamaga kelamiz:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \tag{42}$$

Bu tenglamaning nomi - Legendre tenglamasi. Uni boshqa formada ham yozib olishimiz mumkin:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0. \tag{43}$$

Agar o'zining kelib chiqishi bo'yicha $x = \cos\theta$ ekanligini eslasak, Legendre tenglamasi quyidagi formaga keladi:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n(\cos\theta)}{d\theta}\right) + n(n+1)P_n(\cos\theta) = 0. \tag{44}$$

§2.3. Xususiy hollar

(36)-da $x = 1$ deb olaylik:

$$\frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{1-t} = \sum_{n=0}^{\infty} P_n(1)t^n.$$

Ikkinchi tomondan

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n.$$

Demak,

$$P_n(1) = 1$$

ekan. Endi $(x,\, t) \to (-x,\, -t)$ almashtirish bajaraylik:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} P_n(-x)(-1)^n t^n.$$

Demak,

$$P_n(-x) = (-1)^n P_n(x)$$

ekan. Xususan,

$$P_n(-1) = (-1)^n$$

bo'ladi.

8-bobdagi VIII.1-misolda $P_n(0)$ ning qiymati kerak bo'ladi. Uni topaylik.

$$g(0,t) = \frac{1}{(1+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(0)t^n$$

dan kelib chiqadiki, uning yoyilmasida t ning toq darajalari uchramaydi. Demak,

$$P_{2n+1}(0) = 0. \tag{45}$$

Binomial koeffisientlarning ta'rifidan

$$\frac{1}{(1+t^2)^{1/2}} = \sum \frac{(-1/2)!}{(-1/2-n)!n!} t^{2n} = \sum \frac{\Gamma(1/2)}{\Gamma(1/2-n)\Gamma(n+1)} t^{2n}.$$

Gamma-funksiyalar uchun quyidagi ([9], 18-bet)

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)} \qquad \text{va} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

formulalardan foydalanib qatordagi koeffisientni quyidagi ko'rinishga keltiramiz:

$$\frac{\Gamma(1/2)}{\Gamma(1/2-n)\Gamma(n+1)} = \frac{\cos(n\pi)\Gamma\left(\frac{1}{2}+n\right)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

Quyidagi Legendrening ikkilash formulasi deyiladigan ([9], 19-bet)

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(\frac{1}{2}+z\right)$$

va gamma-funksiyaning $z\Gamma(z) = \Gamma(z+1)$ hossasi ko'rilayotgan koeffisientni quyidagicha ifodalashga imkon beradi:

$$\frac{\cos(n\pi)\Gamma\left(\frac{1}{2}+n\right)}{\sqrt{\pi}\,\Gamma(n+1)} = (-1)^n\frac{2\Gamma(2n)}{2^{2n}\Gamma(n)\Gamma(n+1)} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}.$$

Demak,

$$P_{2n}(0) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}. \qquad (46)$$

§2.4. Ortogonallik

Ortogonallik munosabatlari maxsus funksiyalar uchun juda muhim rol o'ynaydi. (43)-ni $P_m(x)$ ga ko'paytiraylik:

$$P_m(x)\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_m(x)P_n(x) = 0.$$

Shu tenglamaning o'zini $n \leftrightarrow m$ almashtirib yana bir marta yozamiz va ularning birini ikkinchisidan ayiramiz:

$$P_m\left[(1-x^2)P_n'(x)\right]' - P_n\left[(1-x^2)P_m'(x)\right]' = -P_nP_m[n(n+1) - m(m+1)].$$

Chap tomon quyidagi xossaga ega:

$$P_m\left[(1-x^2)P_n'(x)\right]' - P_n\left[(1-x^2)P_m'(x)\right]' =$$

$$= \frac{d}{dx}\left[P_m\left[(1-x^2)P_n'(x)\right] - P_n\left[(1-x^2)P_m'(x)\right]\right].$$

Olingan munosabatni x bo'yicha -1 dan +1 gacha integrallaymiz, bunda uning chap tomoni nologa teng bo'ladi (ixtiyoriy n, m lar uchun), o'ng tomoni esa faqat $n \neq m$ dagina nolga teng:

$$\int_{-1}^{1} dx P_n(x) P_m(x) = 0, \qquad n \neq m. \tag{47}$$

Bu munosabatni sferik koordinat sistemasida ham yozib olishimiz mumkin:

$$\int_{0}^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = 0, \qquad n \neq m. \tag{48}$$

$n = m$ holda yuqoridagi tenglikning chap va o'ng tomonlari $0 = 0$ ko'rinishga ega. Shuning uchun bu holni boshqacha yo'l bilan ko'rib chiqamiz. Hosil qilish funksiyasining kvadratidan integral hisoblaylik:

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \int_{-1}^{1} dx \left(\sum_{n=0}^{\infty} P_n(x) t^n \right)^2 = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} dx\, P_n^2(x). \tag{49}$$

Bu munosabatni olishda biz (47)-ni ishlatdik. Chap tomondagi integralni hisoblash qiyin emas:

$$\int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \frac{1}{t} \ln \frac{1+t}{1-t}.$$

Ikkinchi tomondan

$$\frac{1}{t} \ln \frac{1+t}{1-t} = 2 \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} + \cdots \right) = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}. \tag{50}$$

($\ln(1+t) = t - t^2/2 + t^3/3 - t^4/4 \cdots$). (49)-ning ohirgi qismi bilan (50)-ni solishtirsak Legendre polinomlarining "normasi" ning kvadratini topgan bo'lamiz:

$$\int_{-1}^{1} dx\, P_n^2(x) = \frac{2}{2n+1}. \tag{51}$$

§2.5. Integral tasavvur (Shläfli integrali)

Yana hosil qilish funksiyasiga qaytib kelaylik - (36)-ga. Kompleks analiz qoidalari bo'yicha undan quyidagi integral formulani olamiz:

$$P_n(z) = \frac{1}{2\pi i}\oint_C \frac{g(z,\zeta)}{\zeta^{n+1}}d\zeta.$$

Kontur C - $\zeta = z$ nuqtani o'z ichiga olgan ixtiyoriy kontur. ζ–kompleks o'zgaruvchi. Bu formulani qulayroq ko'rinishga keltirish uchun

$$\sqrt{1-2x\zeta+\zeta^2} = 1-\zeta\eta$$

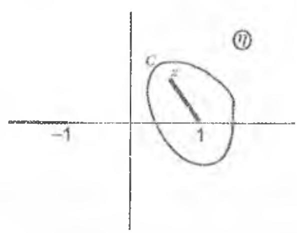
almahtirish bajaramiz, bu yerda η - yangi o'zgaruvchi. Bu holda

$$\zeta = 2\frac{\eta - z}{\eta^2-1}, \qquad d\zeta = 2\frac{1-\zeta\eta}{\eta^2-1}d\eta$$

bo'ladi va integral quyidagicha formaga keltiriladi:

$$P_n(z) = \frac{1}{2\pi i\cdot 2^n}\oint_C \frac{(\eta^2-1)^n}{(\eta-z)^{n+1}}d\eta. \tag{52}$$

Bu integral ***Schläfli***[8] integrali deyiladi. n butun bo'lmaganda integral osti funksiyada uchta tarmoqlanish nuqtasi bor - $\eta = z$, ± 1. Shu sababdan konturda ikkita kesma bo'lishi kerak - (-1) dan $-\infty$ gacha va 1 dan z gacha - (I.2)-rasmga qarang.



I.2-rasm: Schlafli integrali uchun kontur

[8] Ludwig Schläfli (1814-1895) - shveytsar matematigi

§2.6. Rodrigues formulasi

Cauchy teoremasi

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz$$

ni eslab Schlafli integralidan

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{53}$$

formulani olamiz. Bu - Rodrigues[9] formulasi deyiladi.

1.18-mashq. Bevosita Rodrigues formulasidan quyidagilarni keltirib chiqaring:

$$P_0(x) = 1, \quad P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x, \quad P_2(x) = \frac{1}{2^2 2}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1).$$

§2.7. Laplace tenglamasi sferik sistemada

Laplace[10] operatorining sferik sistemadigi ko'rinishini quyidagicha:

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$$

Laplace tenglamasi

$$\Delta u = 0$$

ning yechimini sferik sistemada o'zgaruvchilarni ajratish metodi bilan qidiraylik:

$$u(r,\theta,\varphi) = R(r)Y(\theta,\varphi).$$

Bu holda,

$$\Delta u = \frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} = 0 \tag{54}$$

tenglamani olamiz. Agarda shu tenglamani r^2 ga ko'paytirsak va RY ga bo'lsak

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = \lambda$$

tenglamani olamiz. Bu tenglamaning chap tomoni faqat r ga bog'liq, o'ng tomoni esa (θ,φ) ning funksiyasi. Demak, tenglikning na chap, na o'ng tomoni hech qanday o'zgaruvchi emas, konstanta ekan, shu sababdan biz o'ng

[9] Benjamin Olinde Rodrigues (1795-1851) - fransuz matematigi

[10] Pierre-Simon Laplace (1749-1827) - fransuz matematigi. Rus tilida - Лаплас.

tomonning oxirida hozircha noma'lum konstanta λ kiritdik. Shu sababdan ushbu tenglama ikkita tenglamalar sistemasiga aylanadi:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \lambda R(r) = 0; \qquad (55)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} + \lambda Y(\theta,\varphi) = 0. \qquad (56)$$

Ikkinchi tenglamada yana bir marta o'zgaruvchilarni ajratish mumkin:

$$Y(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi). \qquad (57)$$

Bu holda (56)-tenglama

$$\frac{\Phi(\varphi)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \frac{\Theta(\theta)}{\sin^2\theta}\frac{d^2\Phi(\varphi)}{d\varphi^2} + \lambda\Theta(\theta)\Phi(\varphi) = 0$$

ko'rinishga keladi. Uni $\sin^2\theta$ ga ko'paytiramiz va $\Theta\Phi$ ga bo'lamiz:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi(\varphi)}{d\varphi^2} = \mu.$$

Chap tomon faqat θ ga bog'liq, o'ng tomon - faqat φ ga, tenglik ixtiyoriy θ,φ larda bajarilishi uchun ikkala tomon ham konstanta bo'lishi kerak. O'sha konstantani μ harfi bilan belgiladik. Natijada bitta xususiy hosilali tenglama o'rniga ikkita oddiy hosilali tenglamaga ega bo'lamiz:

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + \mu\Phi(\varphi) = 0; \qquad (58)$$

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + (\lambda\sin^2\theta - \mu)\Theta(\theta) = 0. \qquad (59)$$

μ konstantani aniqlaylik. Buning uchun (58)-ning umumiy yechimini yozib olamiz:

$$\Phi(\varphi) = c_1\cos\sqrt{\mu}\varphi + c_2\sin\sqrt{\mu}\varphi.$$

φ– burchak, burchak 2π ga o'zgarganida biz yana o'sha nuqtaga qaytib kelamiz. Demak,

$$\Phi(\varphi + 2\pi) = \Phi(\varphi)$$

bo'lishi kerak. Bu shart bajarilishi uchun esa

$$\mu = m^2, \quad m = 0,\ 1,\ 2,\ 3,\ldots$$

bo'lishi kerak. Demak,

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{\pm im\varphi} \tag{60}$$

ekan. Koeffisient shunday tanlab olindiki,

$$\int\limits_0^{2\pi} d\varphi\, \Phi_m^*(\varphi)\Phi_n(\varphi) = \delta_{mn}$$

munosabat bajarilsin.

λ konstantani aniqlaylik. Buning uchun (59)-tenglamada o'zagruvchini quyidagicha almashtiramiz: $\cos\theta = x$ ((59)-tenglamani (43)- va (44)-tenglamalar bilan solishtiring). Bu almashtirishdan keyin (59)-tenglama quyidagi ko'rinishni oladi:

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta(x)}{dx}\right) + \left(\lambda^2 - \frac{m^2}{1-x^2}\right)\Theta(x) = 0. \tag{61}$$

λ - konstanta, u o'zining kiritilishi bo'yicha m ga bog'liq emas. Shuning uchun ishni yengillashtirish maqsadida $m = 0$ deb olamiz. Hosil bo'lgan tenglama

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta(x)}{dx}\right) + \lambda^2\Theta(x) = 0 \tag{62}$$

ning yechimini silindrik tenglamani yechganimizdek Frobenius metodi bilan izlaymiz ($x = 0$ nuqta atrofida):

$$\Theta(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots . \tag{63}$$

Oydinki,

$$\Theta'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots ,$$

$$\Theta''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots .$$

Topilganlarni (62)-ga olib borib qo'yamiz va x ning bir xil darajalari oldidagi koeffisientlarni yig'ib chiqamiz:

$$\sum_{n=0}^{\infty} \left[n(n-1)c_n x^{n-2} + (\lambda^2 - n(n+1))c_n x^n\right] = 0.$$

Bu tenglik bajarilishi uchun x ning bir xil darajalari oldidagi koeffisientlar yig'indisi nolga teng bo'lishi kerak, buning uchun

$$c_{n+2} = -\frac{\lambda^2 - n(n+1)}{(n+1)(n+2)} c_n$$

bo'lishi kerak. Ko'rinib turibdiki, (63)-qator $|x| = 1$ nuqtalarda yaqinlashuvchi bo'lmaydi, chunki bu hollarda

$$\left|\frac{c_{n+2}}{c_n}\right|_{n\to\infty} \to 1.$$

Demak, qatorni uzib, uni polinomga aylantirish kerak. Buning uchun

$$\lambda = n(n+1), \qquad n = 0, 1, 2, 3, ... \tag{64}$$

desak, yetarlidir. Bu holda (62)-tenglama Legandre tenglamasi (43)-ning o'zi bo'ladi:

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta(x)}{dx}\right) + n(n+1)\Theta(x) = 0$$

va $\Theta(x) = P_n(x)$ bo'ladi.

λ uchun topilgan qiymatni $R(r)$ uchun (55)-tenglamaga olib borib qo'yamiz:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - n(n+1)R(r) = 0.$$

Bu tenglamaning yechimini $R \sim r^s$ ko'rinishda qidirsak, $s = n$ va $s = -n-1$ bo'lib chiqadi, ya'ni

$$R(r) = Ar^n + Br^{-n-1}, \qquad A,\ B - \text{const.} \tag{65}$$

Agar ko'rilayotgan masala $r = 0$ nuqtani o'z ichiga olgan bo'lsa,

$$R(r) \sim r^n$$

deb olish kerak. Agar tashqi chegaraviy masala (VII-bobga qarang) ko'rilayotgan bo'lsa,

$$R(r) \sim r^{-n-1}$$

bo'ladi.

§2.8. Umumlashgan Legendre polinomlari

(61)-tenglamadagi λ aniqlangandan keyin, uning ko'rinishi quyidagicha bo'ladi:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta(x)}{dx}\right]+\left(n(n+1)-\frac{m^2}{1-x^2}\right)\Theta(x)=0. \qquad (66)$$

Bu tenglama $m = 0$ holda Legendre polinomlari uchun (43)-tenglamaning o'zidir. Uning ixtiyoriy butun m dagi yechimi ***umumlashgan Legendre polinomi*** deyiladi va u quyidagicha yoziladi:

$$\Theta(x) = P_n^m = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \qquad (67)$$

Umumiy holda bu yechimni tekshirish bir muncha hisob-kitobni talab qiladi.

1.19-mashq. Legendre polinomlari $P_n(x)$ uchun (43)-tenglamani bir marta differensiallab

$$P_n^1 = \sqrt{1-x^2}\frac{d}{dx}P_n(x)$$

belgilash kiritilsa tenglama

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n^1(x)}{dx}\right]+\left(n(n+1)-\frac{1}{1-x^2}\right)P_n^1(x)=0$$

ko'rinishga kelishini ko'rsating. Olingan tenglama (66)-da $m = 1$ deb olishga teng.

Ushbu mashqdagi amalni m marta bajarsak, (67)-formulaning to'g'riligiga ishonch hosil qilish mumkin. Ko'rish qiyin emaski,

$$P_n^0(x) = P_n(x), \quad P_1^1(x) = (1-x^2)^{1/2} = \sin\theta,$$

$$P_2^1(x) = 3x(1-x^2)^{1/2} = 3\cos\theta\sin\theta \quad \text{va} \quad \text{h.k.}$$

Agar (67)-formulada $P_n(x)$ uchun Rodrigues formulasini ishlatsak,

$$P_n^m(x) = \frac{1}{2^n n!}(1-x^2)^{m/2}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$$

munosabatni olamiz, undan ko'rinib turibdiki,

$$-n \le m \le n$$

bo'lishi kerak.

Umumlashgan Legendre polinomlarining xossalari juda ko'p, ulardan faqat ba'zi-birlarini misol sifatida keltiraylik:

$$P_n^m(-x) = (-1)^{n+m}P_n^m(x), \quad P_n^m(\pm 1) = 0, \; m > 0, \quad \text{va h.k.}$$

Umumlashgan Legendre polinomlari uchun quyidagi ortogonallik va norma sharti bor:

$$\int_{-1}^{1} dx\, P_n^m(x) P_k^m(x) = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nk}. \tag{68}$$

Sferik sistemada:

$$\int_0^\pi P_n^m(\cos\theta) P_k^m(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nk}.$$

1.20-mashq.

$$P_n^n(\cos\theta) = (2n-1)!!\sin^n\theta, \qquad n = 0, 1, 2, ...$$

ekanligini ko'rsating. Bu yerda $(2n-1)!! = 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)$. Masalan,

$P_1^1 = (1-x^2)^{1/2} = \sin\theta$, $P_2^2 = 3(1-x^2) = 3\sin^2\theta$, $P_3^3 = 15(1-x^2)^{3/2} = 15\sin^3\theta$ va h.k.

§2.9. Sferik funksiyalar

(57)-formula bizga Laplace tenglamasi yechimining burchak qismi $Y(\theta,\varphi)$ ni beradi. Agar (60)- va (67)-formulalarni hisobga olsak Y ni quyidagicha tanlab olishimiz mumkinligi oydin bo'ladi:

$$Y_n^m(\theta,\varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta)e^{im\varphi}. \tag{69}$$

Bu formuladagi koeffisient shunday tanlab olinganki,

$$\int_0^\pi\int_0^{2\pi} d\Omega\, Y_{n_1}^{m_1 *}(\theta,\varphi) Y_{n_2}^{m_2}(\theta,\varphi) = \delta_{n_1 n_2}\delta_{m_1 m_2}, \quad d\Omega = \sin\theta d\theta d\varphi, \tag{70}$$

bo'lsin. Sferik funksiyalarning bir necha xususiy holini keltiraylik:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}},\ Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi},\ Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta,\ Y_1^{-1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}.$$

Undan tashqari

$$Y_n^0(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta). \tag{71}$$

Shu paytgacha yiqqan bilimlarga asoslanib, Laplace tenglamasining sferik sistemadagi eng umumiy yechimi quyidagi ko'rinishga ega bo'lishi kerak degan xulosaga kelamiz:

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}\left[a_{nm}r^n + b_{nm}r^{-n-1}\right] Y_n^m(\theta,\varphi). \tag{72}$$

Laplace tenglamasiga olib kelgan masala sferik simmetriyaga ega bo'lganda bu tenglama sferik sistemada yechiladi. Olingan yechimning birinchi qismi $r^n Y_n^m(\theta,\varphi)$, $n = 0,1,2,..$ sferaning ichki sohasida $r < R$ ishlatiladi, $r^{-n-1}Y_n^m(\theta,\varphi)$, $n = 0,1,2,..$ qism esa sferaning tashqi qismida ishlatiladi.

Faraz qilaylik sferaning ustida $r = R$ yechim $f(\theta,\varphi)$ ga teng bo'lsin:

$$u(R,\theta,\varphi) = f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n} A_{nm}Y_n^m(\theta,\varphi). \tag{73}$$

Laplace tenglamasi uchun chegaraviy masalalarning aniq qo'yilishi 7-bobda muhokama qilingan, bu yerda bizni A_{nm} koeffisientlarni topish qiziqtiradi. A_{nm} koeffisientlar (70)-munosabatdan foydalanib topiladi:

$$A_{nm} = \int d\Omega\; Y_n^{m*}(\theta,\varphi)f(\theta,\varphi). \tag{74}$$

Bu munosabatning bir xususiy holi keyingi paragrafda muhim rol o'ynaydi. (71)-dan foydalanib quyidagini yozamiz:

$$A_{n0} = \int d\Omega\; Y_n^{0*}(\theta,\varphi)f(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}}\int d\Omega f(\theta,\varphi)P_n(\cos\theta). \tag{75}$$

Ikkinchi tomondan

$$f(0,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n} A_{nm}Y_n^m(0,\varphi) = A_{n0}\sqrt{\frac{2n+1}{4\pi}},$$

chunki $P_n^m(1) = \delta_{m,0}P_n(1) = \delta_{m,0}$ va natijada

$$Y_n^m(0,\varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}P_n^m(1)e^{im\varphi} = \sqrt{\frac{2n+1}{4\pi}}$$

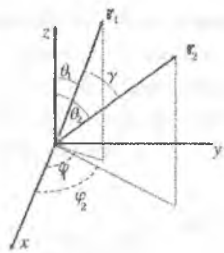
bo'ladi. Demak,

$$\int f(\theta,\varphi)P_n(\cos\theta)\,d\Omega = \frac{4\pi}{2n+1}f(0,\varphi). \tag{76}$$

§2.10. Legendre polinomlari uchun qo'shish teoremasi

Fazoda ikkita vektorlar $\mathbf{r}_1$ va $\mathbf{r}_2$ berilgan bo'lsin, ular orasidagi burchakni γ deb belgilaylik (I.3-rasmga qarang). Agar $\mathbf{r}_1$ ning sferik sistemadagi koordinatlari r_1,θ_1,φ_1 va $\mathbf{r}_2$ ning sferik sistemadagi koordinatlari r_2,θ_2,φ_2 bo'lsa,

$$\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) \tag{77}$$



I.3-rasm: Qo'shish teoremasiga oid

bo'ladi. Bu munosabatni $\mathbf{r}_1$ va $\mathbf{r}_2$ vektorlar orasidagi skalar ko'paytmani ikki xil yo'l bilan ifodalash orqali isbot qilish mumkin. Birinchidan,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_2 r_2 \cos\gamma.$$

Ikkinchi tomondan xuddi shu skalar ko'paytma

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_{1x}r_{2x} + r_{1y}r_{2y} + r_{1z}r_{2z} =$$

$$= r_1 r_2 \left[\sin\theta_1 \sin\theta_2(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2) + \cos\theta_1 \cos\theta_2\right]$$

ga teng. Shu ikkala formulani solishtirish (77)-formulaga olib keladi. γ ga mos keluvchi azimut ψ ni rasmda ko'rsatganimiz yo'q, chunki uning keyingi mulohazalarda ahamiyati yo'q.

Legendre polinomlari uchun qo'shish teoremasi quyidagidan iborat:

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{m=n} Y_n^m(\theta_1, \varphi_1) Y_n^{*m}(\theta_2, \varphi_2). \qquad (78)$$

Buni isbot qilish uchun $P_n(\cos\gamma)$ ni (θ_1, φ_1) burchaklar bo'yicha (73)-qatorga yoyamiz:

$$P_n(\cos\gamma) = \sum_{n'=0}^{\infty} \sum_{m=-n'}^{m=n'} A_{n'm}(\theta_2, \varphi_2) Y_n'^m(\theta_1, \varphi_1).$$

Bu yoyilmada (θ_2, φ_2) burchaklar parametr sifatida qaralyapti. Qatorda haqiqatda faqat $n' = n$ hadgina qoladi, aks holda ifodaning chap va o'ng tomonlari har-xil juftlikka ega bo'lib qolishi mumkin:

$$P_n(\cos\gamma) = \sum_{m=-n}^{m=n} A_{nm}(\theta_2, \varphi_2) Y_n^m(\theta_1, \varphi_1).$$

Koeffisientlar quyidagicha aniqlanadi:

$$A_{nm}(\theta_2, \varphi_2) = \int Y_n^{m*}(\theta_1, \varphi_1) P_n(\cos\gamma)\, d\Omega_{\theta_1,\varphi_1}.$$

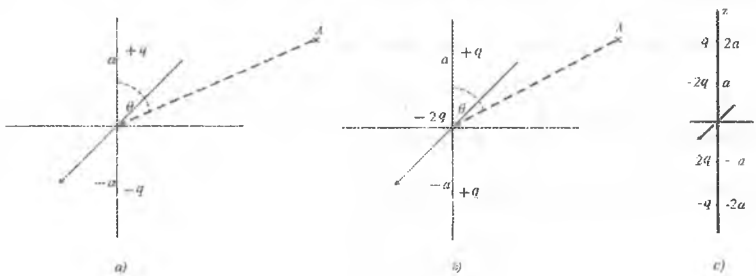
Bu formulaga (76)-ni ishlatsak

$$A_{nm}(\theta_2, \varphi_2) = \frac{4\pi}{2n+1} Y_n^{m*}(\theta_1(\gamma, \psi), \varphi_1(\gamma, \psi))\Big|_{\gamma=0} = \frac{4\pi}{2n+1} Y_n^{m*}(\theta_2, \varphi_2)$$

ekanligi kelib chiqadi. Shu bilan qo'shish teoremasi (78) isbot qilindi.

§2.11. Misollar

1.4-misol. z-o'qida koordinat boshidan a va $-a$ masofada joylashgan $+q$ va $-q$ zaryadlar sistemasi(dipol)ning kuzatish nuqtasi A da hosil qilgan elektrostatik maydonini toping (I.4-rasmning a)-qismi) ($r \gg a$ yaqinlashuvida).



I.4-rasm: Zaryadlar sistemalari

$z = a$ nuqtadagi zaryad hosil qilgan maydon

$$\varphi_{(+q)} = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n = \frac{q}{4\pi\varepsilon_0 r}\left[1 + P_1(\cos\theta)\frac{a}{r} + P_2(\cos\theta)\frac{a^2}{r^2} + \cdots\right].$$

$z = -a$ nuqtadagi zaryad hosil qilgan maydon

$$\varphi_{(-q)} = \frac{-q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{-a}{r}\right)^n = \frac{-q}{4\pi\varepsilon_0 r}\left[1 - P_1(\cos\theta)\frac{a}{r} + P_2(\cos\theta)\frac{a^2}{r^2} - \cdots\right].$$

Superpozitsiya prinsipi bo'yicha to'liq maydon ikkalasining yig'indisiga teng, noldan farqli bo'lgan birinchi had aniqligida (yuqori tartibli hadlarni tashlab yuboramiz, chunki $a/r \ll 1$):

$$\varphi = \varphi_{(+q)} + \varphi_{(-q)} = \frac{2q}{4\pi\varepsilon_0 r} P_1(\cos\theta)\frac{a}{r} + \cdots = \frac{qa\cos\theta}{2\pi\varepsilon_0 r^2} + \cdots$$

Bu formulaning vektor ko'rinishiga o'taylik. Buning uchun avval dipol momenti degan kattalikni kiritamiz: $\mathbf{d} = 2q\mathbf{a}$, bu yerda $\mathbf{a} = \{0, 0, a\}$, shundan keyin formulamiz quyidagi ko'rinishga keladi:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{r}}{r^3}.$$

1.5-misol. I.4-rasmning b)-qismida ko'rsatilgan sistema uchun elektrostatik maydonni toping ($r \gg a$ yaqinlashuvida) (sistemaning nomi - ***chiziqli kvadrupol***).

Uchta maydonni qo'shib chiqishimiz kerak:

$z = a$ nuqtadagi zaryad hosil qilgan maydon

$$\varphi^{(a)}_{(+q)} = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n = \frac{q}{4\pi\varepsilon_0 r}\left[1 + P_1(\cos\theta)\frac{a}{r} + P_2(\cos\theta)\frac{a^2}{r^2} + \cdots\right].$$

$z = -a$ nuqtadagi zaryad hosil qilgan maydon

$$\varphi^{(-a)}_{(+q)} = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{-a}{r}\right)^n = \frac{q}{4\pi\varepsilon_0 r}\left[1 - P_1(\cos\theta)\frac{a}{r} + P_2(\cos\theta)\frac{a^2}{r^2} - \cdots\right].$$

$z = 0$ nuqtadagi zaryad hosil qilgan maydon:

$$\varphi^{(0)}_{(-2q)} = \frac{-q}{2\pi\varepsilon_0 r}.$$

Umumiy maydon:

$$\varphi = \frac{2q}{4\pi\varepsilon_0 r}P_2(\cos\theta)\frac{a^2}{r^2} + \cdots = \frac{qa^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1) + \cdots$$

Uning vektor formasi:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{3(\mathbf{a}\cdot\mathbf{r})^2 - a^2r^2}{r^5}.$$

Quyidagi kattaliklarni kiritaylik:

$$D_{ij} = \sum q(3r_ir_j - \delta_{ij}), \qquad n_i = \frac{r_i}{r}.$$

Kiritilgan kattalik D_{ij} - sistemaning kvadrupol momenti deyiladi, yig'indi hamma zaryadlar bo'yicha, n_i esa birlik vektor. Bu holda

$$\varphi = \frac{1}{8\pi\varepsilon_0}\frac{D_{ij}n_in_j}{r^3}.$$

1.21-mashq. Kuzatish nuqtasi uchun $r < a$ bo'lsa (ya'ni, koordinat boshidan zaryadgacha masofa kuzatish nuqtasigacha masofadan katta bo'lsa) potensial uchun quyidagi ifoda to'g'ri bo'lishini ko'rsating:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{a}\right)^n.$$

Ushbu mashqda olingan natijani (35)-formula bilan bitta formulaga birlashtirish mumkin:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r_>}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_<}{r_>}\right)^n, \qquad r_> > r_<.$$

Bu yerda ikkita masofa kiritilgan - $r_>$ va $r_<$, ularning biri zaryadgacha masofa, ikkinchisi - kuzatish nuqtasigacha masofa. $r_>$ belgi ularning kattasini, $r_<$ belgi esa kichigini bildiradi.

1.22-mashq. I.4-rasmning c) qismida ko'rsatilgan chiziqli oktupol deyiladigan sistema uchun elektr potensialni toping.

§3. Kvant mexanikasida impuls momenti

Bu paragraf asosiy tekstga kirmaydi, uni 6.1-paragrafdan keyin o'qish tavsiya etiladi.

Impuls momenti quyidagicha ta'riflanadi:

$$\mathbf{L} = [\mathbf{r}\mathbf{p}],$$

bu yerda $\mathbf{p} = -i\hbar\nabla$. Impuls momentining komponentalari:

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Momentning kvadrati:

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2.$$

Momentning kvadratini sferik sistemada ifodalaylik. Buning uchun x, y, z va r, θ, φ larni bog'laydigan formulalarni olish kerak:

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arccos\frac{z}{r}, \qquad \varphi = \arctan\frac{y}{x}.$$

Shulardan foydalanib $\partial/\partial x$ ni hisoblaylik. Birinchidan:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}.$$

Ikkinchidan,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\varphi, \qquad \frac{\partial\theta}{\partial x} = \frac{1}{r}\cos\theta\cos\varphi, \qquad \frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$$

Demak,

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}.$$

Quyidagilarni ham xuddi shunday yo'l bilan topish mumkin:

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi};$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}.$$

Bu formulalar yordamida harakat miqdori momenti operatori L komponentalarining sferik sistemadagi ifodalarini topamiz:

$$L_x = i\hbar\left[\sin\varphi\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\cos\varphi\frac{\partial}{\partial\varphi}\right];$$

$$L_y = -i\hbar\left[\cos\varphi\frac{\partial}{\partial\theta} - \operatorname{ctg}\theta\sin\varphi\frac{\partial}{\partial\varphi}\right].$$

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}.$$

Olingan formulalardan foydalanib impuls momentining kvadrati quyidagi ifodaga tengligini ko'rsatish qiyin emas:

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

Laplace operatorining sferik sistemadagi ifodasi (54)-dagi

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{79}$$

qism ***Laplace operatorining burchak qismi*** deyiladi. Demak,

$$\mathbf{L}^2 = -\hbar^2\Delta_{\theta,\varphi}.$$

(56)-tenglamani olingan ma'lumotlar asosida

$$\mathbf{L}^2 Y(\theta,\varphi) = \hbar^2\lambda Y(\theta,\varphi)$$

ko'rinishga keltirish mumkin. Bu esa harakat miqdori momenti operatori kvadrati uchun xususiy qiymatlar masalasidir, bu masala §2.7.-paragrafda yechilgan, uning yechimi (64)-formula orqali ifodalanadi. Demak,

$$\mathbf{L}^2 Y(\theta,\varphi) = \hbar^2 n(n+1)Y(\theta,\varphi), \qquad n = 0, 1, 2, \ldots$$

§4. Hermite polinomlari

§4.1. Hosil qilish fumksiyasi

Hermite[11] polinomlarini boshqa hamma klassik polinomlardek bir necha yo'llar bilan kiritish mumkin. Biz yana hosil qiluvchi funksiya metodidan foydalanamiz:

$$g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{80}$$

Bu formula - Hermite polinomlari $H_n(x)$ ning ta'rifidir. Ta'rifning chap tomonini Taylor qatoriga yoysak,

$$1 - t^2 + 2xt + \frac{1}{2}(-t^2 + 2xt)^2 + \cdots = 1 + 2xt + \frac{t^2}{2!}[4x^2 - 2] + \cdots$$

darrov topishimiz mumkinki

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad \text{va h.k.} \tag{81}$$

(80)-ta'rifdan bevosita ravishda quyidagi xususiy hollarni keltirib chiqarishimiz mumkin:

$$H_{2n}(0) = (-1)^n\frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0, \quad H_n(-x) = (-1)^n H_n(x).$$

§4.2. Rekurrent munosabatlar

Rekurrent munosabatlarga o'taylik.

$$\frac{\partial}{\partial t}g(x,t) = (-2t+2x)e^{-t^2+2xt} = (-2t+2x)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}.$$

Bu tenglikdan

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{82}$$

rekurrent munosabatga kelamiz.

$$\frac{\partial}{\partial x}g(x,t) = 2te^{-t^2+2xt} = 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!},$$

yoki,

$$2nH_{n-1}(x) = H'_n(x). \tag{83}$$

Ikkita rekurrent munosabatni topdik: (82) va (83).

[11] Charle Hermite (1822-1901) - fransuz matematigi. Rus tilida - Шарль Эрмит

§4.3. Rodrigues formulasi

Ta'rif (80)-bo'yicha

$$H_n(x) = \frac{d^n}{dt^n} e^{-t^2+2xt}\Big|_{t=0}.$$

Shu formulani qulai ko'rinishga keltirish uchun

$$e^{-t^2+2xt} = e^{-(t-x)^2+x^2}$$

deb olamiz, unda

$$\begin{aligned} H_n(x) &= \frac{d^n}{dt^n} e^{-t^2+2xt}\Big|_{t=0} = e^{x^2} \frac{d^n}{dt^n} e^{-(t-x)^2}\Big|_{t=0} = \\ &= e^{x^2}(-1)^n \frac{d^n}{dx^n} e^{-(t-x)^2}\Big|_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \end{aligned} \tag{84}$$

formulaga kelamiz. Bu - Hermite polinomlari uchun Rodrigues formulasi.

1.23-mashq. Rodrigues formulasidan foydalanib $H_n(x)$ ni $n = 0, 1, 2$ lar uchun toping va ularni (81)-formulalar bilan solishtiring.

§4.4. Differensial tenglama

(83)-ni (82)-ga olib borib qo'yamiz va hosil bo'lgan munosabatdan x bo'yicha hosila olamiz:

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x) \quad \Rightarrow \quad H_{n+1}'(x) = 2H_n(x) + 2xH_n'(x) - H_n''(x).$$

Bu tenglikning chap tomonida (83)-ni yana bir marta ishlatsak

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{85}$$

tenglamaga kelamiz. Bu - Hermite tenglamasi.

§4.5. Hermite polinomlarining ortogonalligi va normasi

Quyidagi munosabat o'z-o'zidan oydindir:

$$e^{-x^2} g(x,t) g(x,s) = e^{-x^2} e^{-t^2+2xt} e^{-s^2+2xs} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-x^2} H_n(x) H_m(x) \frac{t^n s^m}{n!\, m!}.$$

Qulay ko'rinishga keltiraylik:

$$e^{-x^2} e^{-t^2+2xt} e^{-s^2+2xs} = e^{-(x-(s+t))^2+2st}.$$

Chap va o'ng tomondan x bo'yicha integral olamiz:

$$\int_{-\infty}^{\infty} dx\, e^{-(x-(s+t))^2+2st} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n s^m}{n!\, m!} \int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x).$$

Chap tomondagi integral oson topiladi:

$$\sqrt{\pi} e^{2st} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n s^m}{n!\, m!} \int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x).$$

Eksponentaning ta'rifi bo'yicha

$$e^{2st} = \sum_{n=0}^{\infty} \frac{(2st)^n}{n!}.$$

Demak,

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x) = \begin{cases} 0, & n \neq m; \\ 2^n n! \sqrt{\pi}, & n = m. \end{cases} \tag{86}$$

Kvant mexanikasida garmonik ossillator masalasini yechganimizda to'lqin funksiya Hermite polinomlari orqali ifodalanadi:

$$\psi_n(x) = \frac{e^{-x^2/2} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}. \tag{87}$$

Yuqoridagi formula bilan solishtirsak,

$$\int_{-\infty}^{\infty} dx\, \psi_n(x) \psi_m(x) = \delta_{nm} \tag{88}$$

ekanligini ko'ramiz.

1.6-misol. Yuqorida aytganimizdek Hermite polinomlari chiziqli ossillatorning kvant analizida uchraydi. Bir o'lchamli Schrödinger tenglamasi

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + [U(x) - E]\, \psi(x) = 0$$

ga $U(x) = \frac{1}{2}kx^2$ potensialni kiritamiz. Bunday potensial $F = -U'(x) = -kx$ chiziqli qaytaruvchi kuchga olib keladi. Bu tenglamada $k = m\omega^2$ va $\xi = \sqrt{m\omega/\hbar}x$ almashtirishlar bajarilsa, Schrödinger tenglamasi

$$\psi''(\xi) + \left(\frac{2E}{\hbar\omega} - \xi^2\right) \psi(\xi) = 0$$

ko'rinishga keladi. Olingan tenglamada

$$\psi(\xi) = e^{-\xi^2/2} H(\xi)$$

almashtirish bajarilsa quyidagini olamiz:

$$H''(\xi) - 2\xi H'(\xi) + \left(\frac{2E}{\hbar\omega} - 1\right) H(\xi) = 0. \tag{89}$$

Hosil bo'lgan tenglamaning yechimini Frobenius metodi bo'yicha qidiramiz:

$$H(\xi) = \sum_n c_n \xi^n = c_0 + c_1\xi + c_2\xi^2 + \cdots$$

Qulaylik uchun $a = 2E/(\hbar\omega) - 1$ belgilash kiritilsa, c_n koeffisientlar uchun quyidagi rekurrent munosabat kelib chiqadi:

$$c_{n+2} = \frac{a - 2n}{(n+1)(n+2)} c_n.$$

$|c_n/c_{n+2}|$ nisbat katta n larda cheklangan emas, demak, bu cheksiz qator yaqinlashuvchi bo'lmaydi. Shuning uchun qatorni $a = 2n$ tanlash asosida n-tartibli polinomga aylantiramiz. Bu esa birinchidan, (89)-tenglamani Hermite tenglamasi (85)-ga aylantiradi, ikkinchidan kvantlangan ossillatorning yaxshi ma'lum bo'lgan energetik sathlarini beradi:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

II BOB. IKKINCHI TARTIBLI XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING KLASSIFIKATSIYASI

§1. Ikkita mustaqil o'zgaruvchili hol. Umumiy nazariya

Ikkita mustaqil o'zgaruvchilarni (x, y) noma'lum funksiyani esa $u(x, y)$ deb belgilaymiz. Noma'lum funksiyaning xususiy hosilalarini esa quyidagicha belgilaymiz:

$$u_x = \frac{\partial u(x,y)}{\partial x}, \quad u_y = \frac{\partial u(x,y)}{\partial y}, \quad u_{xx} = \frac{\partial^2 u(x,y)}{\partial x^2},$$

$$u_{xy} = \frac{\partial^2 u(x,y)}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u(x,y)}{\partial y^2}.$$

Noma'lum funksiya, uning hosilalari va mustaqil argumentlar orasidagi quyidagi funksional bog'lanish

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1}$$

ikkita o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglama deyiladi[1]. Agar tenglama

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F_1(x, y, u, u_x, u_y) = 0 \tag{2}$$

ko'rinishga ega bo'lsa (va a_{11}, a_{12} hamda a_{22} koeffisientlar faqat x, y larga bog'liq bo'lsa) bunday tenglama ***yuqori tartibli hosilalarga nisbatan chiziqli tenglama*** deyiladi. Agar koeffisientlar a_{11}, a_{12} va a_{22} noma'lum funksiya u va/yoki u_x, u_y larga bog'liq bo'lsa, tenglama ***kvazichiziqli*** deyiladi.

Quyidagi ko'rinishdagi tenglama

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f(x, y) = 0 \tag{3}$$

agar a_{ij}, b_i, c, f lar faqat x, y larga bog'liq bo'lsa, ***chiziqli tenglama*** deyiladi. Agar $f(x, y) = 0$ bo'lsa, (3)- tenglama ***bir jinsli tenglama*** deyiladi.

[1] Albatta, bu munosabat ixtiyoriy (x, y, u) lar uchun ayniyat bo'lmasa. Masalan, quyidagi tenglik ixtiyoriy (x, y, u) lar uchun ayniyat bo'lib tenglama bo'la olmaydi: $\cos(u_x + u_y) - \cos u_x \cos u_y + \sin u_x \sin u_y = 0$

Bizning maqsadimiz x, y larning o'rniga shunday yangi o'zgaruvchilar

$$\zeta = \varphi(x,y), \qquad \eta = \psi(x,y) \tag{4}$$

kiritishki, natijada ko'rilayotgan tenglama biz uchun qulay bo'lgan ***kanonik*** deb ataladigan ko'rinishga kelsin. Mana shu almashtirishni (2)-tenglamaga qo'llaymiz. Albatta, almashtirish yakobiani noldan farqli bo'lishi kerak:

$$\frac{\partial(\zeta,\eta)}{\partial(x,y)} = \zeta_x\eta_y - \zeta_y\eta_x \neq 0.$$

Hosilalarni almashtirishdan boshlaymiz:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial u}{\partial \zeta} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = \zeta_x u_\zeta + \eta_x u_\eta, \quad u_y = \frac{\partial u}{\partial y} = \zeta_y u_\zeta + \eta_y u_\eta,$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\zeta_x u_\zeta + \eta_x u_\eta\right) = \zeta_x^2 u_{\zeta\zeta} + 2\zeta_x\eta_x u_{\zeta\eta} + \eta_x^2 u_{\eta\eta} + \zeta_{xx}u_\zeta + \eta_{xx}u_\eta,$$

$$u_{xy} = \frac{\partial}{\partial x}\left(\zeta_y u_\zeta + \eta_y u_\eta\right) = \tag{5}$$

$$= \zeta_x\zeta_y u_{\zeta\zeta} + (\zeta_x\eta_y + \zeta_y\eta_x)u_{\zeta\eta} + \eta_x\eta_y u_{\eta\eta} + \zeta_{xy}u_\zeta + \eta_{xy}u_\eta,$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\zeta_y u_\zeta + \eta_y u_\eta\right) = \zeta_y^2 u_{\zeta\zeta} + 2\zeta_y\eta_y u_{\zeta\eta} + \eta_y^2 u_{\eta\eta} + \zeta_{yy}u_\zeta + \eta_{yy}u_\eta.$$

Bu formulalarni (2)-ga olib borib qo'ysak uning ko'rinishi quyidagi holga keladi:

$$\tilde{a}_{11}u_{\zeta\zeta} + 2\tilde{a}_{12}u_{\zeta\eta} + \tilde{a}_{22}u_{\eta\eta} + \bar{F} = 0, \tag{6}$$

Bu yerda

$$\tilde{a}_{11} = a_{11}\zeta_x^2 + 2a_{12}\zeta_x\zeta_y + a_{22}\zeta_y^2, \quad \tilde{a}_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2,$$
$$\tilde{a}_{12} = a_{11}\zeta_x\eta_x + a_{12}\left(\zeta_x\eta_y + \eta_x\zeta_y\right) + a_{22}\zeta_y\eta_y. \tag{7}$$

$\bar{F}$ - noma'lum funksiyaga va uning birinchi tartibli xususiy hosilalariga bog'liqdir.

Endi ζ va η o'zgaruvchilarni shunday tanlab olaylikki, yangi koeffisientlarning bir qismi nolga teng bolib chiqsin. $\bar{a}_{11}$ va $\bar{a}_{22}$ larni nolga tenglashdan boshlaylik. (7)-tenglamaning birinchi va uchinchi qismlarining ko'rinishi bir xildir, ya'ni

$$a_{11}z_x^2 + 2a_{12}z_xz_y + a_{22}z_y^2 = 0. \tag{8}$$

Mana shu tenglamani yechib $z = z(x, y)$ funksiyani topsak va $\zeta = z(x, y)$ deb olsak $\tilde{a}_{11} = 0$ bo'ladi, $\eta = z(x, y)$ deb olsak $\tilde{a}_{22} = 0$ bo'ladi.

Teorema. (8)-tenglamaning yechimi

$$a_{11}dy^2 - 2a_{12}dxdy + a_{22}dx^2 = 0 \tag{9}$$

tenglamaning umumiy integrali $\varphi(x, y) = \text{const}$ ga tengdir.

Isbot.

$$d\varphi = 0 = \varphi_x dx + \varphi_y dy$$

dan

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$$

kelib chiqadi. Bu degani (9)-ni

$$a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 + 2a_{12}\frac{\varphi_x}{\varphi_y} + a_{22} = 0$$

ko'rinishga keltira olamiz. Bu tenglamani

$$a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2 = 0$$

ko'rinishga keltirsak (8)-ning o'zini olamiz($z = \varphi(x, y)$).

(9)-tenglama (2)-ning ***xarakteristik tenglamasi*** deyiladi, uning umumiy integrali esa (2)-ning ***xarakteristikasi*** deyiladi.

(9)-ning ikkita yechimi bor:

$$\begin{aligned} \frac{dy}{dx} &= \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}, \\ \frac{dy}{dx} &= \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}. \end{aligned} \tag{10}$$

Agar $D = a_{12}^2 - a_{11}a_{22}$ belgilash kiritsak, (2)-tenglama D -ning ishorasiga qarab quyidagi uch xil turga bo'linadi:

1. $D > 0$ – giperbolik;
2. $D = 0$ – parabolik;
3. $D < 0$ – elliptik.

Keyin biz ko'ramizki, tenglama o'zining tipiga qarab alohida xususiyatlarga ega bo'ladi - har bir tipdagi tenglama faqat ma'lum tipdagi fizik jarayonlarnigina ifodalaydi. Bundan kelib chiqadiki, D - ning ishorasi (2)-tenglamaning muhim bir xarakteristikasidir. D - ning ishorasi (4)-almashtirishga bog'liq emas:

$$\tilde{a}_{12}^2 - \tilde{a}_{11}\tilde{a}_{22} = (a_{12}^2 - a_{11}a_{22})(\zeta_x\eta_y - \zeta_y\eta_x)^2, \tag{11}$$

ya'ni, tenglamaning tipi (4)-almashtirish bajarilganda o'zgarmaydi.

2.1-mashq. (11)-munosabatni keltirib chiqaring.

Shu uchta holni alohida ko'rib chiqaylik.

§2. Giperbolik hol ($D > 0$)

Bu holda (9)- va (10)-tenglamalarning ikkita har xil yechimi bor:

$$\varphi(x, y) = c_1, \qquad \psi(x, y) = c_2. \tag{12}$$

Shu yechimlardan foydalanib,

$$\zeta = \varphi(x, y), \qquad \eta = \psi(x, y) \tag{13}$$

almashtirish bajaramiz. Natijada $\tilde{a}_{11} = 0$ va $\tilde{a}_{22} = 0$ bo'ladi va (2)-tenglama quyidagi ***kanonik*** ko'rinishga keltiriladi:

$$\frac{\partial^2 u}{\partial\zeta\partial\eta} = u_{\zeta\eta} = \Phi(\zeta, \eta, u, u_\zeta, u_\eta) \tag{14}$$

Bu yerda $\Phi = -\bar{F}/(2\tilde{a}_{12})$. Tenglamamizni yana bir boshqa ko'rinishga keltirishimiz mumkin. Yangi almashtirihs bajaraylik:

$$\zeta = t + z, \qquad \eta = t - z,$$

yoki,

$$t = \frac{\zeta + \eta}{2}, \qquad z = \frac{\zeta - \eta}{2}.$$

Bu holda

$$\frac{\partial u}{\partial\zeta} = \frac{1}{2}\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z}\right), \quad \frac{\partial u}{\partial\eta} = \frac{1}{2}\left(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial z}\right), \quad \frac{\partial^2 u}{\partial\zeta\partial\eta} = \frac{1}{4}(u_{tt} - u_{zz}).$$

Demak, tenglamamiz

$$u_{tt} - u_{zz} = \Phi_1(x, y, u, u_\zeta, u_\eta) \tag{15}$$

ko'rinishga keltirildi. Bu ko'rinish giperbolik tenglamalarning ***ikkinchi kanonik ko'rinishi*** deyiladi, ((14)-esa birinchi kanonik ko'rinish edi).

§3. Parabolik tenglama ($D = 0$)

Bu holda,

$$\frac{dy}{dx} = \frac{a_{12}}{a_{11}}$$

bo'ladi va xarakteristikalarning soni ikkita emas bitta bo'ladi. Mana shu bitta yechimdan foydalanib, yangi ζ o'zgaruvchi kiritamiz, η sifatida esa ixtiyoriy bir funksiya olishimiz mumkin:

$$\zeta = \varphi(x, y), \qquad \eta = \eta(x, y). \tag{16}$$

Bu yerda $\eta(x, y)$ - ixtiyoriy funksiya ($\varphi(x, y)$ ga chiziqli bog'liq bo'lmagan). $D = 0$ dan kelib chiqadigan $a_{12} = \sqrt{a_{11}a_{22}}$ va undan tashqari $\varphi_x dx + \varphi_y dy = \zeta_x dx + \zeta_y dy = 0$ munosabatlardan foydalansak

$$\bar{a}_{11} = \left(\sqrt{a_{11}}\zeta_x + \sqrt{a_{22}}\zeta_y\right)^2 = 0,$$

$$\bar{a}_{12} = \left(\sqrt{a_{11}}\zeta_x + \sqrt{a_{22}}\zeta_y\right)\left(\sqrt{a_{11}}\eta_x + \sqrt{a_{22}}\eta_y\right) = 0$$

ekanligini topamiz. Demak, ***parabolik tenglamaning kanonik ko'rinishi***

$$u_{\eta\eta} = \Phi_3(\zeta, \eta, u, u_\zeta, u_\eta) \tag{17}$$

bo'lar ekan ($\Phi_3 = -\bar{F}/a_{22}$).

§4. Elliptik tenglama ($D < 0$)

Bu holda haqiqiy xarakteristikalar mavjud emas, chunki (10)-ning o'ng tomonlari kompleks funksiyalardir:

$$\frac{dy}{dx} = \lambda(x, y), \quad \frac{dy}{dx} = \bar{\lambda}(x, y), \qquad \lambda(x, y) = \frac{a_{12} + i\sqrt{a_{11}a_{22} - a_{12}^2}}{a_{11}}. \tag{18}$$

Birinchi tenglamaning yechimi $\varphi(x, y) = c$ kompleks funksiyadir, shunga yarasha $\varphi^*(x, y) = c^*$ ikkinchi tenglamaning yechimidir. Shundan foydalanib yangi o'zgaruvchilarni quyidagicha tanlab olamiz:

$$\zeta = \frac{\varphi + \varphi^*}{2}, \qquad \eta = \frac{\varphi - \varphi^*}{2i}. \tag{19}$$

Ya'ni, kompleks funksiya $\varphi(x, y)$ ning haqiqiy qismini ζ deb oldik, mavhum qismini esa η deb oldik. $\varphi(x, y)$ ning ta'rifi bo'yicha

$$a_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2a_{12}\left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial y}\right) + a_{22}\left(\frac{\partial\varphi}{\partial y}\right)^2 = 0.$$

Kompleks tenglikning haqiqiy va mavhum qismlarini alohida nolga tenglashtirishimiz kerak, buning uchun

$$\varphi = \zeta + i\eta, \quad \varphi^* = \zeta - i\eta, \quad \varphi_x^2 = \zeta_x^2 - \eta_x^2 + 2i\zeta_x\eta_x,$$

$$\varphi_y^2 = \zeta_y^2 - \eta_y^2 + 2i\zeta_y\eta_y, \qquad \varphi_x\varphi_y = \zeta_x\zeta_y - \eta_x\eta_y + i\zeta_x\eta_y + i\zeta_y\eta_x$$

munosabatlardan foydalanamiz. Natijada

$$a_{11}\zeta_x^2 + 2a_{12}\zeta_x\zeta_y + a_{22}\zeta_y^2 = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2,$$

ya'ni

$$\tilde{a}_{11} = \tilde{a}_{22}, \tag{20}$$

va

$$a_{11}\zeta_x\eta_x + a_{12}(\zeta_x\eta_y + \zeta_y\eta_x) + a_{22}\zeta_y\eta_y = \tilde{a}_{12} = 0 \tag{21}$$

munosabatlarni olamiz. Demak, ***elliptik tenglamaning kanonik ko'rinishi***

$$u_{\zeta\zeta} + u_{\eta\eta} = \Phi_4(\zeta, \eta, u, u_\zeta, u_\eta) \tag{22}$$

bo'lar ekan ($\Phi_4 = -\bar{F}/\tilde{a}_{11}$).

Xulosa qilib olingan natijalarni bir joyga yig'aylik. Xususiy hosilali ikkinchi tartibli ikki o'zgaruvchili (2)-tenglamani quyidagi uch xil ko'rinishga keltirish mumkin ekan (kanonik ko'rinishga keltirib olganimizdan keyin ixtiyoriy o'zgaruvchilarni ishlatishimiz mumkin):

- giperbolik tip: $u_{xx} - u_{yy} = \Phi_1$ yoki $u_{xy} = \Phi_2$;
- parabolik tip: $u_{xx} = \Phi_3$;
- elliptik tip: $u_{xx} + u_{yy} = \Phi_4$.

Bu tenglamalarning ixcham va sodda ko'rinishi ularni ***kanonik*** deb atashga sabab bo'lgan. Bunday klassifikatsiya nuqtaga bog'liq: a_{ij} koeffisientlar tekisliktadi (x, y) - nuqtaning funksiyasi bo'lgani uchun D ning ishorasi bir nuqtadan ikkinchisiga o'tganda o'zgarishi mumkin va demak, tenglamaning kanonik ko'rinishi ham o'zgarishi mumkin.

2.1-misol. $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$.

Koeffisientlarni topamiz: $a_{11} = 1, a_{12} = -1, a_{22} = -3$.

Diskriminant $D = 4 > 0$, demak, tenglamamiz giperbolik tipga tegishli ekan. Xarakteristik tenglama:

$$\frac{dy}{dx} = -1 \pm 2.$$

Xarakteristikalar:

$$\zeta = x - y, \qquad \eta = 3x + y.$$

Demak, $\zeta_x = 1$, $\zeta_y = -1$, $\eta_x = 3$, $\eta_y = 1$. Hosilalarni hisoblaylik:

$$u_x = u_\zeta + 3u_\eta, \quad u_y = -u_\zeta + u_\eta, \quad \text{va h.k.}$$

Tenglamaning kanonik ko'rinishi:

$$u_{\zeta\eta} + \frac{1}{16}(u_\eta - u_\zeta) = 0.$$

2.2-misol. $yu_{xx} + u_{yy} = 0$.
Bu tenglamaning nomi - Trikomi tenglamasi. U aerodinamikada uchraydi.

Koeffisientlar: $a_{11} = y, a_{12} = 0, a_{22} = 1$. Diskriminant $D = -y$, ya'ni, tenglama

- $y < 0$ sohada giperbolik;
- $y > 0$ sohada elliptik;

a) $y < 0$ giperboliklik soha. Xarakteristik tenglama:

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{-y}}.$$

Xarakteristikalar:

$$\zeta = \frac{3}{2}x + \sqrt{-y^3}, \qquad \eta = \frac{3}{2}x - \sqrt{-y^3}.$$

Tenglamaning kanonik ko'rinishi:

$$u_{\zeta\eta} + \frac{1}{6(\zeta - \eta)}(u_\zeta - u_\eta) = 0.$$

b) $y > 0$ elliptiklik sohasi. Xarakteristik tenglamalar:

$$\frac{dy}{dx} = \pm i\frac{1}{\sqrt{y}}.$$

Ularning umumiy integrallari:

$$\varphi = \frac{3}{2}x \pm i\sqrt{y^3}.$$

Yangi o'zgaruvchilar:

$$\zeta = \frac{3}{2}x, \qquad \eta = -\sqrt{y^3}.$$

Tenglamaning kanonik ko'rinishi:

$$u_{\zeta\zeta} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0.$$

2.3-misol. $xu_{xx} - 2\sqrt{xy}u_{xy} + yu_{yy} + \frac{1}{2}u_y = 0$.

Koeffisientlar: $a_{11} = x$, $a_{12} = -\sqrt{xy}$, $a_{22} = y$. Demak, $D = 0$, tenglama parabolik tipga tegishli. Xarakteristik tenglama:

$$\frac{dy}{dx} = -\sqrt{y/x}.$$

Uning umumiy integrali:

$$\zeta = \sqrt{x} + \sqrt{y}.$$

Ikkinchi mustaqil o'zgaruvchi sifatida ixtiyoriy (lekin ζ ga chiziqli bog'liq bo'lmagan) o'zgaruvchini olishimiz mumkin. Masalan, $\eta = \sqrt{x}$. Kerakli hosilalarni hisoblab berilgan tenglamaga olib borib qo'ysak

$$u_{\eta\eta} - \frac{1}{\eta}(u_\zeta + u_\eta) = 0$$

ko'rinishdagi parabolik tenglamaga kelamiz.

2.2-mashq. Kanonik ko'rinishga keltiring: $u_{xx} - 6u_{xy} + 10u_{yy} + u_x - 3u_y = 0$.
2.3-mashq. Kanonik ko'rinishga keltiring: $4u_{xx} + 4u_{xy} + u_{yy} - 2u_y = 0$.
2.4-mashq. Kanonik ko'rinishga keltiring: $u_{xx} - xu_{yy} = 0$.
2.5-mashq. Kanonik ko'rinishga keltiring: $u_{xx} - yu_{yy} = 0$.
2.6-mashq. Kanonik ko'rinishga keltiring: $xu_{xx} + yu_{yy} = 0$.
2.7-mashq. Kanonik ko'rinishga keltiring: $y^2u_{xx} + x^2u_{yy} = 0$.
2.8-mashq. Kanonik ko'rinishga keltiring: $x^2u_{xx} + y^2u_{yy} = 0$.
2.9-mashq. Kanonik ko'rinishga keltiring: $x^2u_{xx} - y^2u_{yy} = 0$.
2.10-mashq. Kanonik ko'rinishga keltiring: $y^2u_{xx} - x^2u_{yy} = 0$.
2.11-mashq. Kanonik ko'rinishga keltiring: $(1+x^2)u_{xx}+(1+y^2)u_{yy}+xu_x+yu_y-2u = 0$.
2.12-mashq. Kanonik ko'rinishga keltiring: $x^2u_{xx} - 2xu_{xy} + u_{yy} = 0$.
2.13-mashq. Kanonik ko'rinishga keltiring: $y^2u_{xx} + 2yu_{xy} + u_{yy} = 0$.
2.14-mashq. Kanonik ko'rinishga keltiring: $y^2u_{xx} + 2xyu_{xy} + x^2u_{yy} = 0$.

§5. n ta mustaqil o'zgaruvchili hol

Mustaqil o'zgaruvchilarni $x_1, x_2, x_3, ..., x_n$ deb belgilaymiz. Noma'lum funksiyaning argumentida esa bu n ta o'zgaruvchini qisqalik uchun bitta x harfi bilan belgilaymiz: $u(x) = u(x_1, x_2, x_3, ..., x_n)$. Yuqori hosilalarga nisbatan chiziqli bo'lgan ikkinchi tartibli xususiy hosilali differensial tenglamaning ko'rinishi quyidagicha bo'ladi:

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu(x) = f(x). \tag{23}$$

Bu yerda a_{ij}, b_i va c koeffisientlar uzliksiz bo'lib koordinatalarga bog'liq bo'lishi mumkin: $a_{ij} = a_{ij}(x)$, $b_i = b_i(x)$, $c = c(x)$. Yozilgan tenglamani kanonik ko'rinishga keltiramiz. Buning uchun x koordinatlar ustida

$$x_i \to \zeta_i = \zeta_i(x), \qquad i = 1, 2, ..., n, \qquad \zeta_i \in C^2(R^n), \qquad \det\left(\frac{\partial \zeta_i}{\partial x_j}\right) \neq 0 \tag{24}$$

almashtirish bajaramiz. Almashtirish determinanti noldan farqli bo'lgani uchun $x = x(\zeta)$ ni topishimiz mumkin (determinant noldan farqli bo'lgan hamma nuqtalarda). Shuni hisobga olib $\tilde{u}(\zeta) = u(x(\zeta))$ deb belgilaymiz. Hosilalarni hisoblashga o'tamiz:

$$\frac{\partial u}{\partial x_i} = \sum_{l=1}^{n} \frac{\partial \zeta_l}{\partial x_i}\frac{\partial \tilde{u}}{\partial \zeta_l},$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\sum_{l=1}^{n} \frac{\partial \zeta_l}{\partial x_i}\frac{\partial \tilde{u}}{\partial \zeta_l} = \sum_{l,k=1}^{n} \frac{\partial \zeta_l}{\partial x_i}\frac{\partial \zeta_k}{\partial x_j}\frac{\partial^2 \tilde{u}}{\partial \zeta_k \partial \zeta_l} + \sum_{l=1}^{n} \frac{\partial^2 \zeta_l}{\partial x_i \partial x_j}\frac{\partial \tilde{u}}{\partial \zeta_l}.$$

Topilgan hosilalarni (23)-tenglamaga olib borib qo'yamiz:

$$\sum_{l,k=1}^{n}\left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial \zeta_l}{\partial x_i}\frac{\partial \zeta_k}{\partial x_j}\right)\frac{\partial^2 \tilde{u}}{\partial \zeta_k \partial \zeta_l} + \sum_{l=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial \zeta_l}{\partial x_i} b_i + \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 \zeta_l}{\partial x_i \partial x_j}\right)\frac{\partial \tilde{u}}{\partial \zeta_l} +$$

$$+c\tilde{u}(\zeta) = \tilde{f}(\zeta).$$

Ikkinchi tartibli hosilalarning oldidagi yangi koeffisientnlarni quyidagicha belgilaymiz:

$$\tilde{a}_{lk}(\zeta) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial \zeta_l}{\partial x_i}\frac{\partial \zeta_k}{\partial x_j}. \tag{25}$$

Qolgan hadlarning hammasini bitta $\tilde{\Phi}(\zeta, \tilde{u}, \partial\tilde{u}/\partial\zeta)$ harf bilan belgilasak (tenglamaning kanonik tipiga ularning daxli yo'qligini bilamiz) yangi o'zgaruvchilar tilida berilgan tenglama quyidagi ko'rinishga keladi:

$$\sum_{l,k=1}^{n} \tilde{a}_{lk}(\zeta)\frac{\partial^2 \tilde{u}}{\partial \zeta_k \partial \zeta_l} + \tilde{\Phi}(\zeta, \tilde{u}, \partial\tilde{u}/\partial\zeta) = 0. \tag{26}$$

Tenglamaning klassifikatsiyasi nuqtaga bog'liqligini avvalgi paragraflarda ko'rdik, shuning uchun ma'lum bir x_0 nuqtaga o'tamiz. Bu nuqtada $\zeta_0 = \zeta(x_0)$ bo'ladi. Keyingi mulohazalarni yaxshiroq tushunish va soddalashtirish uchun (25)-formulada

$$q_{li} = \left.\frac{\partial \zeta_l}{\partial x_i}\right|_{x=x_0} \tag{27}$$

deb belgilaymiz, unda (25)-formula

$$\tilde{a}_{lk}(\zeta_0) = \sum_{i,j=1}^{n} a_{ij}(x_0) q_{li} q_{kj} \tag{28}$$

ko'rinishni oladi. Bu yerdagi har bir ikki indeksli kattalikni $n \times n$ o'lchamli matritsa deb qarash qulaydir. O'zining ta'rifi bo'yicha q_{li} matritsa $x \to \zeta$ koordinat almashtirish matritsasidir, matritsaning indekslarining o'rnini almashtirsak transponirlangan matritsaga o'tgan bo'lamiz: $q_{kj} = q^T_{jk}$, shuning uchun (28)-formula matrik formada quyidagi ko'rinishni oladi[2]:

$$\tilde{a} = q a q^T. \tag{29}$$

(23)-differensial tenglamani kanonik ko'rinishga keltirish shunday q_{li} matritsaga olib keladigan $x \to \zeta$ koordinat almashtirishni bajarishki natijada $\tilde{a}$ matritsa diagonal ko'rinishga kelsin va uning diagonalida faqat +1, -1 yoki 0 sonlar bo'lsin: $\tilde{a}_{kl} = \alpha_k \delta_{kl}$, $\alpha_k = \pm 1, 0$. Bu holda (26)-tenglamaning ikkinchi hosilali hadida faqat $k = l$ bo'lgan hadlar qoladi. Chiziqli algebra kursida bunday almashtirishni hamma vaqt bajarish mumkinligi isbot qilinadi. Bu masala

$$\sum_{i,j=1}^{n} a_{ij} \alpha_i \alpha_j \tag{30}$$

kvadratik formani

$$\alpha_i = \sum_{k=1}^{n} p_{ik} \beta_k, \quad \det(p) \neq 0 \tag{31}$$

almashtirish yordamida

$$\sum_{i,j=1}^{n} \tilde{a}_{ij} \beta_i \beta_j$$

[2]Matritsalarning ko'paytirish qoidasini eslatib o'taylik: $n \times n$ bo'lgan A va B matritsalar berilgan bo'lsa, ularning ko'paytmasi quyidagicha aniqlanadi: $(AB)_{ij} = \sum\limits_{k=1}^{n} A_{ik} B_{kj}$.

formaga keltirish masalasi bilan bir xil. Chiziqli algebra kursida isbot qilinadiki, (30)-formani hamma vaqt

$$\sum_{i=1}^{k} \beta_i^2 - \sum_{i=k+1}^{m} \beta_i^2, \qquad m \leq n \tag{32}$$

ko'rinishga keltirib olish mumkin.

(30)- va (31)-larni (28)- va (29)-formulalar bilan solishtirilsa p va q matritsalar o'zaro transponirlangan ekanligi ko'rinadi: $q = p^T$.

Ma'lumki, k va m sonlar (31)-almashtirishga bog'liq emas (bu tasdiq kvadratik formalarning inersiya qonuni deyiladi). Demak, differensial tenglamaning kanonik ko'rinishi faqatgina a_{ij} koeffisientlarning x_0 nuqtadagi qiymatigagina bo'g'liq ekan.

(24)-almashtirish natijasida (23)-tenglama quyidagi kanonik ko'rinishga kelsin:

$$\sum_{i=1}^{k} \frac{\partial^2 u}{\partial \zeta_i^2} - \sum_{i=k+1}^{m} \frac{\partial^2 u}{\partial \zeta_i^2} + \Phi(\partial u/\partial \zeta, u, \zeta) = 0.$$

Agarda $k = n$ yoki $k = 0, m = n$ bo'lsa, olingan tenglama ***elliptik tenglama*** deyiladi. Bu holda tenglamadagi ikkinchi tartibli hosilali hadlarning hammasi bir xil ishorali bo'ladi. Agar $m = n$ bo'lib $1 \leq k \leq n-1$ bo'lsa, tenglama ***giperbolik*** deyiladi (xususan, agar $k = 1$ yoki $k = n-1$ bo'lsa, tenglama ***normal giperbolik*** deyiladi). Va nihoyat, agar $m < n$ bo'lsa, tenglama ***parabolik*** (xususan, $m = n-1$ bo'lib $k = 1$ yoki $k = n-1$ bo'lsa, normal parabolik) deyiladi.

2.4-misol. $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} = 0$ tenglamani kanonik ko'rinishga keltiring.

(30)-bo'yicha

$$Q = \alpha_1^2 + 2\alpha_1\alpha_2 + 2\alpha_2^2 + 4\alpha_2\alpha_3 + 5\alpha_3^2$$

forma tuzib olamiz. Bu formani darhol

$$Q = (\alpha_1 + \alpha_2)^2 + (\alpha_2 + 2\alpha_3)^2 + \alpha_3^2$$

ko'rinishga keltirish mumkin. Ko'rinib turibdiki,

$$\beta_1 = \alpha_1 + \alpha_2, \qquad \beta_2 = \alpha_2 + 2\alpha_3, \qquad \beta_3 = \alpha_3 \tag{33}$$

belgilashlar kiritilsa boshlang'ich forma

$$Q = \beta_1^2 + \beta_2^2 + \beta_3^2$$

ko'rinishni qabul qiladi. (33)-formulalardan

$$\alpha_1 = \beta_1 - \beta_2 + 2\beta_3, \qquad \alpha_2 = \beta_2 - 2\beta_3, \qquad \alpha_3 = \beta_3$$

kelib chiqadi, bularni

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

matrik ko'rinishda olsak, p_{ij} matritsa topilgan bo'ladi:

$$p = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\det p = 1$ ekanligi ko'rinib turibdi. q matritsa p ga transponirlangan bo'lishi kerak:

$$q = p^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$

(transponirlash - satrlar va ustunlarning o'rnini almashtirish). Olingan matritsa (27)-formulaning ma'nosi bo'yicha ζ_i va x, y, z o'zgaruvchilarni bog'laydigan matritsadir:

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Ya'ni,

$$\zeta_1 = x, \qquad \zeta_2 = y - x, \qquad \zeta_3 = 2x - 2y + z.$$

Hosilalarning hammasini hisoblab chiqib tenglamaga olib borib qo'ysak, u kanonik ko'rinishi elliptik bo'lgan tenglama ekanligini topamiz:

$$u_{\zeta_1\zeta_1} + u_{\zeta_2\zeta_2} + u_{\zeta_3\zeta_3} = 0.$$

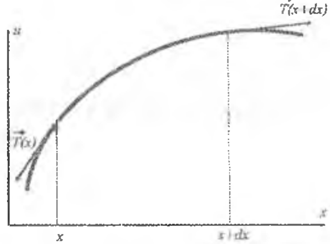
2.15-mashq. $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} + 3u_x = 0$ tenglamani kanonik ko'rinishga keltiring.

2.16-mashq. $u_{xx} - 2u_{xy} - 2u_{xz} + 3u_{yy} - 2u_{yz} + 3u_{zz} = 0$ tenglamani kanonik ko'rinishga keltiring.

III BOB. GIPERBOLIK TENGLAMALARGA OLIB KELADIGAN FIZIK JARAYONLAR

§1. Torning ko'ndalang tebranishlari

Uzunligi l bo'lgan ingichka torning ko'ndalang tebranishlari masalasini ko'rib chiqaylik. Uning bosh va oxirgi nuqtalarini a va b deb berlgilaymiz. Torning muvozanat holatidan siljishini kichik deb qaraymiz, ya'ni, biz ***kichik tebranishlar*** masalasini ko'ramiz. Torning x koordinatali nuqtasining t vaqt momentida o'z muvozanat holatidan siljishini $u(x,t)$ deb belgilaymiz.



III.1-rasm: Torning ko'ndalang tebranishiga doir

Siljish kichik deganimiz tgα $=$ $\partial u(x,t)/\partial x$ ham kichik bo'ladi deganimizga tengdir. Bu holda siljish natijasida torning uzunligi o'zgarmaydi deb olishimiz kerak (chunki uning o'zgarishi ikkinchi tartibli kichik son bo'lib chiqadi):

$$l = \int_a^b \sqrt{du^2 + dx^2} =$$

$$= \int_a^b \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx \simeq b - a$$

Tor bo'yicha taqsimlangan tashqi kuch zichligini $F(x,t)$ deb belgilaylik, bu kuch har bir nuqtada torga perpendikular yo'nalgan bo'lsin. Torning tarangligini $\mathbf{T}(x)$ deb belgilaymiz, albatta $\mathbf{T}(x)$ nutadan nuqtaga o'tganda o'z yo'nalishini o'zgartiradi, lekin uning son qiymati $T = |\mathbf{T}|$ o'zgarmaydi. Bu tasdiq torning uzunligi o'zgarmasligidan kelib chiqadi - torning uzunligi o'zgarmas ekan uning tarangligi ham o'zgarmaydi. Tor massasi zhichligini $\rho(x)$ deb belgilaymiz, ya'ni $\rho(x)dx$ - torning x va $x + dx$ nuqtalari orasidagi massadir. Torning mana shu kichik dx elementi uchun harakat tenglamasi - Newton tenglamasini tuzamiz:

$$T \sin\alpha|_{x+dx} - T\sin\alpha|_x + F(x,t)dx = \rho(x)dx\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

(Newton qonuni: ta'sir qilayotgan kuch=massa×tezlanish). Siljishlar kichik bo'lgani uchun

$$\sin\alpha = \frac{\mathrm{tg}\alpha}{\sqrt{1+\mathrm{tg}^2\alpha}} \simeq \mathrm{tg}\alpha = \frac{\partial u}{\partial x}.$$

Ya'ni,

$$\sin\alpha|_{x+dx} - \sin\alpha|_x \simeq \frac{\partial u(t, x+dx)}{\partial x} - \frac{\partial u(x,t)}{\partial x} \simeq \frac{\partial^2 u(x,t)}{\partial x^2}dx.$$

Demak,

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u(x,t)}{\partial x^2} + F \tag{2}$$

torning kichik ko'ndalang tebranishlari tenglamasi ekan. Bu tenglamani quyidagicha yozib olamiz:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{3}$$

bu yerda $f = F/\rho$, $a^2 = T/\rho$. Uni bizga ma'lum bo'lgan kanonik ko'rinishda ham yozib olishimiz mumkin:

$$u_{tt} - a^2 u_{xx} = f. \tag{4}$$

Ko'pincha bu tenglama ***bir o'lchamli to'lqin tenglamasi*** deyiladi.

Keltirib chiqarish jarayonida taranglikni o'zgaruvchan deb olsak, (2)-ning o'rniga

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + F \tag{5}$$

tenglamani olgan bo'lar edik.

§2. Sterjenning bo'ylanma tebranishlari

Bizga bir sterjen berilgan bo'lsin. Bir o'lchamli bo'ylanma tebranishlari haqida gapirar ekanmiz, sterjenning har bir kesimi deformatsiyasiz x o'qi bo'yicha o'z muvozanat holatidan siljiydi, deb qaraymiz - (III.2)-rasmga qarang. Tashqi kuch (agar mavjud bo'lsa) x o'qi bo'yicha yo'nalgandir. $u(x,t)$ funksiya (x) nuqtaning t vaqt momentidagi o'z muvozanat holati (x) dan siljish kattaligini ifodalaydi. $u(t, x+dx)$ funksiya esa $(x+dx)$ nuqtaning t vaqt momentidagi o'z muvozanat holati $(x+dx)$ dan siljish kattaligini ifodalaydi. Sterjenning boshlang'ich uzunligi dx bo'lgan bir bo'lagini olamiz - x va $x+dx$

koordinatalar orasidagi. Kichik tebranishlar haqida gap ketayotgani uchun hamma yoyilmalarda dx ning birinchi darajasi bilan cheklanamiz:

$$u(t, x + dx) \simeq u(x, t) + u_x(x, t)dx.$$

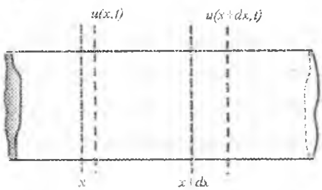
Bu formuladan ko'rinib turibdiki, sterjenning nisbiy cho'zilishi (sterjen bo'lakchasining t vaqt momentidagi uzunligining $t = 0$ vaqt momentidagi uzunligiga nisbati)

$$\frac{u(t, x + dx) - u(x, t)}{dx} \simeq u_x$$

ga teng. Hooke[1] qonuni bo'yicha

$$T = ESu_x.$$

Bu yerda E - Young moduli, S - sterjen kesimi.



III.2-rasm: Sterjenning kesimlari

Bu holda taranglik T ning son qiymati x ga bog'liq bo'ladi chunki sterjenning uzunligi o'zgaruvchandir. Yana harakat tenglamasini tuzaylik:

$$T(x + dx) - T(x) + F(x, t)Sdx =$$

$$= \rho dx S\frac{\partial^2 u(x,t)}{\partial t^2}.$$

Bu yerda $F(x, t)$ - sterjen bo'yicha taqsimlanga kuchning xajm zichligi, S - sterjenning kesim sirti. Chap tomonni qatorga yoyamiz:

$$T(x + dx) - T(x) = ES\left(u_x(t, x + dx) - u_x(x, t)\right) \simeq ES\frac{\partial^2 u}{\partial x^2}dx. \qquad (6)$$

Olingan tenglama

$$E\frac{\partial^2 u}{\partial x^2} + F(x, t) = \rho\frac{\partial^2 u}{\partial t^2} \qquad (7)$$

yoki,

$$u_{tt} - a^2 u_{xx} = f; \qquad a^2 = \frac{E}{\rho}, \quad f = \frac{F}{\rho} \qquad (8)$$

ko'rinishga ega bo'ladi. Biz yana giperbolik tipdagi tenglamani oldik: (4)- va (8)-tenglamalar faqatgina a^2 ta'rifi bilan farq qiladi. Biz keyin ko'ramizki, a - to'lqinning muhit bo'yicha tarqalish tezligini beradi.

Olingan tenglamalar - (4) va (8) - to'lqin tenglamasining bir o'lchamli ko'rinishi. Ikki o'lchamli to'lqin tenglamasi

$$u_{tt} - a^2\left(u_{xx} + u_{yy}\right) = f. \qquad (9)$$

[1] Robert Hooke (1635-1703) - buyuk ingliz olimi. Rus tilida - Роберт Гук.

ko'rinishga ega bo'ladi. Bunday tenglamaga, biror ikki o'lchamli sirtning (masalan, membrananing) tebranishlari masalasini ko'rsak, kelar edik. Quyidagi tenglama esa

$$u_{tt} - a^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = f, \tag{10}$$

yoki,

$$u_{tt} - a^2 \Delta u = f \tag{11}$$

uch o'lchamli to'lqin tenglamasi deyiladi.

§3. Giperbolik tenglamalar uchun chegaraviy va boshlang'ich shartlar

Masalaning bir qiymatli yechimini topish uchun shu masalaga mos keluvchi boshlang'ich va chegaraviy shartlarni berishimiz kerak. To'lqin tenglamasi vaqt bo'yicha ikkinchi tartibli tenglama bo'lgani uchun noma'lum funksiya $u(x,t)$ va uning vaqt bo'yicha birinchi tartibli hosilasi boshlang'ich vaqt momenti $t = 0$ berilgan bo'lishi kerak:

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x).$$

Bunday shartlar ***boshlang'ich shartlar*** yoki, ***Caushy*[2] *shartlari*** deyiladi. Agar masalada faqat boshlang'ich shartlar berilgan bo'lsa, bunday masala ***Caushy masalasi*** deyiladi. Boshlang'ich shartlarning fizik manosiga to'xtalib o'taylik. Masalan, $\varphi(x) \neq 0$ va $\psi(x) = 0$ bo'lsin. Bu - tor (sterjen) nuqtalarining boshlang'ich siljishi noldan farqli va boshlang'ich tezligi nolga teng degani (dutor, rubob, gitara va shunga o'xshash asboblarda uchraydigan boshlang'ich shart). $\varphi(x) = 0$ va $\psi(x) \neq 0$ bo'lgan hol esa tebranishning boshida torning (sterjenning) hamma nuqtalarida muvozanat holatida turibdi, lekin ularga boshlang'ich tezlik berilgan (masalan, bolg'acha bilan urib) degani. Bunday boshlang'ich shartlar pianino, do'mbira va shunga o'xshagan asboblarga mos keladi.

Chegaraviy shartlarga o'taylik. Agar tor yoki sterjenning uzunligi chekli yoki yarim chekli bo'lsa, unga chegaraviy shartlar qo'yishimiz kerak. Ular quyidagi turlarga bo'linadi.

1. Uzunligi l bo'lgan tor(sterjen)ning boshi va oxiri mustahkam biriktirilgan (masalan, devorga):

$$u(t,0) = u(t,l) = 0.$$

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[2]Augustin-Louis Cauchy (1789-1957) - fransuz matematigi. Rus tilida - Огюстен-Луи Коши

Boshqacha yozsak:

$$u(x,t)|_{x=0} = u(x,t)|_{x=l} = 0.$$

Umumiy holda chegaraviy nuqtalar berilgan qonun bo'yicha harakat qiladi:

$$u(x,t)|_{x=0} = \mu_1(t), \quad u(x,t)|_{x=l} = \mu_2(t).$$

Bu yerda $\mu_1(t)$ va $\mu_2(t)$ funksiyalar - berilgan funksiyalar.

2. Sterjenning (torning) x_0 nuqtasiga berilgan $\nu(t)$ kuch ta'sir qilayotgan bo'lsin:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_0} = u_x\Big|_{x=x_0} = \frac{\nu(t)}{T}.$$

Haqiqatan ham,

$$T\frac{\partial u}{\partial x}|_{x=x_0} \simeq T\sin\alpha|_{x=x_0} = \nu(t).$$

Agar shu chegaraga hech qanday kuch ta'sir qilmasa, ya'ni shu chegara ozod bo'lsa,

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_0} = u_x\Big|_{x=x_0} = 0,$$

deb yozishimiz kerak.

3. Tor(sterjen)ning chegarasida elastik kuch ta'sir qilsin. Chap chegarada: $(Tu_x - ku)_{x=0} = 0$, yoki $u_x|_{x=0} = hu|_{x=0}$, $h = k/T$. O'ng hegarada: $(-Tu_x - ku)_{x=l} = 0$, yoki $u_x|_{x=l} = -hu|_{x=l}$. Ishoralarni quyidagicha tushunish mumkin. Chap chegarada taranglik kuchi manfiy yo'nalishga ega, o'ng chegarada taranglik kuchi musbat yo'nalishga ega.

Umumiy holda uchinchi chegaraviy shart

$$u_x|_{x=0} = h(u - \theta(t))$$

ko'rinishda yozilishi kerak, bu yerda $\theta(t)$ - berilgan funksiya, u torning shu chegarasining berilgan harakatini ifodalaydi.

Masalada ham boshlang'ich, ham chegaraviy shartlar berilgan bo'lsa bunday masala ***aralash masala*** deyiladi.

3.1-misol. Ikkala uchi mahkamlangan tor berilgan. Tor nuqtalarining boshlangich tezligi nolga teng, boshlang'ich siljish esa $\varphi(x) = \alpha x(x-l)$ ko'rinishga ega.

Yechim. Tenglama:

$$u_{tt} - a^2 u_{xx} = 0.$$

Masalaning shartida tashqi kuch haqida hech narsa deyilmagan, shuning uchun tenglama bir jinsli.

Boshlang'ich shartlar:

$$u(x,0) = \varphi(x) = \alpha x(x-l), \quad u_t(x,0) = \psi(x) = 0.$$

Chegaraviy shartlar:

$$u(0,t) = u(l,t) = 0.$$

O'zgaruvchilarning o'zgarish sohasi:

$$0 \le x \le l, \quad 0 \le t < \infty.$$

3.2-misol. Erkin tushayotgan liftning shipiga l uzunlikdagi og'ir sterjen osib qo'yilgan. Lift uning tezligi v_0 ga erishganda keskin to'xtaydi. Sterjenning tebranishlari masalasi qo'yilsin

Yechim. x - o'qini liftning shipidan pastga qarab yo'naltiramiz. g - erkin tushish tezlanishi. Bu holda tenglama:

$$u_{tt} - a^2 u_{xx} = g.$$

Boshlang'ich shartlar:

$$u(x,0) = \varphi(x) = 0, \quad u_t(x,0) = \psi(x) = v_0.$$

Chegaraviy shartlar:

$$u(0,t) = 0, \quad u_x(l,t) = 0.$$

O'zgaruvchilarning o'zgarish sohasi:

$$0 \le x \le l, \quad 0 \le t < \infty.$$

3.1-mashq. Ideal gaz bilan to'ldirilgan bir uchi ochiq truba o'z o'qi yo'nalishida v tezlik bilan ilgarilanma harakat qilaypti. $t = 0$ vaqtda truba to'satdan to'xtaydi. Trubaning yopiq uchidan x masofadagi gazning muvozanat vaziyatidan siljishi masalasini qo'ying.

3.2-mashq. Ikki uchi mahkamlangan tor uchun ko'ndalang tebranishlar masalasini qo'ying. Tor qarshiligi tezlikka proportsional bo'lgan muhitda joylashgan.

3.3-mashq. Bir uchi mahkamlangan ikkinchisi o'z tezligiga proportsional bo'lgan kuch ostida bo'lgan bir jinsli elastik sterjenning bo'ylanma tebranishlari masalasini qo'ying. Muhit qarshiligi hisobga olinmasin.

3.4-mashq. Og'ir sterjen o'zining har bir nuqtasi muvozanat holatga keltirilib, siqib qo'yilgan holatda vertikal ravishda bir uchidan osib qo'yilgan. $t = 0$ vaqtda sterjen siquvchi kuchdan ozod bo'ladi. Sterjenning majburiy tebranishlari masalasini qo'ying.

3.5-mashq. $t = 0$ vaqtdan boshlab elastik sterjenning bir uchi berilgan qonun $\mu(t)$ bo'yicha tebranayapti, ikkinchi uchiga uning o'qi bo'yicha yo'nalgan $\Phi(t)$ kuch qo'yilgan. $t = 0$ vaqtda sterjenning ko'ndalang kesimlari o'z muvozanat holatida qo'zg'olmasdan turgan. Sterjenning tebranishlari masalasini qo'ying.

3.6-mashq. III.2-misoldagi sterjenning quyi uchiga og'irligi P bo'lgan yuk osib qo'yilgan bo'lsin. Masalaning qo'yilishi qay darajada o'zgaradi?

§4. Tebranish energiyasi

Tebranayotgan tor yoki sterjenning energiyasini topaylik. Boshlang'ich shartlar quyidagicha bo'lsin: $u(x,0) = u_t(x,0) = 0$. Torning uzunligi l. To'liq energiya $E = K + U$, K - kinetik energiya, U potensial energiya.

Energiyani tor(sterjen)ning kichik elementi uchun aniqlashdan boshlaymiz. dx uzunlikdagi torning kinetik energiyasi

$$\frac{1}{2}dmv^2 = \frac{1}{2}\rho dx\, u_t^2$$

ekanligini hisobga olsak, butun torning kinetik energiyasi

$$K = \frac{1}{2}\int_0^l dx\rho\, u_t^2$$

ga teng bo'ladi.

Ta'rif bo'yicha "potensial energiya—$-\int$ kuch $\times$ siljish elementi" ga teng. dx elementga ta'sir qilayotgan kuch $Tu_{xx}dx$ ga teng ($T\sin\alpha|_{x+dx} - T\sin\alpha|_x \simeq Tu_{xx}dx$), dt vaqt ichidagi siljish $u_t dt$ ga teng, demak

$$-U = \int_0^l\int_0^t Tu_{xx}dxdtu_t = Tu_tu_x\Big|_0^l dt - \int_0^l Tu_xu_{xt}dt =$$

$$= -\frac{1}{2}T\int_0^t dt\frac{d}{dt}\int_0^l u_x^2\,dx = -\frac{1}{2}\int_0^l Tu_x^2(x,t)dx.$$

Shundy qilib to'liq energiya quyidagiga teng:

$$E = \frac{1}{2}\int_0^l dx\left(\rho(x)u_t^2 + T_0u_x^2\right). \tag{12}$$

Energiya uchun ifodani umumlashtirish maqsadida torning ko'ndalang tebranishlari tenglamasi (5)- va sterjenning bo'ylanma tebranishlari tenglamasi (7)-larni umumlashtirib quyidagi ko'rinishda yozib olamiz:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + q(x)u(x,t) = F(x,t). \qquad (13)$$

Bu yerda

$$p(x) > 0, \qquad q(x) \geq 0, \qquad \rho(x) > 0.$$

Boshlang'ich va chegaraviy shartlar o'zgarmasin.

Quyidagi kattalik mana shu tenglamaning energiya integrali deyiladi:

$$E = \frac{1}{2}\int_0^l dx\left(\rho(x)u_t^2 + p(x)u_x^2 + qu^2\right).$$

(12)-bilan solishtirganda bu ifodada paydo bo'lgan qo'shimcha had qu^2 ning ma'nosi (bu had (13)-tenglamadagi qu qo'shimcha had bilan bog'liq) ko'rinib turibdi: bu had potensial energiyaga qo'shilgan hissa.

Energiyadan vaqt bo'yicha hosila hisoblaymiz:

$$\frac{dE}{dt} = \int_0^l dx\left[\rho u_t u_{tt} + p u_x u_{xt} + q u u_t\right].$$

Ikkinchi hadni bo'laklab integrallaymiz:

$$\int_0^l dx p u_x u_{xt} = p u_x u_t\Big|_0^l - \int_0^l dx u_t \frac{\partial}{\partial x}(p u_x).$$

Natijani energiyaning hosilasiga olib borib qo'yib (13)-tenglamani hisobga olsak,

$$\frac{dE}{dt} = \int_0^l dx u_t F(x,t) + p u_x u_t\Big|_0^l$$

ni olamiz. Birinchi had tashqi kuch F bajargan ishni ifodalaydi, ikkinchi had boshlang'ich shartlarni hisobga olganda nolga teng bo'lib ketadi. Agar tashqi kuch bo'lmasa

$$\frac{dE}{dt} = 0$$

bo'ladi. Bu - energiyaning saqlanish qonuni. Tashqi kuch mavjud holda

$$\frac{dE}{dt} = \int_0^l dx u_t F(x,t)$$

ga egamiz.

§5. Aralash masala yechimining yagonaligi

Eng umumiy ko'rinishdagi giperbolik tenglamaga qaytaylik:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + q(x)u(x,t) = F(x,t). \qquad (14)$$

Bu yerda

$$p(x) > 0, \qquad q(x) \geq 0, \qquad \rho(x) > 0.$$

Tenglamaga quyidagi boshlang'ich va chegaraviy shartlar qo'yilgan bo'lsin:

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t). \qquad (15)$$

Faraz qilaylik, masalaning yechimi ikkita bo'lsin: $u_1(x,t)$ va $u_2(x,t)$. Bu yechimlarning farqini

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

deb belgilaymiz. (14)-tenglama bilan bog'liq bo'lgan aralash masalalar yechimlarining yagonaligini isbot qilish $v(x,t) = 0$ ekanligining isbotiga tengdir. Noma'lum $v(x,t)$ funksiya uchun masala quyidagicha qo'yilgan:

$$\rho(x)\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial v}{\partial x}\right) + q(x)v(x,t) = 0,$$
$$v(x,0) = 0, \qquad v_t(x,0) = 0, \qquad v(0,t) = 0, \qquad v(l,t) = 0. \qquad (16)$$

Bu tenglama uchun energiya integralini yozib olamiz:

$$E = \frac{1}{2}\int_0^l dx\left(\rho(x)v_t^2 + p(x)v_x^2 + qv^2\right).$$

Bu ifodadan vaqt bo'yicha hosilani hisoblaymiz. (16)-tenglamada tashqi kuchning yo'qligi va boshlang'ich hamda chegaraviy shartlarning birjisnliligi

$$\frac{dE}{dt} = 0, \quad \text{ya'ni,} \quad E(t) = \text{const},$$

ga olib keladi (avvalgi paragrafning oxiridagi hisob bilan solishtiring). Demak,

$$E(t) = E(0)$$

ekan. (16)-dagi boshlang'ich shartlarni hisobga olsak

$$E(t) = E(0) = \frac{1}{2}\int_0^l dx\,\left(\rho(x)v_t^2 + p(x)v_x^2 + qv^2\right)\Big|_{t=0} = 0$$

ekanligiga kelamiz. Ammo energiya integralidagi har bir had - musbat had, musbat hadlarning yig'indisi nolga teng bo'lishi uchun ularning har biri nolga teng bo'lishi kerak. Buning uchun esa $v(x,t) = 0$ bo'lishi kerak. Demak, (14)-tenglamaning (15)-shartlar bilan aniqlanuvchi yechimi yagonadir.

IV BOB. PARABOLIK TENGLAMALARGA OLIB KELADIGAN JARAYONLAR

Molekulalar orasidagi to'qnashuv jarayonlari parabolik tenglamalar tilida ifodalanadi. Buni quyidagi ikkita misolda ko'raylik.

§1. Issiqlik tarqalishi masalasi

Issiqlik tarqalishi tenglamasini keltirib chiqaraylik. Quyidagi belgilashlardan foydalanamiz:

- $u(\mathbf{r}, t)$ muhitning $\mathbf{r} = (x, y, z)$ nuqtasidgi t vaqt momentidagi temperatura;
- $\rho(\mathbf{r}, t)$ - muhit zichligi, uni izotrop deb qaraymiz;
- $c(\mathbf{r})$- muhitning issiqlik sig'imi;
- $k(\mathbf{r})$- issiqlik o'tkazish koeffisienti;
- $F(\mathbf{r}, t)$-issiqlik manbasi zichligining intensivligi.

Issiqlik o'tkazish koeffisienti k issiqlik oqimi $\mathbf{q}(\mathbf{r}, t)$ (birlik sirtdan birlik vaqt ichida o'tgan issiqlik miqdori) va temperatura gradientini bog'laydigan koeffisient:

$$\mathbf{q}(\mathbf{r}, t) = -k\nabla u.$$

Bu munosabat Fourier[1] qonuni deyiladi. Bu qonundagi minus ishora issiqlik oqimining temperatura gradientiga qarama-qarshi yo'nalganligi bilan bog'lik. Albatta, bu chiziqli munosabat faqat temperatura gradienti kichik bo'lgandagina o'rinlidir, umumiy holda temperatura gradientining yuqori darajalari ham kirishi kerak, ammo biz ushbu gradient kichik deb olamiz. Bu holda chiziqli qonunning o'zi yetarlidir.

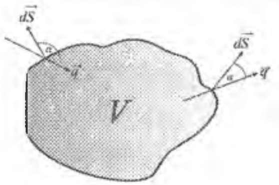
Ixtiyoriy V hajm uchun issiqlik balansini tuzaylik.

Issiqlik oqimi natijasida $(t, t + dt)$ vaqt ichida shu hajm ichidagi issiqlik miqdorining o'zgarishi

$$Q_1 = -\int\limits_S \mathbf{q} \cdot d\mathbf{S}dt = \int\limits_S k\nabla u \cdot d\mathbf{S}dt = \int\limits_S k\frac{\partial u}{\partial n}dSdt$$

[1] Joseph Fourier (1768-1830) - fransuz fizigi va matematigi. Рус тилида - Жозеф Фурье

ga teng. Oxirgi tenglikka o'tishda biz sirtga normal birlik vektor $\mathbf{n}$ tushunchasini kiritdik. $\mathbf{n}$ vektor bilan $d\mathbf{S}$ vektorlar bir xil yo'nalishga ega, shu sababli u yerda skalar ko'paytma belgisini ham yozmadik.



IV.1-rasm: Issiqlik oqimlari

Integral oldidagi minus ishoraga to'xtalaylik. (IV.1)-rasmdan ko'rinib turibdiki, agar oqim tashqariga yo'nalgan bo'lsa, $\mathbf{q}$ va $d\mathbf{S}$ vektorlari orasidagi burchak o'tkir, demak, ularning skalar ko'paytmasi musbat bo'ladi. Oqim tashqariga bo'lgan holda hajm ichidagi issiqlik miqdori kamayishi kerak, shu sababdan integral oldiga minus ishorasini qo'ydik. Agar oqim ichkariga bo'lgan holni tahlil qilsak, rasmdagi ikkita vektorning skalar ko'paytmasi manfiy son bo'lishini ko'ramiz, integral oldidagi minus ishora bu gal hajm ichidagi issiqlik miqdorining o'sishini ta'minlab beradi.

Gauss teoremasi bo'yicha

$$Q_1 = -\int_S \mathbf{q}\cdot d\mathbf{S}dt = -\int_V \operatorname{div}\mathbf{q}dVdt = \int_V \operatorname{div}(k\nabla u)dVdt.$$

Issiqlik manbai hisobiga paydo bo'lgan issiqlik miqdori:

$$Q_2 = \int_V F(t,\mathbf{r})dVdt.$$

Mana shu ikki sabab bo'yicha temperaturaning $(t, t+dt)$ vaqt ichidagi o'zgarishi:

$$u(t+dt,\mathbf{r}) - u(t,\mathbf{r}) \simeq u_t dt. \tag{1}$$

Temperaturaning bunday o'zagarishga mana shu V hajm ichidagi issiqlik miqdorining quyidagi o'zgarishi mos keladi:

$$Q_3 = \int_V c\rho\frac{\partial u}{\partial t}dVdt.$$

Issiqlik balansi:

$$Q_3 = Q_1 + Q_2,$$

ya'ni,

$$\int dVdt\left(c\rho\frac{\partial u}{\partial t} - \operatorname{div}(k\operatorname{grad}u) - F\right) = 0.$$

Hajm va vaqt ixtiyoriy bo'lgani uchun

$$c\rho\frac{\partial u}{\partial t} = \operatorname{div}(k\operatorname{grad} u) + F(\mathbf{r}, t) \qquad (2)$$

tenglamani olamiz. Bu - ***issiqlik tarqalishi tenglamasidir.*** Agar muhit bir jinsli bo'lsa, ya'ni c, ρ, k lar o'zgarmas bo'lsa, tenglamaning ko'rinishi

$$\frac{\partial u}{\partial t} = a^2\Delta u + f(\mathbf{r}, t) \qquad (3)$$

bo'ladi($a^2 = k/c\rho, f = F/c\rho$). Bir o'lchamli holda

$$u_t = a^2 u_{xx} + f.$$

§2. Diffuziya masalasi

Diffuziya tenglamasini ham xuddi avvalgi paragrafdagidek balans prinsipidan, bu gal modda balansi prinsipidan keltirib chiqaramiz. Bu gal gap modda balansi haqida ketadi. Ushbu masalada $u(t, \mathbf{r})$ - moddaning konsentratsiyasini bildiradi. Modda oqimi uchun quyidagi ***Fick***[2] ***qonuni*** o'rinlidir:

$$\mathbf{q} = -D\nabla u. \qquad (4)$$

Bu formulada $\mathbf{q}$ - modda oqimi zichligi, D - diffuziya koeffisienti. Shu oqim borligi natijasida dt vaqt ichida S sirt ichidagi hajmda modda miqdorining o'zgarishi

$$N_1 = -\int_S \mathbf{q}\cdot d\mathbf{S}dt = \int_V \operatorname{div}(D\nabla u)dVdt$$

bo'ladi. Integral oldidagi minus ishora yuqorida issiqlik tarqalishi masalasida muhokama qilingan. Hajmning ichida $F(\mathbf{r}, t)$ zichlik intensivligiga ega bo'lgan modda manbasi bo'lsin. Uning hisobiga hajm ichidagi modda miqdorining o'zgarishi

$$N_2 = \int F(\mathbf{r}, t)dVdt$$

bo'ladi. Konsentratsiyaning shu vaqt ichida o'zgarishi

$$u(\mathbf{r}, t + dt) - u(\mathbf{r}, t) \simeq u_t dt, \qquad (5)$$

V hajm ichidagi modda miqdorining o'zgarishi

$$N_3 = \int_V u_t dVdt$$

[2] Adolf Fick (1829-1901) - nemis fizigi.

bo'ladi. Modda balansini tuzaylik:

$$N_3 = N_1 + N_2.$$

Undan biz quyidagi tenglamaga kelamiz:

$$\frac{\partial u}{\partial t} = \text{div}(D\nabla u) + F.$$

Bir o'lchamli holda

$$u_t = (Du_x)_x + F.$$

Agar $D = \text{const}$ bo'lsa, uch o'lchamli holda

$$u_t = D\Delta u + F \tag{6}$$

bo'ladi, bir o'lchamli holda esa

$$u_t = Du_{xx} + f$$

tenglamani olamiz.

Olingan diffuzuya tenglamasining ko'rinishi issiqlik tarqalishi tenglamasidan farq qilmaydi. Sababi nimada? Sababi shundaki, ikkala jarayonlar asosida molekular to'qnashuvlar yotadi. Issiqlik tarqalishi - bu energiyasi kattaroq bo'lgan molekulalarning to'qnashuvlar orqali o'z energiyasini energiyasi kamroq bo'lgan molekulalarga tarqatishi bo'lsa diffuziya jarayoni bir modda molekulalarining ikkinchi modda molekulalari ichiga o'zaro to'qnashuvlar asosida tarqalishi yotadi.

Issiqlik tarqalishi va diffuziya tenglamalari parabolik tipdagi tenglamalardir.

§3. Parabolik tenglamalar uchun chegaraviy va boshlang'ich masalalar

Issiqlik tarqalishi va diffuziya tenglamalari vaqt bo'yicha birinchi tartibli tenglama bo'lgani uchun, bitta boshlang'ich shart - temperaturaning (konsentratsiyaning) muhitdagi boshlang'ich taqsimoti berilishi kerak:

$$u(x, 0) = \varphi(x).$$

Chegaraviy shartlar quyidagi uchxil turga bo'linadi:

1. Chegarada ma'lum temperatura (konsentratsiya berilgan)

$$u|_S = u_0(\mathbf{r}, t).$$

Masalan, uzunligi l ga teng bo'lgan $(0 \leq x \leq l)$ sterjenning temperaturasini aniqlash masalasi haqida gap ketayotgan bo'lsa chegaraviy shartlar

$$u(0,t) = u_1, \quad u(l,t) = u_2$$

ko'rinishida beriladi.

2. Chegarada ma'lum issiqlik (modda) oqimi berilgan:

$$-k\frac{\partial u}{\partial n}\bigg|_S = q(t,S).$$

Sterjen haqida gap ketgan xususiy holda, uning chap va o'ng chegaralarida

$$ku_x(0,t) = q_1, \quad ku_x(l,t) = -q_2$$

deb yozamiz. Birinchi shartda ishoraning o'zgarishi chap chegarada normal bo'yicha hosilaning x koordinata bo'yicha hosilaga teskariligidan.

3. Chegarada Newton qonuni bo'yicha issiqlik (modda) almashinishi ro'y berayapti:

$$\left(k\frac{\partial u}{\partial n} + h(u - u_0)\right)\bigg|_S = 0,$$

bu yerda h - issiqlik (modda) almashinishi koeffisienti deyiladi.

Parabolik tenglamalar uchun ham ko'proq aralash masalalarni yechishga to'g'ri keladi.

4.1-misol. Boshlang'ich temperaturasi u_0 bo'lgan sterjenning chap uchida o'zgarmas u_1 temperatura ushlanib turibdi. Sterjenning o'ng uchida o'zgarmas issiqlik oqimi q berilgan. Issiqlik tarqalishi masalasi qo'yilsin.

Yechim.

$$u_t - a^2u_{xx} = 0, \quad 0 \leq x \leq l, \quad 0 \leq t < \infty,$$
$$u(x,0) = u_0, \quad u(0,t) = u_1, \quad u_x(l,t) = q/k.$$

4.2-misol. Ingichka sterjenning boshlang'ich temperaturasi $\varphi(x)$. Ikkala uchining temperaturasi o'zgarmasdir:

$$u(0,t) = u_1, \quad u(l,t) = u_2, \quad 0 \leq t < \infty.$$

Sterjenning yon sirti orqali temperaturasi u_0 bo'lgan tashqi muhit bilan Newton qonuni bo'yicha issiqlik almashinishi ro'y berayapti. Shu sterjen uchun issiqlik tarqalishi masalasini qo'ying.

Yechim.

Masalaning shartidagi "ingichka sterjen" ni shu darajada ingichka deb qaraymizki, uning yon sirti bo'yicha tashqi muhit bilan bo'layotgan issiqlik almashinishi natijasidagi issiqlik oqimining zichligi

$$q = -k\frac{\partial u}{\partial n} = -\alpha(u - u_0)$$

ni butun sterjen bo'yicha uzluksiz taqsimlangan manbaning ta'siri deb qarashimiz mumkin bo'lsin. Ya'ni, tenglamani

$$u_t = a^2 u_{xx} + f$$

ko'rinishda qidiramiz. Manba intensivligi $F = fc\rho$ ni Gauss teoremasidan topamiz. Sterjenning uzunligi l, $S = pl$ - sirt yuzasi (p - perimetr), shu sirt yuzasidan 1 sek da o'tgan issiqlik miqdori qS manba intensivligi F dan hajm bo'yicha olingan integralga teng bo'lishi kerak:

$$qpl = fc\rho Sl.$$

Demak, $f = \frac{qp}{c\rho S} = -\frac{\alpha}{c\rho l}(u - u_0)$. Shularni hisobga olib masalaning qo'yilishi quyidagicha ekanligiga ishonch hosil qilamiz:

$$u_t - a^2 u_{xx} = -\frac{\alpha}{c\rho l}(u - u_0),$$

$$u(x, 0) = \varphi(x), \quad u(0, t) = u_1, \quad u(l, t) = u_2, \quad 0 \le t < \infty, \quad 0 \le x \le l.$$

§4. Konvektiv oqimni hisobga olish

Issiqlik tarqalishi va diffuziya tenglamalarini keltirib chiqarganda issiqlik tarqalayotgan muhitda va diffuziya ro'y berayotgan muhitda konvektiv harakat yo'q deb faraz qilingan. Agar muhit nuqtalari $\mathbf{v}(\mathbf{r}, t)$ tezlikka ega bo'lgan konvektiv oqimlarda ishtirok etsa, issiqlik (modda) bir nuqtadan ikkinchi nuqtaga mana shu oqimlar yordamida ham ko'chiriladi. Agar aniqlik uchun modda ko'chishi jarayoni haqida gapirsak, modda ikkita sabab bo'yicha ko'chirilayapti - molekular to'qnashuvlar (ular Fick qonuni (4)-da hisobga olingan) va konvektiv oqim hisobiga. Konvektiv oqimni quyidagicha hisobga olish mumkin. Muhitning t vaqtda $\mathbf{r}$ koordinatali nuqtasi $\mathbf{v}(\mathbf{r}, t)$ tezlik bilan harakat qilayotgan oqim bilan ko'chgani uchun, bu nuqtaning $t + dt$ vaqtdagi koordinatasi

$$\mathbf{r}(t + dt) \simeq \mathbf{r}(t) + (d\mathbf{r}/dt)dt = \mathbf{r}(t) + \mathbf{v}(\mathbf{r}, t)dt$$

bo‘ladi. Shuni hisobga olib, (1)- va (5)-formulalarni boshqacha hisoblash kerak:

$$u(\mathbf{r}(t+dt), t+dt) - u(\mathbf{r}(t), t) \simeq u(\mathbf{r}(t) + \mathbf{v}(\mathbf{r}, t)dt, t+dt) - u(\mathbf{r}(t), t) \simeq$$

$$\simeq (u_t + \mathbf{v} \cdot \nabla u)\, dt.$$

Natijada, issiqlik tarqalishi tenglamasi (3) da qo‘shimcha - konvektiv - had paydo bo‘ladi:

$$u_t + \mathbf{v} \cdot \nabla u - a^2 \Delta = f. \tag{7}$$

Diffuziya tenglamasi (6) ham xuddi shunday o‘zgaradi:

$$u_t + \mathbf{v} \cdot \nabla u - D\Delta = F. \tag{8}$$

Mashqlar.

4.1-mashq. $0 \leq x \leq l$ sterjenning yon sirti issiqlik o’tkazmaydi, ikkala uchi berilgan temperaturada ushlanadi. Sterjenning temperaturasini aniqlash bo’yicha chegaraviy masalani qo’ying.

4.2-mashq. (Bu masalaga 8-bobda delta-funksiyani o‘rganganda keyin qaytib keling.) Ingichka cheksiz termoizolyatsiyalangan sterjen bo‘yicha $t = 0$ vaqtdan boshlab o‘ng tomonga v tezlik bilan issiqlik manbai harakat qilayapti. Uning quvvati q ga teng. Sterjen bo‘yicha issiqlik tarqalishi masalasi qo‘yilsin.

4.3-mashq. Radiusi R bo‘lgan bir jinsli sharning ichida $t = 0$ dan boshlab o‘zgarmas zichlik Q bilan taqsimlangan issiqlik manbalari ta’sir qila boshlaydi. Shar nuqtalarining boshlang‘ich temperaturasi faqat markazgacha bo‘lgan masofaga bog‘liq deb issiqlik tarqalishi masalasini quyidagi chegaraviy shartlarda qo‘ying:

a) shar sirtida nolga teng temperatura ushlanib turibdi;

b) shar sirtida temperaturasi nolga teng bo‘lgan tashqi muhit bilan Newton qonuni bo‘yicha konvektiv issiqlik almashinishi ro’y berayapti.

4.4-mashq. Radiusi R va boshlang‘ich temperaturasi nolga teng bol‘gan bir jinsli shar berilgan. Shar sirtining hamma nuqtalari o‘zgarmas q oqim bilan isitilyapti. Shar ichidagi temperatura taqsimoti masalasini qo‘ying.

4.5-mashq. Asosining radiusi a va balandligi h bo’lgan bir jinsli silindr berilgan. Quyidagi hollarda silindrning barqaror taqsimlangan (vaqtga bog‘liqlik yo‘q) temperaturasini topish bo‘yicha chegaraviy masala qo‘yilsin:

a) Quyi asos va yon sirt temperaturalari nolga teng, yuqori asos temperaturasi faqat r ning funksiyasi;

b) Quyi asos temperaturasi nolga teng, yon sirti issiqlik o’tkazmaydi, yuqori sirti temperaturasi $u_0(r)$;

c) Quyi asos temperaturasi nolga teng, yon sirti esa temperaturasi nolga teng tashqi muhit bilan sovutilyapti, yuqori sirti temperaturasi $u_0(r)$.

V BOB. TARQALAYOTGAN TO'LQIN METODI

§1. Cheksiz tor: erkin tebranishlar masalasi

Cheksiz tor uchun quyidagi Cauchy masalasini yechamiz:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \\ 0 \le t < \infty, \quad -\infty < x < \infty. \end{array}\right\} \tag{1}$$

Bu tenglamani yechish uchun xarakteristika metodidan foydalanamiz. Xarakteristikalar:

$$\frac{dx}{dt} = \pm a.$$

Bundan kelib chiqib,

$$\xi = x + at, \qquad \eta = x - at \tag{2}$$

almashtirish bajarsak,

$$u_{\xi\eta} = 0 \tag{3}$$

tenglamani olamiz. Bu tenglamaning yechimini umumiy holda

$$u(\xi, \eta) = h(\xi) + g(\eta) \tag{4}$$

ko'rinishda qidiramiz. Bu yerda $h(\xi)$ va $g(\eta)$ funksiyalar o'z argumentining ixtiyoriy ikki marta differensiallanuvchi funksiyasidir. Demak,

$$u(x,t) = h(x+at) + g(x-at). \tag{5}$$

Boshlang'ich shartlarni qanoatlantirish qoldi:

$$u(x,0) = h(x) + g(x) = \varphi(x), \quad a(h'(x) - g'(x)) = \psi(x). \tag{6}$$

Shartlarning ikkinchisini

$$h(x) - g(x) = \frac{1}{a}\int\limits_{x_0}^{x} dz\,\psi(z) + c$$

ko'rinishga keltirib quyidagilarni olamiz:

$$
\begin{aligned}
h(x) &= \frac{1}{2}\varphi(x) + \frac{1}{2a}\int\limits_{x_0}^{x} dz\,\psi(z) + \frac{c}{2} \\
g(x) &= \frac{1}{2}\varphi(x) - \frac{1}{2a}\int\limits_{x_0}^{x} dz\,\psi(z) - \frac{c}{2}.
\end{aligned} \tag{7}
$$

Demak, (1)-Cauchy masalasining yechimi

$$
u(x,t) = \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at)\right) + \frac{1}{2a}\int\limits_{x-at}^{x+at} dz\,\psi(z) \tag{8}
$$

ko'rinishga ega ekan.

Olingan natijaning ma'nosiga kelaylik. $f(\xi) = f(x+at)$ - ning argumenti o'zgarmasligi uchun $dx/dt = -a$ bo'lishi kerak. Demak, chapga $-a$ tezlik bilan harakat qilayotgan sistemaga o'tsak to'lqinimiz $f(\xi) = f(x+at)$ shu sistemada o'zgarmasdan turar ekan. Bu esa $f(\xi)$ ko'rinishdagi to'lqin chap tomonga $-a$ tezlik bilan harakat qiladi degani. Xuddi shunday, argumenti $x - at$ bo'lgan funksiya o'ng tomonga a tezlik bilan harakat qilayotgan to'lqinga mos keladi. $u(x,t)$ funksiya esa boshlang'ich g'alayonning ikkiga parchalanib chap va o'ng tomonga tarqalayotgan to'lqinlar superpositsiyasi ekan.

§2. Cheksiz tor: majburiy tebranishlar masalasi

Endi quyidagi ko'rinishdagi Cauchy masalasini yechaylik:

$$
\left.
\begin{array}{l}
u_{tt} - a^2 u_{xx} = f(x,t), \\
u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \\
0 \le t < \infty, \quad -\infty < x < \infty.
\end{array}
\right\} \tag{9}
$$

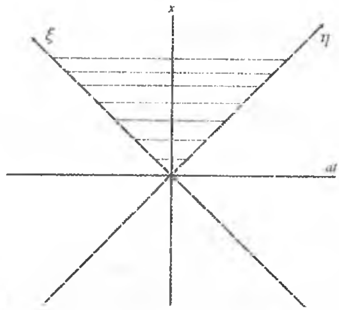
Bu yerdagi $f(x,t)$ funksya tor bo'yicha taqsimlangan tashqi kuchni bildiradi. Xarakteristikalardan foydalanib, yana (2)-almashtirish bajarsak, quyidagi tenglamani olamiz:

$$
-4a^2 u_{\xi\eta} = f(\xi,\eta). \tag{10}
$$

Bir jinslimas tenglamaning to'liq yechimi birjinsli tenglamaning umumiy yechimi va birjinslimas tenglamaning xususiy integralidan iboratdir. Bir jinsli tenglamaning umumiy yechimi (8)-orqali ifodalangan. Bir jinslimas tenglamaning xususiy integralini topish qiyin emas, uni bevosita (10)-dan keltirib chiqaramiz:

$$\bar{u}(\xi,\eta) = -\frac{1}{4a^2}\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta} d\xi d\eta f(\xi,\eta). \tag{11}$$

Almashtirish yakobiani $d\xi d\eta = 2adxdt$ ekanligini va (V.1)-rasmda ko'rsatilgan



V.1-rasm: Integrallash chegaralarini aniqlashga doir

integrallash sohasini hisobga olsak, (9)-ning yechimi

$$u(x,t) = \frac{1}{2}(\varphi(x+at)+\varphi(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} dz\psi(z) +$$

$$+\frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} dy f(y,\tau) \tag{12}$$

ekanligini topamiz. Bu formula ***D'Alembert*[1]*formulasi*** deyiladi.

§3. Bir tomondan cheklangan tor. Akslantirish metodi

Bir uchi mahkamlangan va ikkinchi tomoni cheksiz bo'lgan tor berilgan bo'lsin. Torning mahkamlanish nuqtasini $x = 0$ deb olsak, mahkamlanganlik sharti

[1] Jean-Baptiste le Roud D'Alembert (1717-1783) - fransuz olimi. Rus tilida - *Даламбер*

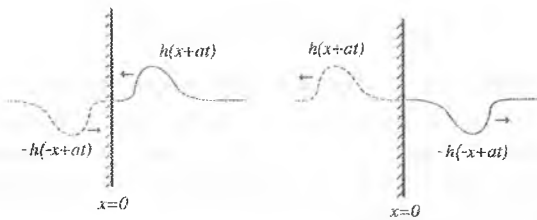
$u(0,t) = 0$ ko'rinishga ega bo'ladi. To'lqin tarqalayotgan soha $x > 0$ bo'lsin. Masalani quyidagicha qo'yish kerak:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad u(0,t) = 0; \\ 0 \le t < \infty, \quad 0 \le x < \infty. \end{array}\right\} \qquad (13)$$

To'lqin tenglamasining 5-yechimiga yuqoridagi chegaraviy shartni qo'llasak,

$$0 = h(at) + g(-at)$$

munosabatni olamiz. Demak,



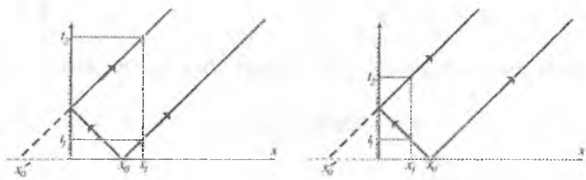
V.2-rasm: Bir uchi mahkamlangan tor bo'yicha to'lqin tarqalishi

$$u(x,t) = h(x+at) - h(at-x) \qquad (14)$$

ekan. Bu yerda $h(x + at)$ - o'ngdan chapga harakat qilayotgan to'lqin, $-h\,(-\,x + at\,)$ esa chapdan o'ngga harakat qilayotgan to'lqin, ular V.2-rasmda ko'rsatilgan. Fizik jarayon $x > 0$ sohada ro'y berayapti, lekin (V.2)-rasmda qulaylik uchun $x < 0$ soha ham ko'rsatilgan. Shunday qilinsa 14-formulaning talqini yengillashadi, uni butun $-\infty < x < \infty$ sohada o'rinli deb qarash mumkin. Rasmdan ko'rinib turibdiki, bu holda $t = 0$ vaqtda boshlang'ich $h(x)$ to'lqindan tashqari (nofizik) $x < 0$ sohada $-h(-x)$ ko'rinishdagi to'lqin ham berilgan, uning vazifasi $x = 0$ nuqtada $u(0,t) = 0$ chegaraviy shartning bajarilishini ta'minlash. Bu soha masalaga kirmagan soha bo'lgani uchun, u yerdagi to'lqinlar rasmda shtrixlab ko'rsatilgan.

$x > 0$ o'qida o'ngdan chap tomonga harakat qilayotgan to'lqin $h(x + at)$ chegaraviy $x = 0$ nuqtada devordan akslanib, ishorasini o'zgartirib, o'ng tomonga harakat qila boshlaydi. Agar biron $x_0 > 0$ nuqtada $t = 0$ vaqtda g'alayonlanish hosil qilinsa yetarli darajada katta bo'lgan $t > 0$ vaqt ichida ixtiyoriy $x_1 > 0$ nuqtaga ikkita to'lqin galma-galdan yetib keladi: $x_1 < x_0$

bo'lsa, boshlang'ich to'lqinning chap tomonga ketgan qismi va akslangan teskari ishorali to'lqin, $x_1 > x_0$ bo'lsa boshlang'ich to'lqinning o'ng tomonga ketgan qismi va akslangan teskari ishorali to'lqin. Xarakteristikalardan foydalanib bu holatlarni (V.3)-rasmdagidek tasavvur qilishimiz mumkin. To'lqin tenglamasi



V.3-rasm: Bir uchi mahkamlangan tor

uchun xarakteristikalar $x \pm at = \text{const}$ formula orqali aniqlanadi. $t = 0$ da x_0 nuqtada boshlangan to'lqin uchun const $= x_0$ ga teng. Demak, $x - at = x_0$. Akslanib qaytayotgan to'lqin uchun esa $x - at = -x_0$. G'alayonlangan nuqtadan chapga ketgan to'lqin uchun $x + at = x_0$. Mos keluvchi xarakteristikalar rasmda strelkali chiziqlar bilan ko'rsatilgan.

Kuzatish nuqtasi $x_1 > x_0$ holni ko'raylik, unga (V.3)-rasmning birinchi qismi mos keladi. x_0 nuqtadan o'ng tomonga ketgan to'lqin x_1 nuqtaga $t_1 = (x_1 - x_0)/a$ vaqtda keladi, xuddi shu x_1 nuqtaga chap tomonga ketib $x = 0$ chegarada akslanib qaytgan to'lqin $t_2 = (x_1 + x_0)/a$ vaqtda yetib keladi. Demak, x_1 nuqtadan t_1 va t_2 momentlarda ikkita to'lqin o'tar ekan - biri to'g'ri to'lqin, ikkinchisi akslangan to'lqin.

(V.3)-rasmning ikkinchi qismi $x_1 < x_0$ holga mos keladi. Bu holda ham x_1 nuqtadan ikkita to'lqin galma-gal o'tayapdi: $t_1 = (x_0 - x_1)/a$ vaqtda x_0 dan chapga ketgan to'lqin, $t_2 = (x_0 - x_1)/a + 2x_1/a = (x_0 + x_1)/a$ vaqtda chegaradan akslanib qaytgan to'lqin.

Har gal ham akslangan to'lqin go'yoki (fiktiv bo'lgan) $x_0' = -x_0$ nuqtadan chiqib kelayotgandek ko'rinadi. Shuning uchun akslangan to'lqin uchun $x - at = -x_0$.

Yechim qidirilayapgan soha $x > 0$, ammo formal nuqtai nazardan formulalarni $-\infty < x < \infty$ soha uchun yozganimiz qulayroq. Yechimning butun x o'qiga davomini $\widetilde{u}(x,t)$ deb belgilaymiz. (14)-dan kelib chidagiki, bu funksiya toq bo'lishi kerak: $\tilde{u}(x,t) = -\tilde{u}(-x,t)$. $\tilde{u}(x,t)$ ning toqligi $\tilde{u}(0,t) = 0$ chegaraviy shartning avtomatik ravishda bajarilishiga olib keladi. Boshlang'ich shartlarni ham butun x o'qiga toq ravishda davom ettiramiz:

$$\varphi(x,t) \to \tilde{\varphi}(x,t) = -\tilde{\varphi}(-x,t), \qquad \psi(x,t) \to \tilde{\psi}(x,t) = -\tilde{\psi}(-x,t).$$

Albatta, ko'rilayotgan funksiyalarning sinfi o'zgarmasligi kerak: $\tilde{u} \in C^2(R^2)$, $\tilde{\varphi} \in C^2(R)$, $\tilde{\psi} \in C^1(R)$. Yangi kiritilgan $\tilde{u}, \tilde{\varphi}, \tilde{\psi}$ funksiyalar tilida (13)-masalaning qo'yilishi (1)-masaladan farq qilmaydi. Shuning uchun ko'rilyapgan masalaning yechimini darhol yozib olish mumkin:

$$\tilde{u}(x,t) = \frac{1}{2}\left(\widetilde{\varphi}(x+at) + \widetilde{\varphi}(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} dz\, \widetilde{\psi}(z). \tag{15}$$

Bu formulani haqiqiy to'lqinlarga keltirish uchun $x - at \geq 0$ va $x - at < 0$ sohalarni alohida ko'rish kerak (har gal ham $x > 0$).

a) $x - at \geq 0$: Bu sohada $\widetilde{\varphi}(x-at) = \varphi(x-at)$, $\widetilde{\psi}(z) = \psi(z)$. Demak,

$$u(x,t) = \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} dz\, \psi(z), \qquad x \geq at.$$

b) $x - at < 0$: Bu holda $\widetilde{\varphi}(x-at) = -\varphi(-x+at)$, $\widetilde{\psi}(z) = -\psi(-z), z < 0$. Demak,

$$u(x,t) = \frac{1}{2}\left(\varphi(x+at) - \varphi(-x+at)\right) + \frac{1}{2a}\int_{-x+at}^{x+at} dz\, \psi(z), \qquad x < at.$$

Xuddi shu yo'l bilan bir uchi ozod bol'gan

$$u_x(0,t) = 0$$

yarim cheksiz tor masalasini ham yechish mumkin. Bu holda ham to'lqin $x = 0$ nuqtada akslanadi, ammo bu holda u o'z ishorasini o'zgartirmaydi:

$$h'(at) + g'(-at) = 0, \qquad \Rightarrow \qquad h(at) = g(-at) + \text{const}.$$

Bu holga to'g'ri keluvchi yechim:

$$u(x,t) = h(x+at) + h(at-x) + \text{const}.$$

§4. Akslantirish metodi: cheklangan tor (sterjen)

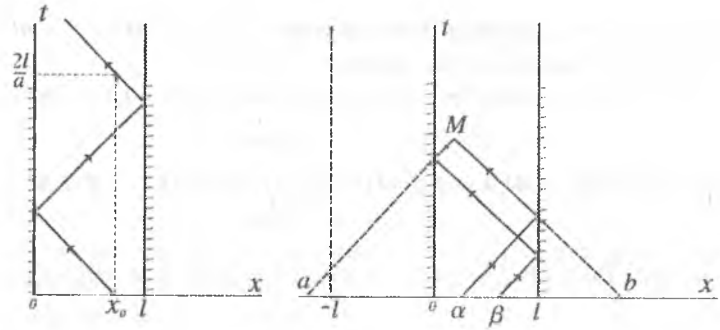
Tor(sterjen)ning ikkala uchi mahkamlangan bo'lsin:

$$u(0,t) = u(l,t) = 0.$$

Bu shartlarning birinchisi (14)-ga olib kelgan edi, ikkinchisidan esa

$$h(x) = h(x + 2l)$$

bo'lishi kerakligi kelib chiqadi. Demak, $h(x)$ - davri $2l$ bo'lgan davriy funksiya ekan. (V.4)-rasmning birinchi qismida x_0 nuqtadan chap tomonga ketgan to'lqin ko'rsatilgan, uning vaqt bo'yicha $2l/a$ davrli funksiya ekanligi ko'rinib turibdi. Ikki chegarada akslanishi natijasida bitta nuqtaga bir necha



V.4-rasm: Ikki uchi mahkamlangan tor

nuqtalardan chiqqan to'lqinlar bir vaqtda kelishi mumkin. (V.4)-rasmning ikkinchi qismida (x, t) tekisligidagi bir M nuqtaga α va β nuqtalardan chiqqan to'lqinlarning kelishi ko'rsatilgan. α nuqtadan chiqqan to'lqin o'ng devordan bir marta akslanib, β nuqtadan chiqqan to'lqin esa bir marta o'ng devordan, bir marta chap devordan akslanib M nuqtaga kelgan. Bu to'lqinlarni fiktiv bo'lgan a va b nuqtalardan chiqib, to'g'ri yetib kelgan, deb qarashimiz mumkin.

D'Alembert formulasini bu holga moslash uchun unga kirgan hamma funksiyalarni $2l$ davrli funksiyalarga davom ettirish kerak. Bu masala [3]-kitobda ko'rib chiqilgan.

Mashqlar.

5.1-mashq. [2]-kitobning II-bobidagi 52-57 sonli masalalarni yechib chiqing.

5.2-mashq. $u_{xx} - 2u_{xy} - 3u_{yy} = 0$ tenglamaning umumiy yechimini toping.

5.3-mashq. $3u_{xx} - 5u_{xy} - 2u_{yy} + 3u_x + u_y = 0$ tenglamaning umumiy yechimini toping.

5.4-mashq. $u_{xy} + au_x + bu_y + abu = 0$ tenglamaning umumiy yechimini toping.

5.5-mashq. Avvalgi masalaning natijasidan foydalanib

$$u_{xy} - 2u_x - 3u_y + 6u = 2e^{x+y}$$

tenglamaning umumiy yechimini toping.

5.6-mashq. $x^2u_{xx} - y^2u_{yy} = 0$ tenglamaning umumiy yechimini toping.

5.7-mashq. Cheksiz tor bo'yicha $f(x - at)$ to'lqin harakat qilayapti. Shu to'lqinni boshlang'ich shart sifatida olib $t > 0$ da tor bo'yicha tarqalayotgan to'lqinni toping.

VI BOB. FOURIER METODI

§1. Xususiy funksiyalar va xususiy qiymatlar masalasi

Klassik matematik fizikaning deyarli hamma tenglmalarining fazoviy qismi quyidagi ko'rinishga egadir:

$$-\mathrm{div}(p \text{ grad } u) + \mathrm{q}u = \lambda u. \tag{1}$$

Bir o'lchamli holda bu tenglamaning ko'rinishi quyidagichadir:

$$-(pu')' + qu = \lambda u. \tag{2}$$

Agar (1)-dagi operatorni

$$L = -\mathrm{div}(p \text{ grad }) + \mathrm{q} \tag{3}$$

deb belgilab olsak, (1)- tenglama bilan bog'liq bo'lgan chegaraviy masalalarni quyidagi ko'rinishda yozib olishimiz mumkin:

$$\begin{aligned} &Lu = \lambda u, \\ &\left(\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}}\right)_S = 0. \end{aligned} \tag{4}$$

Bu yerda S-tenglama o'rinli bo'lgan sohaning chegarasi, **n**-shu chegaraga tashqi normal , α va β lar chegaraviy shartlarni aniqlab beradilar, $\alpha + \beta > 0$. Albatta, funksiya u tenglama berilgan G sohada va uning chegarasi S da kerakli bo'lgan silliqlik, ya'ni, uzluksiz hosilalarga ega bo'lish xossalariga ega, deb olamiz. Ushbu masalaning yechimi u funksiya operator L ning ***xususiy funksiyasi*** va λ son esa L ning ***xususiy qiymati*** deyiladi. Masala λ ning shunday qiymatlarini topishdan iboratki, bunda berilgan tenglamaga va chegaraviy shartlarga bo'ysunuvchi u funksiya mavjud bo'lsin. Bunday masala ***xususiy qiymatlar masalasi*** deyiladi. Bir o'lchamli tenglama (2)-haqida gap ketganda bunday masala ***Sturm-Liouville***[1] ***masalasi*** deyiladi. Odatda xususiy funksiyalar va xususiy qiymatlar soni ko'p bo'ladi va har bir xususiy qiymat λ_n ga o'zining xususiy funksiyasi u_n mos keladi. Shuning uchun

$$Lu_n = \lambda_n u_n, \qquad n - \text{to'plam} \tag{5}$$

[1] Charle-Francois Sturm (1803-1855) va Joseph Liouville (1809-1882) - fransuz matematiklari. Rus tilida - Шарль-Франсуа Штурм ва Жозеф Лиувиль

deb yozamiz. Xususiy qiymatlarning to'plami $\{\lambda_n\}$ *L operatorning spektri* deyiladi.

Xususiy qiymatlar masalasi bilan bir necha misollarda tanishamiz.

6.1-misol. (2)-da $p = 1$, $q = 0$ va $0 < x < l$ bo'lsin:

$$u'' + \lambda u = 0. \tag{6}$$

Bu deganimiz, biz $L = -\dfrac{d^2}{dx^2}$ operatorning xususiy qiymatlarini qidiryapmiz:

$$u'' + \lambda u = 0 \quad \Rightarrow \quad Lu = \lambda u, \qquad L = -\frac{d^2}{dx^2}.$$

Chegaraviy shartlarni quyidagicha tanlab olamiz:

$$u(0) = u(l) = 0, \tag{7}$$

ya'ni, $\beta_1 = \beta_2 = 0$ va $\alpha_1 = \alpha_2 = 1$.

Xususiy qiymatlar $\lambda < 0, \lambda = 0, \lambda > 0$ bo'lishi mumkin. Qo'yilgan chegaraviy shartlarga faqatgina $\lambda > 0$ mos kelishini isbot qilaylik.

1. $\lambda < 0$ bo'lsin. Bu holda (6)-tenglamaning umumiy yechimi

$$u(x) = c_1 e^{x\sqrt{|\lambda|}} + c_2 e^{-x\sqrt{|\lambda|}}$$

bo'ladi. Birinchi chegaraviy shart $u(0) = 0$ dan

$$c_1 + c_2 = 0$$

kelib chiqadi. Ikkinchi chegaraviy shart $u(l) = 0$ dan

$$c_1 e^{l\sqrt{|\lambda|}} + c_2 e^{-l\sqrt{|\lambda|}} = 0$$

kelib chiqadi. Bu tenglamalarning yechimi $c_1 = c_2 = 0$ bo'ladi. Demak, $\lambda < 0$ bo'lishi mumkin emas.

2. $\lambda = 0$ bo'lsin. Bu holda (6)-tenglamaning umumiy yechimi

$$u(x) = c_1 + c_2 x$$

bo'ladi. Chegaraviy shartlar yana $c_1 = c_2 = 0$ ga olib keladi.

3. $\lambda > 0$ bo'lsin. Bu holda (6)-tenglamaning umumiy yechimi

$$u(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \tag{8}$$

bo'ladi. Birinchi chegaraviy shartdan

$$u(0) = c_1 = 0$$

kelib chiqadi. Ikkinchi chegaraviy shartni ishlatamiz:

$$u(l) = c_2 \sin(\sqrt{\lambda} l) = 0.$$

$c_2 \neq 0$ deb

$$\sqrt{\lambda_n} l = n\pi, \qquad n = 1, 2, 3, \ldots$$

yoki

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad n = 1, 2, 3, \ldots$$

deb olishimiz kerak. Masalaning xususiy qiymatlari ixtiyoriy butun son n ga bog'liq bo'lib chiqqani uchun xususiy qiymatlarga ham indeks n ni biriktirib qo'ydik. Demak, (6)-(7) xususiy qiymatlar masalasining yechimi quyidagicha ekan:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad u_n(x) = c_2 \sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, 3, \ldots \tag{9}$$

6.2-misol. Yana $p = 1$, $q = 0$ va $0 \leq x \leq l$ bo'lsin:

$$u'' + \lambda u = 0. \tag{10}$$

Ammo chegaraviy shartlarning birini o'zgartiramiz:

$$u(0) = u'(l) = 0, \tag{11}$$

yoki, $\beta_1 = 0$, $\beta_2 = 1$ va $\alpha_1 = 1$, $\alpha_2 = 0$ bo'lsin.

Bu holda ham $\lambda < 0$ va $\lambda = 0$ variantlar chegaraviy shartlarga mos kelmasligini tekshirib chiqish qiyin emas. Demak, $\lambda > 0$.

Tenglamaning umumiy yechimi o'sha:

$$u(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x). \tag{12}$$

Birinchi chegraviy shartdan yana $c_1 = 0$ kelib chiqadi. Ikkinchi chegaraviy shartni qanoatlantirish uchun

$$\sqrt{\lambda} l = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, 3, \ldots$$

deb olishimiz kerak. Demak, (10)-(11) xususiy qiymatlar masalasining yechimi

$$\lambda_n = \frac{\left(n + \frac{1}{2}\right)^2 \pi^2}{l^2}, \qquad u_n(x) = c_2 \sin\left(\frac{\left(n + \frac{1}{2}\right)\pi}{l}\right), \quad n = 0, 1, 2, 3, \ldots \tag{13}$$

ekan.

6.3-misol. Yana $p = 1$, $q = 0$ va $0 < x < l$ bo'lsin. Chegaraviy shartlar esa:

$$u'(0) = u'(l) = 0, \tag{14}$$

$\beta_1 = \beta_2 = 1$ va $\alpha_1 = \alpha_2 = 0$ bo'lsin. Yana yuqoridagidek tahlil qilib $\lambda < 0$, $\lambda = 0$ hollar chegaraviy shartlarga mos kelmasligini topishimiz mumkin. Tenglamaning umumiy yechimi yana o'sha:

$$u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x). \tag{15}$$

Chegaraviy shartlarni qo'llasak, xususiy qiymatlar masalasining (14)-chegaraviy shartlarga mos keluvchi yechimi

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad u_n(x) = c_1 \cos\left(\frac{n\pi}{l}x\right), \quad n = 0, 1, 2, 3, ... \tag{16}$$

ekanligini topamiz.

Bir narsaga ahamiyat berish kerak: uchchala misolda differensial operator bir xil edi, ammo har gal bitta chegaraviy shartni o'zgartirib turdik. Bu misollar xususiy qiymatlar masalasi uchun chegaraviy chartlarning ahamiyatini ko'rsatadi. Albatta, shunday bo'lishi kerak ham - tabiatdagi hamma to'lqin jarayonlar o'sha bitta to'lqin tenglamasi bilan ifodalanadi, ammo har gal har xil to'lqin kelib chiqishiga sabab har xil chegaraviy va boshlang'ich shartlardir (keyin ko'ramizki, boshlang'ich shartlar (1)-dagi q ga ta'sir qiladi).

6.4-misol. (56)- va (64)-formulalarni solishtirsak, sferik funksiyalar $Y_n^m(\theta, \varphi)$ Laplace operatorining burchak qismi bo'lgan

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

operatorning (minus ishora bilan) $n(n+1)$ xususiy qiymatiga mos keluvchi xususiy funksiyalari ekan:

$$-\Delta_{\theta,\varphi} Y_n^m(\theta, \varphi) = n(n+1) Y_n^m(\theta, \varphi).$$

Buning to'liq isboti I.§3.-paragrafda berilgan.

§2. Funksiyalarning ortogonalligi va normasi

Bizga kerakli bo'lgan yana bir necha tushunchani kiritaylik. Buning uchun yaxshi ma'lum bo'lgan ba'zi bir tushunchalarni umumlashtiramiz.

Bizga ikkita uch o'lchamli vektor berilgan bo'lsin - $\vec{f}$ va $\vec{g}$ (bu yerda vektorlarni strelkalar bilan belgilaymiz, shunisi qulayroq). Ularning skalar ko'paytmasi quyidagicha aniqlanadi:

$$\vec{f}\cdot\vec{g} = f_1 g_1 + f_2 g_2 + f_3 g_3 = \sum_{i=1}^{3} f_i g_i.$$

Agar n o'lchamli fazoning elementlari bo'lgan vektorlar $\vec{f}$ va $\vec{g}$ berilgan bo'lsa, bu holda, ularning skalar ko'paytmasi

$$\vec{f}\cdot\vec{g} = \sum_{i=1}^{n} f_i g_i \tag{17}$$

ko'rinishda aniqlanadi. Ko'pincha skalar ko'paytma uchun quyidagi belgi ishlatiladi:

$$\vec{f}\cdot\vec{g} = \left(\vec{f}, \vec{g}\right).$$

Ikkita vektorning skalar ko'paytmasi tushunchasini umumlashtirib ikkita haqiqiy funksiya $f(x)$ va $g(x)$ larning skalar ko'paytmasi tushunchasini kiritamiz:

$$(f, g) = \int_a^b f(x) g(x) dx.$$

Biz bunda avvalgi formuladagi vektor indekslar bo'yicha yig'indini funksiyalarning argumentlari bo'yicha uzluksiz yig'indi - integralga almashtirdik. Agar ko'riyapgan funksiyalarimiz kompleks bo'lsa, ularning skalar ko'paytmasi quyidagicha ta'riflanadi:

$$(f, g) = \int_a^b f^*(x) g(x) dx.$$

Agar $\vec{f}\cdot\vec{g} = \left(\vec{f}, \vec{g}\right) = 0$ bo'lsa, bunday vektorlar o'zaro ortogonal deyilar edi, xuddi shunday, agar

$$(f, g) = 0$$

bo'lsa, $f(x)$ va $g(x)$ funksiyalar ***o'zaro ortogonal*** deyiladi. Masalan, avvalgi paragrafdagi $(-d^2/dx^2)$ operatorining xususiy funksiyalari (9)-larni olaylik. Ular uchun:

$$(u_n, u_m) = c_2^2 \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) = c_2^2 \frac{l}{2} \delta_{mn}. \tag{18}$$

Ko'rinib turibdiki, $n \neq m$ holda $u_n(x)$ va $u_m(x)$ funksiyalar o'zaro ortogonal bo'lar ekan:

$$(u_n, u_m) = 0, \qquad n \neq m.$$

Ya'ni, $\{u_n, n = 1, 2, 3, ...\}$ funksiyalar to'plami o'zaro ortogonal funksiyalar to'plamini tashkil qilar ekan.

Qulay tushunchalardan biri - ***norma*** tushunchasi. U quyidagicha kiritiladi:

$$||f|| = \sqrt{(f, \overline{f})}.$$

Ko'rinib turibdiki, oddiy uch o'lchamli fazoga qaytsak, bu formula vektorlarning normasi, yani, uzunligining o'zi bo'ladi. Agar $||f|| = 1$ bo'lsa, funksiyaning normasi birga teng deyiladi.

(9)-sistemaga qaytib qo'shimcha ravishda

$$(u_n, u_m) = \delta_{mn}, \qquad n = 0, 1, 2, ...$$

bo'lishini talab qilsak, bunday normalari birga teng va o'zaro ortogonal funksiyalar to'plami $\{u_n, n = 0, 1, 2, ...\}$ ***ortonormal sistema*** deyiladi. (9)-funksiyalar to'plamini ortonormal sistemaga aylantirish uchun $c_2 = \sqrt{\frac{2}{l}}$ deb qabul qilishimiz kerak. Shunda quyidagi cheksiz ketma-ketlik

$$u_n(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, ...$$

$0 < x < l$ intervalda ortonormal sistemani tashkil qiladi. Bu sistemaning elementlari o'zaro ortogonalligini yuqorida ko'rdik, har bir elementining normasi esa birga teng:

$$||u_n||^2 = (u_n, u_n) = 1. \qquad (19)$$

Quyidagini ko'rsatish qiyin emas:

$$\int_0^l dx \cos\frac{n\pi x}{l} \cos\frac{m\pi x}{l} = \frac{l}{2}\delta_{mn}.$$

Ushbu misoldan ko'rinib turibdiki,

$$\bar{u}_n(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{n\pi x}{l}\right), \qquad n = 0, 1, 2, ...$$

funksiyalar to'plami ham $0 < x < l$ intervalda ortonormal sistemani tashkil qiladi.

Ortonormal sistemalarning ahamiyati nimadan iborat? Oddiy misol - uch o'lchamli fazodagi o'zaro perpendikular ortlar sistemasi

$$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\} = \{\vec{e}_i, i = 1, 2, 3\}, \qquad (\vec{e}_i, \vec{e}_j) = \delta_{ij}.$$

Bu sistema uch o'lchamli fazoda ***ortonormal bazis*** rolini o'ynaydi. Bu degani, uch o'lchamli fazodagi ixtiyoriy vektor $\vec{A}$ ni mana shu ortonormal sistema bo'yicha qatorga yoyishimiz mumkin:

$$\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z = \sum_{i=1}^{3} A_i\vec{e}_i. \tag{20}$$

Matematik fizika tenglamalarining yechimlari bo'lgan funksiyalar avvalgi paragrafda ko'rsatilganidek, cheksiz ketma-ketliklarni tashkil qiladi. Bu cheksiz ketma-ketliklar ortonormal sistemalarga aylantirilgandan keyin mos keluvchi cheksiz funksional fazolarda ortonormal bazis rolini o'ynaydi.

Biror bir funksional fazoda (ya'ni, elementlari funksiyalardan iborat bo'lgan fazoda) bizga bir to'plam G va ortonormal sistema $\{\varphi_n\} \in G$ berilgan bo'lsin. Yuqorida (20)-formula orqali ixtiyoriy uch o'lchamli vektorni $\{\vec{e}_i,\ i = 1, 2, 3\}$ ortonormal sistema bo'yicha qatorga yoyganimizdek ixtiyoriy $f \in G$ funksiyani ham ortonormal sitema $\{\varphi_n\} \in G$ bo'yicha qatorga yoyishimiz mumkin:

$$f(x) = \sum_n c_n\varphi_n(x). \tag{21}$$

Bu qator $f(x)$ funksiyasining ***Fourier qatori*** deyiladi. $\{\varphi_n\}$ ning ortonormalligidan $c_n = (f, \varphi_n)$ ekanligi kelib chiqadi:

$$(f, \varphi_n) = \sum_m c_m(\varphi_m, \varphi_n) = \sum_m c_m\delta_{mn} = c_n.$$

Sistema $\{\varphi_n\}$ G to'liq deyiladi, qachonki $f \in G$ uchun uning (21)-qatori shu fazoning normasi bo'yicha tekis yaqinlashsa:

$$\|f - f_n\| \longrightarrow 0, \quad n \longrightarrow \infty, \tag{22}$$

Bu yerda

$$f_n = \sum_{m=1}^{n} c_m\varphi_m.$$

Boshqacha so'z bilan aytganda, $\{\varphi_n\}$ G da to'liq bo'lsa, G da noldan farqli va hamma φ_n larga ortogonal bo'lgan funksiya topilmaydi.

6.5-misol. $[-\pi, \pi]$ intervalda davriy va $f(-\pi) = f(\pi)$ shartga bo'ysunadigan funksiyalar to'plamini ko'raylik va quyidagi sistemani kiritaylik:

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} \exp(inx),\ n = 0, \pm 1, \pm 2, ...$$

φ_n lar shu davriy funksiyalar to'plamiga kiradi va to'liq ortogonal sistemani tashkil qiladi:

$$(\varphi_n, \varphi_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(ix(n-m))dx = \delta_{mn}.$$

Ixtiyoriy $f(x)$ ni shu sistema bo'yicha qatorga yoyamiz:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum c_n \exp(inx).$$

Bu qator f ning Fourier qatoridir. Qator koeffisientlari uchun ma'lum formulani olamiz:

$$c_n = (f, \varphi_n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \exp(inx)dx.$$

Yuqorida ko'rsatilgan ediki, $\sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right)$, $n = 1, 2, ...$ va $\sqrt{\frac{2}{l}} \cos\left(\frac{n\pi x}{l}\right)$, $n = 0, 1, 2, ...$ funksiyalar sistemalari ham ortonormal bazisni tashkil qiladi, demak, ular bo'yicha ham $0 < x < l$ intervalda tegishli juftlik hossasiga ega bo'lgan funksiyalarni sinus va cosinus Fourier-qatorlariga yoyish mumkin.

Xususiy qiymatlar masalasidagi L operatorimizga qaytib kelaylik. Uning spektri va xususiy funksiyalarining asosiy xossalari quyidagi tasdiqda mujassamlashgandir:

(1)-dagi L operatorning xususiy qiymatlari manfiy bo'lmagan, sanoqli cheksiz, cheksizlikka intiluvchi va karraligi chekli bo'lgan sonlar to'plamini hosil qiladi:

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty.$$

Xususiy funksiyalar $\{u_n\}$ o'zaro ortogonal, to'liq va haqiqiy funksiyalar to'plamini hosil qiladi.

Biz bu tasdiqning isbotini keltirib o'tirmaymiz, uning isbotini [3] kitobda topish mumkin. Faqat bir narsani aytib ketamiz: $\{u_n\}$ to'plamga kirgan funksiyalarni har doim normalashtirishimiz mumkin, bu degani, ixtiyoriy to'liq ortogonal sistemadan to'liq ortonormal sistemani (bazisni) olishimiz mumkin.

Matematik fizikaning harxil sohalarida (ayniqsa, kvant mexanikasida) uchraydigan operatorlarning hammasi (1)-ko'rinishga ega bo'lavermaydi, ularning xususiy funksiyalari ham shunga yarasha haqiqiy funksiya bo'lavermaydi.

§3. O'zgaruvchilarni ajratish metodi - Fourier metodi. Giperbolik tenglamalar

Matematik fizikada eng ko'p qo'llaniladigan metodlardan biri - o'zgaruvchilarni ajratish metodi. Uning mohiyati quyidagicha. Biz yechayotgan tenglamaga kirgan funksiya $u(x, y, z, t)$ faqatgina bir o'zgaruvchining funksiyasi bo'lgan funksiyalar ko'paytmasi sifatida izlanadi. Masalan, dekart koordinat sistemasida

$$u(x, y, z) = X(x)Y(y)Z(z)T(t),$$

sferik sistemada:

$$u(r, \theta, \varphi, t) = R(r)\Theta(\theta)\Phi(\varphi)T(t),$$

va h.k. Natijada, xususiy hosilali differensial tenglama to'liq hosilali differensial tenglamalar sistemasiga keltiriladi, ularni yechish esa ko'p marta osonroqdir. Afsuski, bu yo'l hamma masalalarda ham o'tavermaydi - faqat ma'lum differensial operatorlar ma'lum koordinat sistemalaridagina o'zgaruvchilarni ajratishga yo'l qo'yadi.

Bizning kursimizga oid bo'lgan shunday misollardan bir nechtasi keyingi paragraflarda ko'rsatilgan.

§3.1. Erkin tebranishlar masalasi

Quyidagi masalani ko'raylik:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = 0, \\ u(0, t) = u(l, t) = 0, \\ u(x, 0) = \varphi(x);\ u_t(x, 0) = \psi(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{23}$$

Bu - ikkala uchi mahkam biriktirilgan l uzunlikdagi torning (sterjenning) erkin tebranishlari masalasi. Ham boshlang'ich, ham chegaraviy shartlar berilgan masala ***aralash masala*** deyiladi. Yechimni

$$u(x, t) = X(x)T(t) \tag{24}$$

ko'rinishda qidiramiz. Buni tenglamaga qo'ysak,

$$X(x)T''(t) - a^2X''(x)T(t) = 0 \tag{25}$$

ga kelamiz. Boshqacha so'z bilan,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2T(t)}. \tag{26}$$

Tenglamaning chap tomoni x ning funksiyasidir, o'ng tomoni esa t ning funksiyasi. Agar x ni (t ni) o'zgartira boshlasak tenglikning o'ng (chap) tomoni o'zgarmaydi, demak, haqiqatda tenglikning chap (o'ng) tomoni ham x ga (t ga) bog'liq emas ekan. Ya'ni, tenglikning ikkala tomoni ham bir o'zgarmas songa teng ekan, shu sonni $-\lambda$ deb belgilaylik:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2T(t)} = -\lambda. \tag{27}$$

Natijada, biz boshidagi bitta xususiy hosilali differensial tenglamaning o'rniga ikkita oddiy differensial tenglamalar sistemasiga egamiz:

$$\begin{aligned} &X''(x) + \lambda X(x) = 0, \qquad X(0) = X(l) = 0; \\ &T''(t) + \lambda a^2T(t) = 0. \end{aligned} \tag{28}$$

Hosil bo'lgan tenglamalarning birinchisiga masaladagi chegaraviy shartlarni ko'chirdik, chunki chegaraviy shartlar masalaning fazoviy qismiga qo'yilgan shartdir. (28)-sistemaning birinchisi yuqorida muhokama qilingan xususiy qiymatlar masalasi (6)-(7) ning o'zidir, uning yechimlari (9) ham bizga ma'lum:

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad X_n(x) = c_2 \sin\left(\frac{n\pi x}{l}\right), \qquad n = 0, 1, 2, 3, ... \tag{29}$$

$\lambda > 0$ bo'lib chiqdi, (27)-dagi ishora tushunarli bo'ldi. Agar $\lambda < 0$ bo'lsa, ikkala chegaraviy shartlarni qanoatlantira olmas edik (§1.-paragrafdagi muhokamani eslang).

(28)-ning ikkinchisining yechimini topish qiyin emas (λ-ning qiymatlarini xususiy qiymatlar masalasidan olamiz):

$$T_n(t) = a_n \sin\frac{n\pi at}{l} + b_n \cos\frac{n\pi at}{l}, \quad n = 0, 1, 2, 3, ... \tag{30}$$

Chiziqli tenglamaning to'liq yechimi uning xususiy eychimlarining superpozitsiyasidir:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(a_n \sin\frac{n\pi at}{l} + b_n \cos\frac{n\pi at}{l}\right). \tag{31}$$

Bu yechim o'zining fazoviy qismi orqali chegaraviy shartlarni qanoatlantiradi. Boshlang'ich shartlarni qanoatlantirish qoldi. Buning uchun Fourier qatorlari nazariyasini (yoki ortonormal qatorlar haqidagi nazariyani) eslasak yetarlidir:

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \tag{32}$$

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty} a_n \frac{n\pi a}{n} \sin\frac{n\pi x}{l}. \tag{33}$$

Ikkala qatorni $\sin(m\pi x/l)$ ga ko'paytiramiz, 0 dan l gacha integrallaymiz va

$$\int_0^l dx \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} = \frac{l}{2}\delta_{mn} \tag{34}$$

ekanligini eslaymiz. Natijada,

$$a_n = \frac{1}{na\pi}\int_0^l dx\psi(x)\sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l dx\varphi(x)\sin\frac{n\pi x}{l} \tag{35}$$

formulalarni olamiz.

Topilgan yechimni yana bir qulay holga keltirib olishimiz mumkin:

$$u_n(x,t) = X_n(x)T_n(t) = N_n \sin\frac{n\pi x}{l} \sin\left(\frac{n\pi a l}{l} + \alpha_n\right), \tag{36}$$

bu yerda

$$N_n^2 = a_n^2 + b_n^2, \ \ \mathrm{tg}\alpha_n = b_n/a_n.$$

Yechimning bu tasavvuri shu bilan qulayki, undan shu yechimning fizik ma'nosini bevosita aniqlash mumkin: (36)-formula ***xususiy chastotasi*** $\omega_n = \frac{n\pi a}{l}$, (maksimal) amplitudasi N_n, tugunlari soni ($0 < x < l$ da) $(n-1)$ bo'lgan ***turg'un to'lqinni*** ifodalaydi - (VI.1)-rasmga qarang. Har bir ω_n chastotali tebranish (turg'un to'lqin) ***garmonika*** deyiladi. Ba'zi bir hollarda garmonika so'zining o'rniga ***moda*** so'zi sihlatiladi - ***tebranish modasi*** degan termin ham bor.

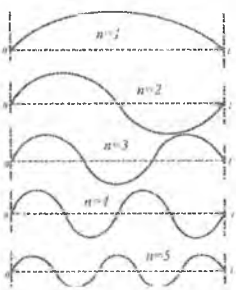
Quyidagi terminologiya ham keng tarqalgan: $\omega_1 = \frac{\pi a}{l}$ chastotali tebranish ***asosiy ton*** deyiladi, qolgan tebranishlar T_2X_2, T_3X_3, ... ***oberton***lar ketma-ketligini taskil qiladi. Torning xususiy chastotalari uning fizikaviy xoss ,lari bilan bog'langan:

$$\omega_n = \frac{n\pi a}{l} = \frac{n\pi}{l}\sqrt{\frac{T}{\rho}}.$$

(31)- va (36)- formulalardan foydalanib tebranishlarni ochib yozaylik:

$$u(x,t) = N_1 \sin\frac{\pi x}{l}\sin\left(\frac{\pi a t}{l} + \alpha_1\right) + N_2 \sin\frac{2\pi x}{l}\sin\left(\frac{2\pi a t}{l} + \alpha_2\right) + \\ + N_3 \sin\frac{3\pi x}{l}\sin\left(\frac{3\pi a t}{l} + \alpha_3\right) + ... \quad (37)$$

Bu (VI.1)-rasmda ko'rsatilgan garmonikalarning birinchi uchtasi. Qatorni ochib yozganimizdan maqsad garmonikalarning amplitudalari N_n larning rolini muhokama qilish. Muayyan misollar shuni ko'rsatadiki, n oshib borishi bilan, N_n kamaya boradi. Ya'ni, har bir keyingi garmonikaning umumiy tovushga qo'shgan hissasi kamroq bo'ladi. Ammo mana shu tebranishlarning yig'indisi tovush ***tembr***ini tashkil qiladi.



VI.1-rasm: Turg'un to'lqinlar

n-garmonikaning energiyasini (12)-formuladan topishimiz mumkin:

$$E_n = \frac{1}{2}\int_0^l \left\{\rho\left(\frac{\partial u_n}{\partial t}\right)^2 + T\left(\frac{\partial u_n}{\partial x}\right)^2\right\} dx =$$

$$= \frac{1}{4}M\omega_n^2 N_n^2 = \frac{1}{4}M\omega_n^2\left(a_n^2 + b_n^2\right). \quad (38)$$

Bu yerda $M = \rho l$ - torning to'liq massasi.

6.6-misol. Ikki uchi mahkam biriktirilgan tor kengligi 2δ bo'lgan bolg'acha zarbi ostida quyidagi boshlang'ich tezlik bilan muvozanatdan siljitilgan:

$$u_t(x,0) = \psi(x) = \begin{cases} 0, & 0 \le x < x_0 - \delta, \\ v_0, & x_0 - \delta \le x \le x_0 + \delta, \\ 0, & x_0 + \delta < x \le l. \end{cases}$$

Shu torning erkin tebranishlarini toping.

Yechim. Torning ikkala uchi biriktirilganligi $u(0,t) = u(l,t) = 0$ ekanligiga teng. Boshlang'ich siljishning yo'qligi: $u(x,0) = \varphi(x) = 0$, $0 \le x \le l$.

Tenglama bir jinsli bo'lishi kerak, chunki masalada tor bo'yicha taqsimlangan kuch berilmagan:

$$u_{tt} - a^2 u_{xx} = 0.$$

Masala qo'yildi. Uni yechishga o'taylik.

Noma'lum funksiya $u(x,t)$ da o'zgaruvchilarni ajratamiz:

$$u(x,t) = X(x)T(t).$$

Natijada, yana o'sha (28)-tenglamalar sistemasiga kelamiz, chegaraviy shartlarni hisobga olsak (29)-formulani olamiz:

$$X_n(x) = c_n \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3, ...$$

Demak, umumiy yechim

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(a_n \sin\frac{n\pi at}{l} + b_n \cos\frac{n\pi at}{l}\right)$$

ekan. Boshlang'ich siljishning yo'qligidan

$$u(x,0) = 0 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} a_n \Longrightarrow b_n = 0, \quad n = 1, 2, 3, ...$$

ekanligi kelib chiqadi. Koeffisient a_n (35)-formulaga asosan

$$a_n = \frac{1}{n\pi a}\int_0^l \psi(x)\sin\frac{n\pi x}{l}dx = \frac{2v_0}{n\pi a}\int_{x_0-\delta}^{x_0+\delta} \sin\frac{n\pi x}{l}dx = \frac{4v_0 l}{n^2\pi^2 a}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi\delta}{l}$$

ga tengdir. Demak, masalamizning hamma chegaraviy va boshlang'ich shartlarni hisobga olgan yechimi

$$u(x,t) = \frac{4v_0 l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi\delta}{l}\sin\frac{n\pi at}{l} =$$

$$= \frac{4v_0 l}{\pi^2 a}\left\{\sin\frac{\pi x}{l}\sin\frac{\pi x_0}{l}\sin\frac{\pi\delta}{l}\sin\frac{\pi at}{l} + \frac{1}{4}\sin\frac{2\pi x}{l}\sin\frac{2\pi x_0}{l}\sin\frac{2\pi\delta}{l}\sin\frac{2\pi at}{l} + \right.$$

$$\left. + \frac{1}{9}\sin\frac{3\pi x}{l}\sin\frac{3\pi x_0}{l}\sin\frac{3\pi\delta}{l}\sin\frac{3\pi at}{l} + ...\right\}$$

ko'rinishga ega ekan.

E'tibor bering: $\omega_2 = \frac{2\pi a}{l}$ chastotali garmonikaning (obertonning) amplitudasi birinchi garmonika $\omega_1 = \frac{\pi a}{l}$ ning (asosiy tonning) amplitudasidan 4 marta kam, undan keyingi garmonikaning amplitudasi esa birinchi garmonikaga nisbatan 9 marta kam va h.k.

Agar shu yechimda $v_0 = \frac{I}{2\rho\delta}$ deb olib $\delta \to 0$ limitga o'tsak, $x = x_0$ nuqtada I impuls beruvchi ko'ndalang zarba olgan torning tebranishlari masalasini yechgan bo'lamiz:

$$u(x,t) = \frac{2I}{\pi\rho a}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\sin\frac{n\pi x_0}{l}\sin\frac{n\pi a t}{l}.$$

§3.2. Majburiy tebranishlar masalasi

Tebranish masalasiga tor bo'yicha taqsimlangan tashqi kuch $f(x,t)$ ni kiritaylik:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = f(x,t), \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x);\ u_t(x,0) = \psi(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{39}$$

Chegaraviy shartlarni hisobga olib, bu masalaning yechimini quyidagi ko'rinishda izlash tabiiydir:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l}. \tag{40}$$

Boshqa hamma funksiyalarni ham xuddi shunday Fourier qatoriga yoyamiz:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{l}, \quad f_n(t) = \frac{2}{l}\int_0^l dx f(x,t)\sin\frac{n\pi x}{l}$$

$$\varphi(x) = \sum_{n=1}^{\infty}\varphi_n\sin\frac{n\pi x}{l}, \quad \varphi_n = \frac{2}{l}\int_0^l dx\varphi(x)\sin\frac{n\pi x}{l}, \tag{41}$$

$$\psi(x) = \sum_{n=1}^{\infty}\psi_n\sin\frac{n\pi x}{l}, \quad \psi_n = \frac{2}{l}\int_0^l dx\psi(x)\sin\frac{n\pi x}{l}.$$

Bu qatorlarni (39)-tenglamaga olib borib qo'sak quyidagi tenglamaga kelamiz:

$$\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \left\{\ddot{u}_n(t) + \omega_n^2 u_n(t) - f_n(t)\right\} = 0, \qquad \omega_n = \frac{n\pi a}{l}. \tag{42}$$

Demak,

$$\ddot{u}_n(t) + \omega_n^2 u_n(t) - f_n(t) = 0, \quad u(0) = \varphi_n, \quad \dot{u}(0) = \psi_n. \tag{43}$$

Umumiy metodga asosan birjinslimas tenglamaning umumiy yechimi bir jinsli tenglamaning umumiy yechimi va bir jinslimas tenglamaning xususiy yechimlaridan iborat:

$$u_n(t) = u_n^{(0)}(t) + u_n^{(1)}(t). \tag{44}$$

Erkin tebranishlar tenglamasining yechimi ma'lum:

$$u_n^{(0)}(t) = a_n \sin(\omega_n t) + b_n \cos(\omega_n t). \tag{45}$$

Bir jinslimas tenglamaning xususiy yechimini topish ham qiyin emas:

$$u_n^{(1)}(t) = \frac{1}{\omega_n} \int_0^t \sin(\omega_n(t-\tau)) f_n(\tau) d\tau. \tag{46}$$

Bularni (44)- va (40)- larga olib borib qo'ysak, (39)-masalaning yechimini topgan bo'lamiz:

$$u(x,t) = \sum_1^{\infty} \sin\frac{n\pi x}{l} \left\{ \left(\varphi_n \sin(\omega_n t) + \frac{1}{\omega_n}\psi_n \cos(\omega_n t)\right) + \right.$$
$$\left. + \frac{1}{\omega_n} \int_0^t \sin(\omega_n(t-\tau)) f_n(\tau) d\tau \right\}. \tag{47}$$

§3.3. Birinchi umumiy chegaraviy masala

Chegaraviy shartlar birinchi turga tegishli umumiy holda bo'lsin:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = f(x,t), \\ u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \\ u(x,0) = \varphi(x); \; u_t(x,0) = \psi(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{48}$$

Bunday masalaning yechimini quyidagi ko'rinishda izlaymiz:

$$u(x,t) = U(x,t) + v(x,t). \tag{49}$$

Agar $U(x,t)$ funksiyani maxsus ko'rinishda tanlab olsak:

$$U(x,t) = \mu_1(t) + \frac{x}{l}(\mu_2(t) - \mu_1(t)) \tag{50}$$

yangi noma'lum funksiya $v(x,t)$ bir jinsli chegaraviy shartlarga bo'sunadigan bo'ladi:

$$v(0,t) = v(l,t) = 0. \tag{51}$$

Natijada, $v(x,t)$ uchun avvalgi paragrafda ko'rib chiqilgan masalani olamiz:

$$\left.\begin{array}{l} v_{tt} - a^2 v_{xx} = \bar{f}(x,t), \\ v(0,t) = v(l,t) = 0, \\ v(x,0) = \bar{\varphi}(x);\ v_t(x,0) = \bar{\psi}(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{52}$$

Bu yerda

$$\bar{f}(x,t) = f(x,t) - U_{tt}(x,t),\ \bar{\varphi}(x) = \varphi(x) - U(x,0),\ \bar{\psi}(x) = \psi(x) - U_t(x,0).$$

§3.4. Statsionar ozod hadli chegaraviy masala

Avvalgi punktdagi masalamizning bir xususiy holini ko'raylik:

$$\left.\begin{array}{l} u_{tt} - a^2 u_{xx} = f(x), \\ u(0,t) = u_1, \quad u(l,t) = u_2, \\ u(x,0) = \varphi(x);\ u_t(x,0) = \psi(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{53}$$

Tenglamamizdagi ozod had va chegaraviy shartlar vaqtga bog'liq emas. Bu holda yechim quyidagicha qidiriladi:

$$u(x,t) = v(x,t) + w(x). \tag{54}$$

Shu tarzda kiritilgan ikkita yangi $v(x,t)$ va $w(x)$ funksiyalarni quyidagi masalalarni yechimlari sifatida izlaymiz:

$w(x)$ uchun masala:

$$a^2 w''(x) + f(x) = 0, \quad w(0) = u_1,\ w(l) = u_2. \tag{55}$$

$v(x,t)$ uchun masala:

$$\left.\begin{array}{l} v_{tt} - a^2 v_{xx} = 0, \\ v(0,t) = v(l,t) = 0, \\ v(x,0) = \varphi(x) - w(x);\ v_t(x,0) = \psi(x), \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \qquad (56)$$

(55)-tenglama to'liq hosilali tenglama, uni yechish qiyin emas, (56)-masalani esa ko'rib chiqqanmiz.

6.7-misol.

Quyidagi masala yechilsin:

$$u_{tt} - u_{xx} = 2b, \quad b = \text{const}, \quad u(0,t) = u(l,t) = 0, \ \ u(x,0) = u_t(x,0) = 0.$$

Yechish

(55)- ga muvofiq

$$w''(x) + 2b = 0, \quad w(0) = w(l) = 0$$

tenglama va chegaraviy shartlarga egamiz, buning yechimi:

$$w(x) = -bx(x-l).$$

Shunda $v(x,t)$ uchun quyidagi masalani olamiz:

$$\left.\begin{array}{l} v_{tt} - v_{xx} = 0, \\ v(0,t) = v(l,t) = 0. \\ v(x,0) = bx(x-l);\ v_t(x,0) = 0, \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \qquad (57)$$

Bu - erkin tebranishlar masalasining o'zi, uning yechimi (31)-formula bo'yicha

$$v(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(a_n \sin\frac{n\pi a t}{l} + b_n \cos\frac{n\pi a t}{l}\right).$$

Bizning holimizda $\psi(x) = 0$ bo'lgani uchun $a_n = 0$ bo'ladi, b_n ni esa quyidagicha topamiz:

$$b_n = \frac{2b}{l}\int_0^l dx\, x(x-l)\sin\frac{n\pi x}{l} = \frac{4bl^2}{n^3\pi^3}\left((-1)^n - 1\right).$$

Ko'rinib turibdiki, $b_2 = b_4 = b_6 = \cdots = 0$. $n = 2k+1$ bo'lganda

$$b_{2k+1} = -\frac{8bl^2}{(2k+1)^3\pi^3}.$$

Demak,

$$u(x,t) = -bx(x-l) + v(x,t),$$

bu yerda

$$v(x,t) = -\frac{8bl^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\sin\frac{(2k+1)\pi x}{l}\cos\frac{(2k+1)\pi at}{l} =$$

$$= -\frac{8bl^2}{\pi^3}\left[\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + \frac{1}{27}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \frac{1}{125}\sin\frac{5\pi x}{l}\cos\frac{5\pi at}{l} + \cdots\right]$$

Ko'rinib turibdiki, asosiy garmonika - birinchi garmonika, keyingi hadlarning amplitudasi (va demak, tovushga qo'shgan hissasi) garmonikaning nomeri oshishi bilan keskin kamayib ketadi.

§3.5. Misollar

Og'ir sterjenning tebranishlari masalasi

Quyidagi masalani yechaylik:

$$u_{tt} - a^2u_{xx} = g, \quad u(0,t) = 0, \quad u_x(l,t) = 0, \quad u(x,0) = kx, \quad u_t(x,0) = 0. \tag{58}$$

Bu masalaning ma'nosi - bir uchidan shipga osib qo'yilgan sterjen ikkinchi (ozod) uchidan elastik ravishda tortilgan va $t = 0$ vaqt momentida qo'yib yuborilgan. Yechimni quyidagicha qidiramiz:

$$u(x,t) = v(x,t) + w(x).$$

Agar $w(x)$ ni quyidagi masalaga bo'ysundirsak,

$$a^2w''(x) + g = 0, \quad w(0) = 0, \quad w'(l) = 0$$

$v(x,t)$ uchun bir jinsli masalani olgan bo'lamiz:

$$v_{tt} - a^2v_{xx} = 0, \quad u(0,t) = 0, \quad u_x(l,t) = 0, \quad u(x,0) = kx - w(x), \quad u_t(x,0) = 0. \tag{59}$$

w ni topish oson:

$$w(x) = \frac{g}{a^2}x\left(l - \frac{x}{2}\right).$$

(59)-ni yechaylik.

$$v(x,t) = X(x)T(t)$$

deb olsak, $X(x)$ uchun

$$X''(x) + \lambda^2 X(x) = 0, \quad X(0) = X'(l) = 0$$

masalani olamiz, uning yechimi bizga ma'lum:

$$X(x) = c_1 \sin\left(\frac{2n+1}{2l}\pi x\right).$$

Demak, umumiy yechim quyidagi ko'rinishga ega:

$$v(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{2n+1}{2l}\pi x\right)\left[a_n \cos\left(\frac{2n+1}{2l}\pi at\right) + b_n \sin\left(\frac{2n+1}{2l}\pi at\right)\right].$$

$v_t(x,0)$ shart bizga $b_n = 0$ ekanligini beradi. Ikkinchi boshlang'ich shartni olaylik:

$$\sum_{n=0}^{\infty} \sin\left(\frac{2n+1}{2l}\pi x\right) a_n = kx - \frac{g}{a^2}x\left(l - \frac{x}{2}\right).$$

Bundan

$$a_n = \frac{2}{l}\int_0^l dx\left[kx - \frac{g}{a^2}x\left(l - \frac{x}{2}\right)\right]\sin\left(\frac{2n+1}{2l}\pi x\right) = \frac{8l}{a^2\pi^2}\frac{(-1)^n a^2 k(2n+1) - gl}{(2n+1)^3}$$

ni topamiz. To'liq yechim:

$$u(x,t) = \frac{g}{a^2}x\left(l - \frac{x}{2}\right) +$$

$$+\frac{8l}{a^2\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n a^2 k(2n+1) - gl}{(2n+1)^3}\sin\left(\frac{2n+1}{2l}\pi x\right)\cos\left(\frac{2n+1}{2l}\pi at\right).$$

Tashqi kuch davriy bo'lgan hol

Quyidagi masalani yechaylik:

$$u_{tt} - u_{xx} = \cos t, \quad 0 \le x \le \pi, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = u_t(x,0) = 0.$$

Chegaraviy shartlar yechimni quyidagi ko'rinishda qidirishni taqazo qiladi:

$$u(x,t) = \sum_n u_n(t) \sin nx.$$

Tenglamamizning ko'rinishi quyidagicha bo'ladi:

$$\sum_{n=1}^{\infty}\left(\ddot{u}_n(t)+n^2u(t)\right)\sin nx=\cos t.$$

Ikkala tomonni $\sin nx$ ga ko'paytirib integrallaymiz (0 dan π gacha):

$$\ddot{u}_n(t)+n^2u(t)=\frac{2}{\pi n}(1-(-1)^n)\cos t. \tag{60}$$

$n=1$ holni alohida ko'rishimiz kerak, chunki bu holda rezonans bor:

$$\ddot{u}_1+u_1=\frac{4}{\pi}\cos t.$$

Bu tenglamaning xususiy yechimi

$$\bar{u}_1(t)=\frac{2}{\pi}t\sin t.$$

Uning umumiy yechimi

$$u_1(t)=\frac{2}{\pi}t\sin t+c_1\sin t+c_2\cos t.$$

Ammo boshlahg'ich shartlardan $c_1=c_2=0$ ekanligi kelib chiqadi. $n=2,3,...$ hollar uchun esa (60)-ning yechimi (boshlang'ich shartlarni hisobga oldik):

$$u_n(t)=\frac{4}{\pi n}\frac{1}{n^2-1}(\cos t-\cos nt),\quad n=2k+1,\quad k=1,2,3,....$$

$n=2k$ juft bo'lganda boshlang'ich shartlarni hisobga olsak yechim trivial bo'ladi. To'liq yechim:

$$u(x,t)=\frac{2}{\pi}t\sin t\sin x+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k(2k+1)(k+1)}(\cos t-\cos(2k+1)t)\sin nx.$$

Albatta, bu yechimning qo'llanilish sohasi kichik t lar bilan cheklangan - vaqt o'tishi bilan birinchi had ckeksiz o'sa boshlaydi va kichik tebranishlar sohasidan chiqib ketiladi.

Mashqlar

O'zgaruvchilarni ajratish metodi bilan quyidagi masalalarni yeching:

6.1-mashq.

$$u_{tt}=u_{xx},\ 0\le x\le l,\ u(0,t)=0,\ u(l,t)=t,\ u(x,0)=u_t(x,0)=0.$$

6.2-mashq.

$$u_{tt} = u_{xx},\ 0 \le x \le 1,\ u(0,t) = t+1,\ u(1,t) = t^2+2, \quad u(x,0) = x+1,\ u_t(x,0) = 0.$$

6.3-mashq.

$$u_{tt} = u_{xx} - 4u,\ 0 \le x \le 1,\ u(0,t) = u(1,t) = 0, \quad u(x,0) = x^2 - x,\ u_t(x,0) = 0.$$

6.4-mashq.

$$u_{tt} = u_{xx} + u,\ 0 \le x \le 2,\ u(0,t) = 2t,\ u(2,t) = 0, \quad u(x,0) = 0,\ u_t(x,0) = 0.$$

6.5-mashq.

$$u_{tt} = u_{xx} + u,\ 0 \le x \le l,\ u(0,t) = 0,\ u(1,t) = t, \quad u(x,0) = 0,\ u_t(x,0) = x/l.$$

6.6-mashq.

$$u_{tt} - a^2 u_{xx} = 0, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = \sin\frac{2\pi x}{l}.$$

6.7-mashq.

$$u_{tt} - a^2 u_{xx} = 0, \quad u(0,t) = u_x(l,t) = 0, \quad u(x,0) = \sin\frac{5\pi x}{2l}, \quad u_t(x,0) = \sin\frac{\pi x}{2l}.$$

6.8-mashq.

$$u_{tt} - a^2 u_{xx} = 0, \quad u(0,t) = u_x(l,t) = 0, \quad u(x,0) = x, \quad u_t(x,0) = \sin\frac{\pi x}{2l} + \sin\frac{3\pi x}{2l}.$$

6.9-mashq.

$$u_{tt} - a^2 u_{xx} = 0, \quad u_x(0,t) = u(l,t) = 0, \quad u(x,0) = \cos\frac{\pi x}{2l}, \quad u_t(x,0) = \cos\frac{3\pi x}{2l} + \cos\frac{5\pi x}{2l}.$$

6.10-mashq.

$$u_{tt} - a^2 u_{xx} = 0, \quad u_x(0,t) = u_x(l,t) = 0, \quad u(x,0) = x,\ u_t(x,0) = 1.$$

§4. Parabolik tenglamalarga Fourier metodini qo'llash

O'zgaruvchilarni ajratish metodini parabolik tenglamalarga qo'llash yo'llarini ham bir necha misollarda ko'rib chiqamiz.

§4.1. Bir jinsli chegaraviy masala

Eng sodda masaladan boshlaymiz: uzunligi l bo'lgan sterjenning ikkala uchida nolga teng temperatura ushlanib turibdi, sterjen bo'yicha temperaturaning boshlang'ich qiymati - $\varphi(x)$. Sterjenning t vaqt momentidagi temperaturasini toping.

Berilgan masalaning matematik ko'rinishi quyidagicha:

$$\left.\begin{array}{l} u_t - a^2 u_{xx} = 0, \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x); \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{61}$$

Yechimni quyidagi ko'rinishda izlaymiz:

$$u(x,t) = X(x)T(t).$$

Demak,

$$T'(t)X(x) = a^2 T(t)X''(x), \tag{62}$$

yoki,

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{63}$$

Ikkinchi tenglikdan keyin noma'lum o'zgarmas son λ ning paydo bo'lishi yana o'sha mantiqiy mulohazalardan keyin kelib chiqadi: galma-galdan t va x o'zgaruvchilarni o'zgartirib chiqsak na chap tomon va na o'ng tomonning o'zgarmasligini ko'ramiz. Demak, ikkala tomon ham o'zgarmas songa teng ekan, bu sonni $-\lambda$ deb belgiladik. $\lambda > 0$ bo'lishi kerak, buning sababi Xuddi (29)-tenglamadan keyingi ko'rsatilgan sababning o'zidir. Natijada, quyidagi ikkita tenglamani olamiz:

$$\begin{array}{ll} X''(x) + \lambda X(x) = 0, & X(0) = X(l) = 0; \\ T'(t) + \lambda a^2 T(t) = 0. & \end{array} \tag{64}$$

Bu sistemaning birinchi qismi bizga yaxshi ma'lum bo'lgan xususiy qiymatlar masalasi, uning yechimi ham bizga ma'lum:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = c_2 \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots \tag{65}$$

Ikkinchi tenglamani yechaylik:

$$\frac{dT}{T} = -\frac{n^2 a^2 \pi^2}{l^2} dt \implies T(t) = b \exp\left(-\frac{n^2 a^2 \pi^2}{l^2} t\right). \tag{66}$$

Ikkala funksiya oldidagi noma'lumlarni birlashtirib bitta xususiy yechimni olamiz:

$$u_n(x,t) = X_n(x)T_n(t) = a_n \sin\frac{n\pi x}{l} \exp\left(-\frac{n^2a^2\pi^2}{l^2}t\right). \tag{67}$$

Xususiy yechimlarning superpozitsiyasi to'liq yechimni beradi:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \exp\left(-\frac{n^2a^2\pi^2}{l^2}t\right). \tag{68}$$

Boshlang'ich shartdan foydalanib, a_n koeffisientni topamiz:

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = \varphi(x) \implies a_n = \frac{2}{l}\int_0^l dx\varphi(x)\sin\frac{n\pi x}{l}. \tag{69}$$

Masala to'liq yechildi.

§4.2. Tashqi manba bo'lgan hol

Quyidagi bir jinslimas masalani ko'raylik:

$$\left.\begin{array}{l} u_t - a^2u_{xx} = f(x,t), \\ u(0,t) = u(l,t) = 0, \\ u(x,0) = \varphi(x); \\ 0 \leq t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{70}$$

Avvalgi masalaga nisbatan o'zgarish bitta - issiqlikning sterjen bo'yicha taqsimlangan tashqi manbasi paydo bo'ldi (agar bu dissuziya masalasi bo'lsa, bu manba - modda manbasi bo'ladi). Bunday masalalarning yechish metodi giperbolik tenglamalarga qo'llangan metod bilan bir xildir. Yechimni sinus-Fourier qatoriga yoyish metodi bilan qidiramiz:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l}. \tag{71}$$

Chegaraviy shartlar bu formulada avtomatik ravishda hisobga olindi. Masaladagi boshqa funksiyalarni ham sinus-Fourier qatoriga yoyamiz:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin\frac{n\pi x}{l}, \qquad f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{l}. \tag{72}$$

Shuni aytish joizki, fransuz matematigi Fourier mana shu masalani yechish davomida o'zining mashhur Fourier qatorlarini kiritgan, yuqorida yozilgan qatorlar shu qatorlaning bir xususiy holidir.

(71)- va (72)- qatorlarni (70)-ga olib borib qo'ysak

$$\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left\{\dot{u}_n(t) + \frac{a^2n^2\pi^2}{l^2}u_n(t) - f_n\right\} = 0$$

tenglamani olamiz. Bundan kelib chiqadiki, u_n koeffisientlar quyidagi masalaning yechimidir:

$$\dot{u}_n(t) + \frac{a^2n^2\pi^2}{l^2}u_n(t) = f_n, \qquad u_n(0) = \varphi_n.$$

Bu tenglamani yechish qiyin emas (masalan, o'zgarmaslarni variatsiyalash yoki operatsion metod bilan). Natijada, (70)-masalaning yechimi sifatida quyidagi formulani olamiz:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[\varphi_n \exp\left(-\frac{a^2n^2\pi^2}{l^2}t\right) + \right.$$
$$\left. + \int_0^t d\tau f_n(\tau)\exp\left(-\frac{a^2n^2\pi^2}{l^2}(t-\tau)\right)\right]. \tag{73}$$

§5. Umumlashgan hollar

Xuddi giperbolik tenglamalardagidek, yana ikkita muhim hollarda to'xtab o'tishimiz kerak, matematik nuqtai nazardan ular giperbolik holdan farq qilmagani uchun ular ustida qisqa to'xtab ketamiz.

§5.1. Birinchi umumiy chegaraviy masala:

$$\left.\begin{array}{l} u_t - a^2u_{xx} = f(x,t), \\ u(0,t) = \mu_1(t),\ u(l,t) = \mu_2(t), \\ u(x,0) = \varphi(x); \\ 0 \le t < \infty, \quad 0 < x < l. \end{array}\right\} \tag{74}$$

Bu holda yechim

$$u(x,t) = U(x,t) + v(x,t), \qquad U(x,t) = \mu_1(t) + \frac{x}{l}(\mu_1(t) - \mu_2(t))$$

ko'rinishda qidiriladi. Natijada, $v(x,t)$ funksiya uchun chegaraviy shartlari bir jinsli bo'lgan masalani olamiz, bunday masalalarni esa yechishni bilamiz.

§5.2. Manba statsionar bo'lgan hol

Tashqi manba statsionar bo'lsin: $f = f(x)$. Bu holda yechimni

$$u(x,t) = v(x,t) + w(x)$$

ko'rinishda qidiramiz va chegaraviy shartlarni (agar ular bir jinsli bo'lmasa) $w(x)$ ga tashlaymiz. Maqsad - $v(x,t)$ uchun bir jinsli chegaraviy shartli masalani olish.

§5.3. Misollar

Sterjenning chap uchi issiqlik o'tkazmaydi, o'ng uchida u_2 temperatura berilgan

Quyidagi masalani yeching:

$$u_t - u_{xx} = 0, \quad u_x(0,t) = 0, \quad u(l,t) = u_2, \quad u(x,0) = \frac{A}{l}x.$$

Yechimni quyidagicha qidiramiz: $u(x,t) = u_2 + v(x,t)$. $v(x,t)$ uchun quyidagi masala olinadi:

$$v_t - v_{xx} = 0, \quad v_x(0,t) = v(l,t) = 0, \quad u(x,0) = \frac{A}{l}x - u_2.$$

Davom etamiz:

$$v(x,t) = X(x)T(t) \Rightarrow X'(0) = X(l) = 0 \Rightarrow X(x) = \cos\frac{2n+1}{2l}\pi x \Rightarrow$$

$$\Rightarrow v(x,t) = \sum_n a_n \cos\left[\frac{2n+1}{2l}\pi x\right]\exp\left(-\frac{(2n+1)^2a^2\pi^2}{4l^2}t\right).$$

$$a_n = \frac{2}{l}\int_0^l dx\left(\frac{A}{l}x - u_2\right)\cos\left[\frac{2n+1}{2l}\pi x\right] = \frac{4[(2n+1)\pi(A-u_2)(-1)^n - A]}{\pi^2(2n+1)^2}.$$

Sterjenning chap uchi issiqlik o'tkazmaydi, o'ng uchida Q issiqlik oqimi berilgan

$$u_t - a^2u_{xx} = 0, \quad u(x,0) = 0, \quad u_x(0,t) = 0, \quad u_x(l,t) = Q/k.$$

Yechimni

$$u(x,t) = Ax^2 + Bx + v(x,t)$$

ko'rinishda qidiramiz. Nima uchun? Ikkita chegaraviy shartlarni ikkita noma'lum A va B lar orqali ifodalamoqchimiz:

$$u_x(0,t) = B + v_x(0,t) = 0 \quad \Rightarrow \quad B = 0, \ v_x(0,t) = 0.$$

$$u_x(l,t) = Q/k = 2Al + v_x(l,t) \quad \Rightarrow \quad A = \frac{Q}{2lk}, \ v_x(l,t) = 0.$$

Demak,

$$u(x,t) = \frac{Qx^2}{2kl} + v(x,t).$$

v uchun esa quyidagi masalaga kelamiz:

$$v_t - a^2 v_{xx} = \frac{Qa^2}{kl}, \quad v_x(0,t) = 0, \ v_x(l,t) = 0, \quad v(x,0) = -\frac{Q}{2kl}x^2.$$

Buning yechimini o'z navbatida

$$v(x,t) = \frac{Qa^2}{kl}t + \tilde{v}(x,t)$$

ko'rinishda qidiramiz. Hamma amallarni bajarsak

$$u(x,t) = \frac{Qx^2}{2kl} + \frac{Qa^2}{kl}t - \frac{Ql}{6k} - \frac{2Ql}{\pi^2 k}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}\exp\left\{-\frac{n^2\pi^2a^2}{l^2}t\right\}$$

yechim topiladi.

Qor ostidagi yerning sovish tezligi

Yerning ustida l qalinlikdagi qor yotibdi, havo temperaturasi T_2 va u juda past. Qor ostidagi yer sirtining boshlang'ich temperaturasi T_1 va qor sovug'i ostida u pasaya boshlaydi. Yerdan ma'lum bir miqdordagi issiqlik oqimi bor q bor. Qancha vaqt ichida yer sirtining temperaturasi T_0 gacha tushadi?

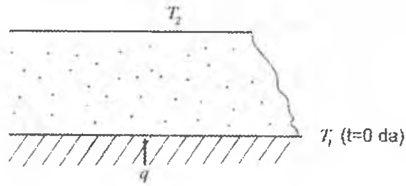
Masalaning qo'yilishi:

$$u_t - a^2 u_{xx} = 0, \quad u_x(0,t) = -\frac{q}{k}, \quad u(l,t) = T_2, \ u(x,0) = T_1 + \frac{x}{l}(T_2 - T_1),$$

$$0 \le x \le l, \qquad 0 \le t < \infty.$$

Bu yerda boshlang'ich temperatura Yer sirtidan qor sirtigacha chiziqli o'zgaradi deyildi. Yechimni

$$u(x,t) = T_2 - \frac{q}{k}(x - l) + v(x,t)$$



VI.2-rasm: Yer sirtining sovishi

ko'rinishda izlaymiz. Bunda $v(x,t)$ uchun quyidagi masalaga ega bo'lamiz:

$$v_t - a^2 v_{xx} = 0, \quad v_x(0,t) = v(l,t) = 0, \quad v(x,0) = (x-l)\left[\frac{q}{k} + \frac{T_2 - T_1}{l}\right].$$

Standart metodlarni qo'llab, quyidagi yechimni olamiz:

$$u(x,t) = T_2 - \frac{q}{k}(x-l) + \frac{8}{\pi^2}\left[T_1 - T_2 - \frac{ql}{k}\right]\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)\pi x}{2l}}{(2n+1)^2}e^{-a^2(2n+1)^2\pi^2 t/(2l)^2}.$$

Tushunarliki, yechimda faqat $n = 0$ had sezilarli bo'lishi mumkin, shuning uchun

$$u(x,t) \simeq T_2 - \frac{q}{k}(x-l) + \frac{8}{\pi^2}\left[T_1 - T_2 - \frac{ql}{k}\right]\cos\left(\frac{\pi x}{2l}\right)e^{-a^2\pi^2 t/(2l)^2}$$

yechim yetarli darajada yaxshi yaqinlashuv bo'ladi. Ko'rinib turibdiki, $u(0,t_0) = T_0$ bo'lishi uchun $(a^2 = k/(c\rho))$

$$t_0 \simeq -\frac{4l^2 c\rho}{\pi^2 k}\ln\left[\frac{\pi^2}{8}\frac{T_0 - T_2 + ql/k}{T_1 - T_2 - ql/k}\right]$$

vaqt kerak.

Manba temperaturaga proporsional

Quyidagi masalani ko'raylik:

$$u_t - u_{xx} = -4u, \quad 0 \le x \le \pi, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x^2 - \pi x.$$

Ikki xil yo'l tutishimiz mumkin. Birinchisi yechimni $u(x,t) = e^{-4t}v(x,t)$ ko'rinishda qidiramiz, shunda $v(x,t)$ uchun masalaning qo'yilishi bizga tanish bo'lgan holga keltiriladi:

$$v_t - v_{xx} = 0, \quad 0 \le x \le \pi, \quad v(0,t) = v(\pi,t) = 0, \quad v(x,0) = x^2 - \pi x.$$

Ikkinchi tomondan birinchi tenglamada bevosita $u(x,t) = X(x)T(t)$ deb olishimiz mumkin, bu holda tenglama

$$XT' - X''T + 4XT = 0$$

ko'rinishga keladi, bu yerda o'zgaruvchilarning ajralishi oydindir, masalan:

$$\frac{T'}{T} + 4 = \frac{X''}{X} = -\lambda^2.$$

X uchun oldin bir necha marta yechgan masalamizning o'zini oldik, T uchun

$$T' + (4 + \lambda^2)T = 0$$

tenglamani olamiz. Chegaraviy shartlarni hisobga olsak

$$X_n(x) = c_1 \sin nx$$

bo'ladi, T uchun esa

$$T(t) = a_n e^{-4t-n^2t}$$

yechimni olamiz. Umumiy yechim

$$u(x,t) = e^{-4t} \sum_n a_n \sin nx e^{-n^2t},$$

$$a_n = \frac{2}{\pi} \int_0^\pi dx(x^2 - \pi x) \sin nx = \frac{4(-1 + (-1)^n)}{\pi n^3}.$$

Shar uchun issiqlik tarqalishi masalasi.

Markazi koordinat boshida, radiusi a, boshlang'ich temperaturasi $f(r,\theta)$ bo'lgan sharning sirt temperaturasi nolga teng qilib ushlanib turilibdi. Shar uchun temperatura taqsimoti masalasini yeching.

Masalaning qo'yilishi:

$$u_t = \Delta u, \quad u(a,\theta,t) = 0, \quad u(r,\theta,0) = v(r,\theta),$$

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le t < \infty.$$

Masalaning sharti bo'yicha unda φ burchakka bog'liqlik yo'q, demak, $u = u(r,\theta,t)$. Yechimni $u(r,\theta,t) = f(r,\theta)T(t)$ ko'rinishda qidiramiz, unda

$$\frac{T'(t)}{T} = \frac{\Delta f(r,\theta)}{f(r,\theta)} = -\lambda$$

munosabatga kelamiz. Demak, birinchidan

$$T(t) = Ce^{-\lambda t},$$

ikkinchidan, masalaning fazoviy qismi uchun Helmholtz tenglamasiga egamiz:

$$\Delta f(r,\theta) + \lambda f(r,\theta) = 0.$$

Albatta, $\lambda \geq 0$, aks holda temperatura o'z-o'zidan o'sib ketishi kerak.

Masala sferik koordinat sistemasida yechilishi kerak:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r,\theta)}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f(r,\theta)}{\partial\theta}\right) + \lambda f(r,\theta) = 0$$

$f(r,\theta) = R(r)\Theta(\theta)$ almashtirish quyidagi munosabatga olib keladi:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = \mu.$$

Bu yerda ikkinchi noma'lum doimiy μ ni kiritishga to'g'ri keldi. Bu noma'lumni aniqlash qiyin emas, buning uchun θ bo'yicha tenglamani yozib olish yetarli:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + \mu\Theta(\theta) = 0.$$

Bu tenglama I-bobdagi §2.7.-paragrafdagi (62)-tenglama bilan bir xil, uni tahlil qilganimizda ko'rsatilgan ediki, $\mu = n(n+1)$, $n = 0,1,2,...$ bo'lishi kerak. Ya'ni olingan tenglama Legendre tenglamasi, uning yechimlari esa bizga ma'lum:

$$\Theta_n(\theta) = P_n(\cos\theta), \qquad n = 0,1,2,...$$

Masalaning faqat radial qismini yechish qoldi:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \left(\lambda r^2 - n(n+1)\right)R(r) = 0. \tag{75}$$

Bu yerda $\sqrt{\lambda}r = x$ va $R(x) = Z(x)/\sqrt{x}$ almshtirishlar bajarilsa

$$x^2 Z''(x) + xZ'(x) + \left(x^2 - \left(n + \frac{1}{2}\right)^2\right)Z(x) = 0$$

tenglama olinadi. Uning yechimi yarim butun indeksli Bessel funksiyasi

$$Z(x) = J_{n+1/2}(x) = J_{n+1/2}(\sqrt{\lambda}r).$$

Shu yerda chegaraviy shart $u(a, \theta, t = 0$ ni qo'llash kerak:

$$J_{n+1/2}(\sqrt{\lambda}a) = 0.$$

Demak, λ son yarim butun Bessel funksiyalarining nollari orqali aniqlanar ekan:

$$\sqrt{\lambda_k}a = \mu_k^{(n+1/2)}, \quad k = 1, 2, 3, ...$$

yoki,

$$\lambda_k = \frac{(\mu_k^{(n+1/2)})^2}{a^2}, \quad k = 1, 2, 3, ...$$

To'liq yechim:

$$u(r, \theta, t) = \sum_{n,k} c_{nk} e^{-\lambda_k t} P_n(\cos\theta) \frac{J_{n+1/2}(\sqrt{\lambda_k}r)}{\sqrt{r}}.$$

Boshlang'ich shartni ishlatish qoldi:

$$u(r, \theta, 0) = \sum_{n,k} c_{nk} P_n(\cos\theta) \frac{J_{n+1/2}(\sqrt{\lambda_k}r)}{\sqrt{r}} = v(\theta, r).$$

Legendre polinomlari uchun ortogonallik va norma shartlari (47)-va (51)-larni, Bessel funksiyalari uchun ortogonallik va norma shartlari (27)- va (28)-larni ishlatish natijasida noma'lum c_{nk} larni aniqlash mumkin[2]:

$$c_{nk} = \frac{2n+1}{\left[aJ'_{n+1/2}(\sqrt{\lambda_k})\right]^2} \int_0^{\pi} d\theta \sin\theta \int_0^{a} dr r^{3/2} v(\theta, r) P_n(\cos\theta) J_{n+1/2}(\sqrt{\lambda_k}r).$$

Mashqlar

6.11-mashq. Yon sirti issiqlik o'tkazmaydigan ingichka sterjen berilgan: $0 \le x \le l$. Sterjenning uchlari issiqlik o'tkazmaydi, boshlang'ich temperaturasi $u_0(x) = A =$ const. $t > 0$ dagi temperatura taqsimotini toping .

6.12-mashq. Yon sirti issiqlik o'tkazmaydigan ingichka sterjen berilgan: $0 \le x \le l$. Sterjenning uchlarida o'zgarmas $u(t, 0) = u_1$; $u(t, l) = u_2$ temperatura ushlab turilgan holda undagi temperatura taqsimotini toping. Boshlang'ich temperatura $u(0, x) = u_0 =$ const.

6.13-mashq. Yon sirti issiqlik o'tkazmaydigan ingichka sterjen berilgan: $0 \le x \le l$. Sterjenning uchlarida o'zgarmas $u(t, 0) = u(t, l) = u_1$ temperatura ushlab turilgan holda undagi temperatura taqsimotini toping. Boshlang'ich temperatura $u(0, x) = Ax(l - x)$, $A =$ const.

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[2] (28)-ni ishlatganda $J_{n+1/2}(\sqrt{\lambda_k}) = 0$ ekanligini unutmang

6.14-mashq. Yon sirti issiqlik o'tkazmaydigan ingichka sterjen berilgan: $0 \le x \le l$. Sterjenning chap uchi issiqlik o'tkazmaydi va o'ng uchi o'zgarmas $u(l,t) = u_2$ temperaturada ushlab turiladi deb undagi temperatura taqsimotini toping. Boshlang'ich temperatura $u(x,0) = \frac{A}{l}x$, $A = \text{const}$.

6.15-mashq. Yon sirti issiqlik o'tkazmaydigan ingichka sterjen berilgan: $0 \le x \le l$. Sterjenning chap uchida $u(0,t) = u_1$ temperatura berilgan, o'ng uchida tashqaridan o'zgarmas q issiqlik oqimi berilib turibdi. Boshlang'ich temperatura $u(x,0) = u_0(x)$. Temperatura taqsimotini toping.

O'zgaruvchilarni ajratish metodi bilan quyidagi masalalarni yeching:

6.16-mashq. $u_t = u_{xx}$, $0 \le x \le l$, $u_x(0,t) = 0$, $u_x(l,t) = 0$, $u(x,0) = x^2 - l^2$.

6.17-mashq. $u_t + u = u_{xx}$, $0 \le x \le l$, $u(0,t) = u(l,t) = 0$, $u(x,0) = 1$.

6.18-mashq. $u_t = u_{xx} - 4u$, $0 < x < \pi$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = x^2 - \pi x$.

6.19-mashq. $u_t - a^2u_{xx} = 0$, $u(0,t) = u(l,t) = 0$, $u(x,0) = Ax$.

6.20-mashq. $u_t - a^2u_{xx} = 0$, $u(0,t) = u_x(l,t) = 0$, $u(x,0) = A(l-x)$.

6.21-mashq. $u_t - a^2u_{xx} = 0$, $u_x(0,t) = u(l,t) = 0$, $u(x,0) = A(l-x)$.

6.22-mashq. $u_t - a^2u_{xx} = 0$, $u_x(0,t) = u_x(l,t) = 0$, $u(x,0) = u_0$.

6.23-mashq. $u_t - a^2u_{xx} = -\beta u$, $u(0,t) = u(l,t) = 0$, $u(x,0) = Ax$.

6.24-mashq. $u_t - a^2u_{xx} = 0$, $u(0,t) = u_1$, $u(l,t) = u_2$, $u(x,0) = 0$.

6.25-mashq. $u_t - a^2u_{xx} = \sin(\pi x/l)$, $u(0,t) = u(l,t) = 0$, $u(x,0) = 0$.

6.26-mashq. O'zgaruvchilarni ajratish metodi bilan IV.2-misolda keltirilgan masalani yeching.

6.27-mashq. $u_t - a^2u_{xx} = 0$, $u_x(0,t) - hu(0,t) = u_x(l,t) = 0$, $u(x,0) = \varphi(x)$.

6.28-mashq. $u_t - a^2u_{xx} = 0$, $u_x(0,t) - hu(0,t) = 0$, $u_x(l,t) + hu(l,t) = 0$, $u(x,0) = \varphi(x)$.

VII BOB. ELLIPTIK TENGLAMALAR UCHUN CHEGARAVIY MASALALAR

§1. Chegaraviy masalalarning qo'yilishi

Elliptik tenglamalarning ichida eng ko'p uchraydiganlari quyidagilardir:

$$\begin{aligned} &\Delta u = 0, && \text{Laplace tenglamasi;} \\ &\Delta u = -f, && \text{Poisson tenglamasi;} \qquad (1) \\ &\Delta u + k^2 u = 0, && \text{Helmholtz tenglamasi.} \end{aligned}$$

Bu tenglamalar to'lqin, issiqlik va modda tarqalishi jarayonlarining statik va statsionar hollariga mos keladi. Undan tashqari, ular elektrostatika, gidrostatika va magnitostatika masalalarida ko'p uchraydi.

Odatda bunday tenglamalar uchun chegaraviy masalalar quyidagicha qo'yiladi :

S chegarali V sohada o'rinli bo'lgan (1)- tenglamaning shunday yechimi $u(x, y, z)$ topilsinki, u shu chegarada quyidagi shartlarning biriga bo'ysunsin:

$u_{|S} = f_1 -$ ***birinchi chegaraviy masala - Dirichlet[1] masalasi***; (2)

$\frac{\partial u}{\partial \vec{n}}|_S = f_2 -$ ***ikkinchi chegaraviy masala - Neumann masalasi***; (3)

$\frac{\partial u}{\partial \vec{n}}|_S - h(u - f_3) = 0 -$ ***uchinchi chegaraviy masala.*** (4)

Chegaraviy shartlarning eng umumiy formasi:

$$\left(\beta \frac{\partial u}{\partial n} + \alpha u\right)_S = f, \ \alpha + \beta > 0, \alpha \geq 0, \beta \geq 0.$$

Bu yerda f_1, f_2, f_3 - berilgan funksiyalar. Chegaraviy masalalar ***ichki*** va ***tashqi*** masalalarga bo'linadi. Ichki masalaning yechimi biror bir cheklangan G sohaning ichida izlanadi, tashqi masalaning yechimi qandaydir cheklangan sohaga tashqi bo'lgan G sohada izlanadi. Ikkinchi holda yechimdan cheksizlikda nolga intilish talab qilinadi: $u \to 0$, $r \to \infty$ (Helmgholtz tenglamasidan tashqari).

u funksiya yopiq soha G da ***garmonik*** deyiladi, qachonki bu sohada $\Delta\, u\ =\ 0$ bo'lsa, u ikkinchi tartibli hosilalari bilan uzliksiz, soha chegarasida uzliksiz bo'lsa.

§2. Chegaraviy masala yechimining yagonaligi

Poisson tenglamasi uchun birinchi ichki masala - Dirichlet masalasidan boshlaylik:

$$\Delta u = f, \quad u|_S = u_0.$$

Faraz qilaylik, bu masalaning yechimi ikkita bo'lsin: u_1 и u_2. Bu holda $\tilde{u} = u_1 - u_2$ uchun

$$\Delta\tilde{u} = 0, \ \ \tilde{u}|_S = 0 \qquad (5)$$

masalaga ega bo'lamiz. Quyidagi oddiy mulohazaga qaraylik:

$$\int\limits_G dV(\nabla\tilde{u})^2 = \int\limits_G dV\nabla(\tilde{u}\nabla\tilde{u}) - \int\limits_G dV\tilde{u}\Delta\tilde{u} = \oint\limits_S \tilde{u}\frac{\partial\tilde{u}}{\partial n}dS = 0. \qquad (6)$$

Oxirgi tenglikka o'tganda biz (5)-dan foydalandik. $(\nabla\tilde{u})^2 \geq 0$ bo'lishigina mumkin, undan olingan integral (integrallash sohasi ixtiyoriy) nolga teng ekan birdan-bir imkoniyat:

$$(\nabla\tilde{u})^2 = 0 \to \tilde{u} = \text{const} \to \tilde{u} = 0, \, G.$$

Demak, ichki Dirichlet masalasining yechimi yagona ekan.

Neumann masalasiga kelaylik:

$$\Delta u = f, \quad \frac{\partial u}{\partial n}\Big|_S = u_1.$$

Bu holda ham (6)-tenglik o'rinli bo'ladi, demak, yana $(\nabla\tilde{u})^2 = 0$ bo'lishi kerak. Ammo bu galda

$$(\nabla\tilde{u})^2 = 0 \to \tilde{u} = \text{const} \to u_1 - u_2 = \text{const} \qquad (7)$$

debgina yoza olamiz. Demak, Neumanning ichki masalasini yechimiga ixtiyoriy o'zgarmas sonni qo'shib qo'yishimiz mumkin ekan - yechim yagona emas.

Ammo Neumanning tashqi masalasi yagona yechimga ega, chunki bu holda cheksizlikda yechimning nolga intilishi kerakligi sharti $u \to 0$, $r \to \infty$ (7)-dagi constantaning nolga teng bo'lishiga olib keladi: const $= 0$.

Neumann masalasi uchun yana bir shartga egamiz (Gauss teoremasidan kelib chiqadi):

$$\oint_S \frac{\partial u}{\partial n} dS = \int_G dV \Delta u = \int_G dV f. \tag{8}$$

Ko'rinib turibdiki, Neumann masalasi uchun chegaraviy shartlar ixtiyoriy bo'lishi mumkin emas, ular (8)-shartga bo'ysunishi kerak. Masalan, Laplace tenglamasi uchun

$$\oint_S \frac{\partial u}{\partial n} dS = \oint_S u_1 dS = 0 \tag{9}$$

bo'lishi kerak, aks holda chegaraviy masala noto'g'ri qo'yilgan bo'ladi.

Helmholtz tenglamasiga o'taylik. Agar cheksiz fazodagi to'lqin tenglamasida yechimning vaqtga bog'liqligini monoxromatik desak tenglama bir jinslimas Helmholtz tenglamasiga aylanadi:

$$\frac{\partial^2 u}{c^2 \partial t^2} - \Delta u = -4\pi\rho \quad \Rightarrow \quad u \sim \exp(\pm ick_0 t) \quad \Rightarrow \quad \Delta u + k_0^2 u = 4\pi\rho.$$

Muammo shundan iboratki, $\sin(kr)/r$ funksiya bir jinsli Helmholtz tenglamasining yechimidir. Bu degani, Helmholtz tenglamasining yechimlariga $u \to 0$, $r \to \infty$ shartning qo'yilishi ularni bir qiymatli aniqlab bera olmaydi.

7.1-mashq. VIII.8-mashqning natijasidan foydalanib, $\sin(kr)/r$ funksiya $\Delta u + k^2 u = 0$ tenglamaning yechimi ekanligini ko'rsating.

Yechimning yagonaligini ta'minlash uchun ular ***Sommerfeldning***[2] ***nurlanish shartlari*** deyiladigan quyidagi qo'shimcha shartlarga bo'ysundiriladi:

$$u(x,y,z) = O(1/r), \quad \frac{\partial u}{\partial r} - iku = o(1/r), \quad r \to \infty$$

- tarqaluvchi to'lqin;

$$u(x,y,z) = O(1/r), \quad \frac{\partial u}{\partial r} + iku = o(1/r), \quad r \to \infty$$

- yig'iluvchi to'lqin[3]. Bu shartlarning kelib chiqishi quyidagicha. Faraz qilaylik, uzoqdan bir chegaralangan sohaga (jismga, nishonga) yassi to'lqin $e^{i\mathbf{k}\cdot\mathbf{r}}$ tushsin, shu nishondan akslanib tarqalgan to'lqin yetarli darajadagi uzoq masofada sferik to'lqin ko'rinishiga ega bo'ladi:

$$f\left(\frac{\mathbf{r}}{r}\right)\frac{e^{ikr}}{r} + o(1/r).$$

[2] Arnold Sommerfeld (1868-1951) - nemis fizigi. Rus tilida - Зоммерфельд.

[3] $O(x)$ va $o(x)$ belgilarning ma'nosi quyidagicha: $O(x)/x \to A < \infty$, $x \to \infty$ va $o(x)/x \to 0$, $x \to \infty$.

Paydo bo'lgan $f(\mathbf{r}/r)$ funksiya sochilish amplitudasi deyiladi. Ko'rinib turibdiki, ushbu tarqalgan to'lqin yagona bo'lishi uchun Sommerfeldning birinchi sharti bajarilishi kerak.

§3. Doira uchun ichki va tashqi chegaraviy masalalar

Doira uchun ichki va tashqi Dirichlet masalalarini ko'rib chiqaylik:

$$\Delta u = 0, \ a \text{ radiusli doiraning ichida}, \quad u|_{\rho=R} = f.$$

Bu yerda $u = u(x,y)$, $f = f(x,y)$. Doira uchun masalani qutb koordinat sistemasida yechish qulaydir. Laplace tenglamasining qutb sistemasidagi ko'rinishi:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{10}$$

Eslatib ketamiz $\rho^2 = x^2+y^2$, $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Yechimni Fourier metodi bo'yicha qidiramiz:

$$u(\rho,\varphi) = R(\rho)\Phi(\varphi). \tag{11}$$

Buni (10)-ga olib borib qo'ysak,

$$\frac{\Phi(\varphi)}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) + \frac{R(\rho)}{\rho^2}\frac{d^2\Phi(\varphi)}{d\varphi^2} = 0 \tag{12}$$

tenglamani olamiz va uni quyidagi ko'rinishga keltiramiz:

$$\frac{\rho}{R(\rho)}\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = \lambda^2 \tag{13}$$

Chap tomondagi λ^2 konstanta yuqorida ko'p marta muhokama qilingan mulohazalar asosida paydo bo'ldi.

Shu bilan (10)-xususiy hosilali tenglamani ikkita to'liq hosilali tenglamalar sistemasiga keltirdik:

$$\begin{aligned} &\rho\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) - \lambda^2 R(\rho) = 0, \\ &\frac{d^2\Phi(\varphi)}{d\varphi^2} + \lambda^2\Phi(\varphi) = 0. \end{aligned} \tag{14}$$

Bu sistemadagi tenglamalarning ikkinchisining yechimi

$$\Phi(\varphi) = A\cos(\lambda\varphi) + B\sin(\lambda\varphi). \tag{15}$$

Masalamizning yechimi bir qiymatli bo'lishi uchun

$$u(\rho, \varphi + 2\pi) = u(\rho, \varphi)$$

bo'lishi kerak, ya'ni,

$$\Phi(\rho, \varphi + 2\pi) = \Phi(\rho, \varphi).$$

Demak,

$$\lambda = n, \quad n = 0, \pm 1, \pm 2, ...$$

bo'lishi kerak. (14)-sistemaning ikkinchi tenglamasi $\lambda = n$ butun sonlarga bog'liq bo'lgan quyidagi ko'rinishli yechimlarga ega bo'lib chiqdi:

$$\Phi_n(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi). \tag{16}$$

Masalaning radial qismiga kelaylik:

$$\rho \frac{d}{d\rho} \left(\rho \frac{dR(\rho)}{d\rho} \right) - n^2 R(\rho) = 0. \tag{17}$$

Uning yechimini $R = \rho^\mu$ ko'rinishda qidirsak $\mu = \pm n$ ekanligini topamiz. Demak, (17)-tenglamaning eng umumiy yechimi

$$R_n(\rho) = C_n \rho^n + D_n \rho^{-n} + E \ln \rho$$

ko'rinishga ega bo'lishi kerak. Bu formuladagi oxirgi had $n = 0$ holga to'g'ri keladi. Ichki masala haqida gap ketayotgan bo'lsa, $D_n = 0$, $E = 0$ ($\rho = 0$ da yechimning cheklanganlik shartidan), tashqi masala haqida gap ketayotgan bo'lsa, $C_n = 0$, $E = 0$ ($\rho = \infty$ da yechimning cheklanganligi shartidan). Topilgan yechimlarning xususiy sistemalarini yozib olaylik:

$$\begin{aligned} u_n(\rho, \varphi) &= \rho^n \left(A_n \cos(n\varphi) + B_n \sin(n\varphi) \right), \quad \rho \leq a; \\ u_n(\rho, \varphi) &= \rho^{-n} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi) \right), \quad \rho \geq a. \end{aligned} \tag{18}$$

Umumiy yechim mana shu xususiy yechimlarning chiziqli superpozitsiyasidan iborat:

$$\begin{aligned} u(\rho, \varphi) &= \sum_{n=0}^{\infty} \rho^n \left(A_n \cos(n\varphi) + B_n \sin(n\varphi) \right), \quad \rho \leq a; \\ u(\rho, \varphi) &= \sum_{n=0}^{\infty} \rho^{-n} \left(A_n \cos(n\varphi) + B_n \sin(n\varphi) \right), \quad \rho \geq a. \end{aligned} \tag{19}$$

A_n va B_n koeffisientlarni chegaraviy shartlardan topamiz:

$$u(a,\varphi) = \sum_{n=0}^{\infty} a^{\pm n}\left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) = f(\varphi). \tag{20}$$

$f(\varphi)$ ni Fourier qatoriga yoyaylik:

$$f(\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi)) \tag{21}$$

Bu yerda

$$\alpha_0 = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)d\varphi,\ \alpha_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos(n\varphi)d\varphi,\ \beta_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin(n\varphi)d\varphi. \tag{22}$$

(19)- va (21)-formulalarni solishtirsak, ichki masala uchun:

$$A_0 = \frac{\alpha_0}{2},\quad A_n = \frac{\alpha_n}{a^n},\quad B_n = \frac{\beta_n}{a^n},\qquad n = 1, 2, 3, ... \tag{23}$$

va tashqi masala uchun:

$$A_0 = \frac{\alpha_0}{2},\quad A_n = \alpha_n a^n,\quad B_n = \beta_n a^n,\qquad n = 1, 2, 3, ... \tag{24}$$

ekanligini topamiz. Shularni hisobga olib, yechimlarni yana bir marta yozib olaylik:

$$\begin{aligned} u(\rho,\varphi) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{a}\right)^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi)),\ \ \rho \le a; \\ u(\rho,\varphi) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a}{\rho}\right)^n (\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi)),\ \ \rho \ge a. \end{aligned} \tag{25}$$

7.1-misol. $\Delta u = 0$, $u|_S = A\cos\varphi$ masalani $\rho = a$ doiraning ichida yeching.

$f(\varphi) = A\cos\varphi$ funksiyani Fourier qatoriga yoysak faqat $\alpha_1 = A$, va boshqa hamma koeffisientlar uchun $\alpha_n = 0$, $\beta_n = 0$ ekanligini topamiz. Demak,

$$u(\rho,\varphi) = \frac{\rho}{a}\cos\varphi = \frac{x}{a}.$$

Neumann masalasiga kelaylik:

$$\Delta u = 0, \qquad \left.\frac{\partial u}{\partial \rho}\right|_{\rho=a} = f.$$

Tenglamaning yechimi o‘sha (18)-formula orqali aniqlanadi, bajarilishi shart bo‘lgan (9)-formula doira uchun

$$\int_0^{2\pi} d\varphi f(a,\varphi) = 0 \tag{26}$$

ko‘rinishga ega bo‘ladi. Bu shartni qanoatlantirmaydigan masala to‘g‘ri qo‘yilmagan masala bo‘ladi, uning yagona yechimi mavjud emas. Ichki masala uchun chegaraviy shart

$$f(a,\varphi) = \sum_{n=1}^{\infty} na^{n-1}\left(A_n \cos(n\varphi) + B_n \sin(n\varphi)\right) = A_1 \cos\varphi + B_1 \sin\varphi +$$

$$+2aA_2\cos(2\varphi) + 2aB_2\sin(2\varphi) + 3a^2A_3\cos(3\varphi) + 3a^2B_3\sin(3\varphi) + \cdots$$

ko‘rinishga ega bo‘lgani uchun A_0 koeffisientni chegaraviy shartdan aniqlab bo‘lmaydi. Bu - (7)-formuladan keyin muhokama qilingan noaniqlikning o‘zidir. Oxirgi formulani (21)-formula bilan solishtirsak, Dirichlet masalasidagi (23)-formulaning o‘rniga

$$A_n = \frac{\alpha_n}{na^{n-1}}, \quad B_n = \frac{\beta_n}{na^{n-1}}, \quad n = 1, 2, 3, ...$$

formulalarni olamiz. Umumiy yechim

$$u(\rho,\varphi) = a\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{a}\right)^n\left(\alpha_n\cos(n\varphi) + \beta_n\sin(n\varphi)\right) + C, \quad \rho \le a$$

ko‘rinishga ega bo‘ladi, bu yerda C- noaniq konstanta.

Tashqi Neumann masalasida bunday noaniqlik yo‘q, $\lim_{\rho\to\infty} u = 0$ sharti $A_0 = 0$ bo‘lishiga olib keladi. Yechim

$$u(\rho,\varphi) = -a\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{\rho}\right)^n\left(\alpha_n\cos(n\varphi) + \beta_n\sin(n\varphi)\right), \quad \rho \ge a$$

ko‘rinishga ega bo‘ladi, bu yerda (24)-formulaning o‘rniga

$$A_n = -\frac{a^{n+1}}{n}\alpha_n, \quad B_n = -\frac{a^{n+1}}{n}\beta_n, \quad n = 1, 2, 3, ...$$

ifodalarni ishlatdik. α_n va β_n lar (22)-Fourier formulalaridan topiladi ($n = 0$ dan tashqari).

Mashqlar.

7.2-mashq. Ichki va tashqi Dirichlet masalalarining yechimlarini quyidagicha birlashtirib:

$$u(\rho,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} t^n \left(\alpha_n \cos(n\varphi) + \beta_n \sin(n\varphi)\right); \quad t = \begin{cases} \rho/a, & \text{ichki masala;} \\ a/\rho, & \text{tashqi masala,} \end{cases}$$

va $\cos n\varphi \cos(n\psi) + \sin(n\varphi)\sin(n\psi) = \cos(n(\varphi - \psi))$ formuladan foydalanib quyidagi, Poisson formulalarini keltirib chiqaring:

$$u(\rho,\varphi) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\psi)\frac{a^2-\rho^2}{\rho^2 - 2a\rho\cos(\varphi-\psi) + a^2}d\psi, & \rho < a; \\ f(\varphi), & \rho = a. \end{cases} \tag{27}$$

$$u(\rho,\varphi) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\psi)\frac{\rho^2-a^2}{\rho^2 - 2a\rho\cos(\varphi-\psi) + a^2}d\psi, & \rho > a; \\ f(\varphi), & \rho = a. \end{cases} \tag{28}$$

7.3-mashq. Birlik aylana ichida quyidagi Dirichlet masalasini yeching:

$$\Delta u = 0, \quad u(1,\varphi) = \cos^2\varphi.$$

7.4-mashq. Birlik aylana ichida quyidagi Dirichlet masalasini yeching:

$$\Delta u = 0, \quad u(1,\varphi) = \cos^4\varphi.$$

7.5-mashq. Birlik aylana ichida quyidagi Dirichlet masalasini yeching:

$$\Delta u = 0, \quad u(1,\varphi) = \sin^3\varphi.$$

7.6-mashq. Birlik aylana ichida quyidagi Dirichlet masalasini yeching:

$$\Delta u = 0, \quad u(1,\varphi) = \sin^4\varphi + \cos^6\varphi.$$

7.7-mashq. R radiusli aylana ichida quyidagi Neumann masalasini yeching:

$$\Delta u = 0, \quad \frac{\partial u}{\partial r}|_{r=R} = A\cos\varphi.$$

(26)-bajarilganmi yo'qmi?

7.8-mashq. R radiusli aylana ichida quyidagi Neumann masalasini yeching:

$$\Delta u = 0, \quad \frac{\partial u}{\partial r}|_{r=R} = A\cos 2\varphi.$$

(26)-bajarilganmi yo'qmi?

7.9-mashq. R radiusli aylana ichida quyidagi Neumann masalasini yeching:

$$\Delta u = 0, \quad \frac{\partial u}{\partial r}|_{r=R} = \sin^3\varphi.$$

(26)-bajarilganmi yo'qmi?

§4. Helmholtz tenglamasi – doira uchun chegaraviy masala

Doira uchun quyidagi xususiy qiymatlar masalasini ko'raylik:

$$-\Delta u = \lambda u, \qquad u\Big|_{\rho=a} = 0. \tag{29}$$

Qutb koordinatlarida masala quyidagicha ko'rinishga ega:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\rho^2} + \lambda u = 0, \qquad u(a,\varphi) = 0.$$

Noma'lum funksiyani $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ ko'rinishda qidiramiz, natijada

$$\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \qquad \Phi(\varphi) = \Phi(\varphi + 2\pi)$$

va

$$\rho^2 R''(\rho) + \rho R'(\rho) + (\lambda\rho^2 - \mu)R(\rho) = 0, \qquad R(a) = 0$$

masalalarga egamiz. Birinchi masalaning yechimi yuqorida muhokama qilingan ((14)-(16) formulalarga qarang), bu yerda u yechimning boshqa formasi qulayroqdir:

$$\Phi_n(\varphi) = \frac{1}{\sqrt{2\pi}}e^{in\varphi}, \qquad \mu = n^2, \quad n = 0, 1, 2, ...$$

Ikkinchi tenglama Bessel tenglamasidir, uning yechimi $J_n(\sqrt{\lambda}\rho)$. Chegaraviy shart $J_n(\sqrt{\lambda}a) = 0$ xususiy qiymatlarni beradi: $\sqrt{\lambda}a = \mu_l^{(n)},$, $l = 1, 2, ...,$ bu yerda $\mu_l^{(n)}$ - Bessel funksiyasi J_n ning nollari: $J_n(\mu_l^{(n)}) = 0$, $l = 1, 2, ...$ Masalan,

$$\mu_1^{(0)} = 2,4048...; \quad \mu_2^{(0)} = 5.5201...; \quad \mu_3^{(0)} = 8.6537...,$$

$$\mu_1^{(1)} = 3,8317...; \quad \mu_2^{(1)} = 7,0156...; \quad \mu_3^{(1)} = 10,1735...$$

va h.k.

Shu bilan ikkinchi masalaning yechimlari ham topildi:

$$R_{nl}(\rho) = c_{nl}J_n\left(\mu_l^{(n)}\frac{\rho}{a}\right).$$

$\{\Phi_n(\varphi)\}$ funksiyalar $(0, 2\pi)$ intervalda to'liq va ortonormal sistemani tashkil qiladi, $R_{nl}(\rho)$ funksiyalarni ortonormal sistemaga aylantirish uchun c_{nk} koeffisientlarni quyidagicha tanlab olish kerak:

$$\frac{1}{c_{nl}} = \sqrt{\int_0^a J_n^2\left(\mu_l^{(n)}\frac{\rho}{a}\right)\rho d\rho} = \frac{a}{\sqrt{2}}\left|J_n'(\mu_l^{(n)})\right|.$$

Bu formulani keltirib chiqarishda (28)-ishlatildi. Shu bilan (29)-masalaning normasi birga keltirilgan yechimlari sistemasi topildi:

$$u_{nl}(\rho, \varphi) = \frac{J_n\left(\mu_l^{(n)}\frac{\rho}{a}\right) e^{in\varphi}}{\sqrt{\pi}a\left|J_n'(\mu_l^{(n)})\right|}, \qquad \lambda_{nl} = \frac{\mu_l^{(n)}}{a^2}, \quad n = 0, 1, 2, ...; \; l = 1, 2, ... \tag{30}$$

Bu funksiyalar to'plami to'liq ortonormal sistemani tashkil qiladi: $(u_{nk}, u_{ml}) = \delta_{nm}\delta_{kl}$.

Yechilgan masalaning fizik ma'nosiga kelaylik. Quyidagi ikki o'lchamli to'lqin tenglamasi

$$u_{tt} - c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = u_{tt} - c^2\Delta u = 0$$

ning doira ichidagi statsionar yechimini topish kerak bo'lsin:

$$u(t, x, y) = e^{-i\omega t}\tilde{u}(x, y).$$

Bu holda $\tilde{u}$ uchun Helmholtz tenglamasini olamiz:

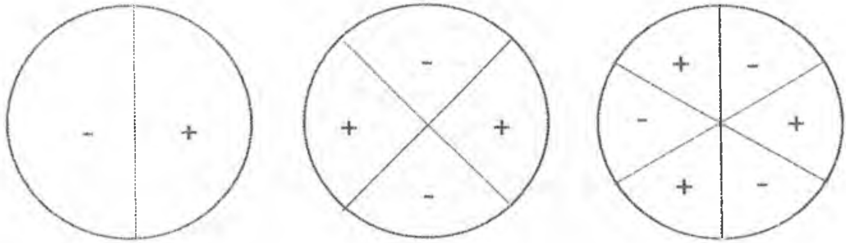
$$\Delta\tilde{u} + k^2\tilde{u} = 0, \qquad k^2 = \frac{\omega^2}{c^2}.$$

Ko'rilyotgan masala - chetlari mahkam biriktirilgan membrananing tebranishlari masalasi, topilgan yechimlar sistemasi (30) - radiusi a bo'lgan membranadagi turg'un to'lqinlar, garmonikalar. Umumiy yechim ($e^{-i\omega t}$ ning haqiqiy va mavhum qismlarini alohida yechim sifatida olamiz)

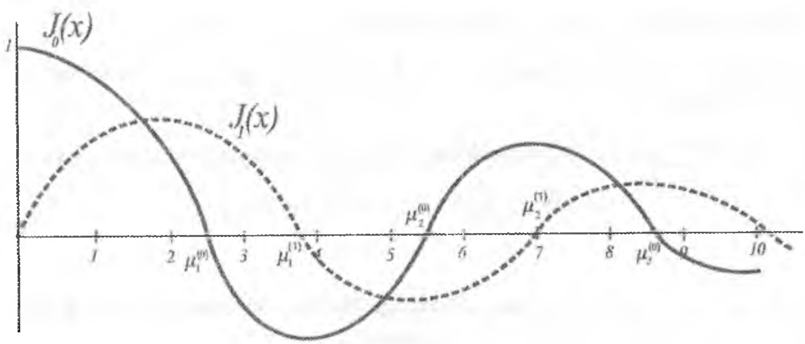
$$u(x, y, t) = \sum_{n,l}^{\infty}(a_{nl}\cos(\omega_{nl}t) + b_{nl}\sin(\omega_{nl}t))\frac{J_n\left(\mu_l^{(n)}\frac{\rho}{a}\right) e^{in\varphi}}{\sqrt{\pi}a\left|J_n'(\mu_l^{(n)})\right|}, \quad \omega_{nl} = \frac{c}{a}\mu_l^{(n)} \tag{31}$$

ko'rinishga ega. a_{nl} va b_{nl} koeffisientlar boshlang'ich shartlardan topiladi.

Garmonikalar (30)-formulada kompleks ko'rinishda berilgan, tebranishlarni o'rganish uchun yechimning haqiqiy qismini olamiz: Reu. VII.1-rasmda $\cos(n\varphi)$ ko'paytuvchi bilan bog'liq bo'lgan manzara $n = 1, 2, 3$ hollar uchun ko'rsatilgan. "+" ishorasi kosinusning musbat bo'lgan sohasi, "-" ishorasi kosinusning manfiy bo'lgan sohasi. Agar membrana sirti muvozanat holatida shu varaq sirti bilan mos tushsa "+" deb belgilangan sohalarda membrana sirti varaq sirtidan yuqoriga ko'tarilgan bo'ladi, "-" ishorali sohada esa teskari - pastga tushgan bo'ladi. To'g'ri chiziqlar sirt tebranishi amplitudasi



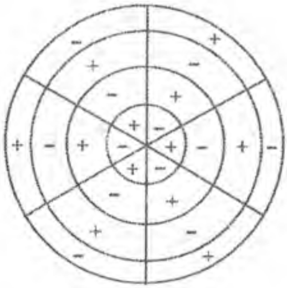
VII.1-rasm: Membrana tebranishlariga oid



VII.2-rasm: J_0 va J_1 ning grafiklari

nolga teng bo'lgan sohalar. Ammo bu hali hammasi emas. J_0 va J_1 Bessel funksiyalarining grafiklari VII.2-rasmda ko'rsatilgan. Ko'rinib turibdiki, Bessel funksiyalari markazdan tarqalayotgan va markazdan uzoqlashgan sari amplitudasi kamaya borayotgan turg'un to'lqinlarni ifodalaydi. Shu VII.1- va VII.2-rasmlarni (garmonikalarning yuqori hadlariga mos keluvchi rasmlarni ham) o'zaro ko'paytirsak, membrananing garmonikalari haqida tasavvur olgan bo'lamiz. VII.3-rasmda $J_3(\mu_4^{(3)}\rho/a)\cos(3\varphi)$ turg'un to'lqinga mos keluvchi manzara ko'rsatilgan. "+" va "-" ishoralarning va to'g'ri chiziqlarning ma'nosi yuqorida tushuntirilgandek. Membrana ω_{34} chastota bilan tebranadi, ya'ni, rasmda ko'ratilgan manzarada "+" va "-" ishoralar ω_{34} chastota bilan o'rin almashinib turadi.

7.2-misol. Radiusi a bo'lgan va cheti mahkamlangan membrana uchun



VII.3-rasm: $J_3(\mu_4^{(3)}\rho/a)\cos(3\varphi)$ turg'un to'lqinga mos keluvchi manzara

tebranishlar masalasi quyidagi boshlang'ich shartlarda yechilsin:

1. Boshlang'ich chetlanish $u(\rho,0) = AJ_0(\mu_l^{(0)}\rho/a)$ ga teng, boshlang'ich tezlik nolga teng.

2. Boshlang'ich chetlanish va boshlang'ich tezliklar faqat ρ ning funksiyasi:

$$u(\rho,0) = f(\rho), \qquad u_t(\rho,0) = F(\rho).$$

Yechim.

1. Boshlang'ich tezlikning nolga tengligi (31)-formulada $b_{nl} = 0$ ga olib keladi. a_{nl} koeffisientlar quyidagicha aniqlanadi:

$$a_{nk} = A\int_0^a d\rho\rho\int_0^{2\pi} d\varphi J_0(\mu_l^{(0)}\rho/a)\frac{J_n\left(\mu_k^{(n)}\frac{\rho}{a}\right)e^{in\varphi}}{\sqrt{\pi}a\left|J_n'(\mu_l^{(n)})\right|} = A\delta_{n,0}\delta_{kl}.$$

Demak, yechim:

$$u(\rho,t) = A\cos\frac{c\mu_l^{(0)}t}{a}J_0\left(\mu_l^{(0)}\frac{\rho}{a}\right).$$

2. Bu holda (31)-formuladagi koeffisientlar quyidagicha aniqlanadi:

$$a_{nl} = \int d^2\rho f(\rho)u_{nl}(\rho,\varphi) = \frac{2}{a^2J_1^2(\mu_l^{(0)})}\int_0^a d\rho\,\rho f(\rho)J_0\left(\mu_l^{(0)}\frac{\rho}{a}\right)\delta_{n,0};$$

$$b_{nl} = \frac{1}{\omega_{nl}}\int d^2\rho F(\rho)u_{nl}(\rho,\varphi) = \frac{2}{a^2\omega_{nl}J_1^2(\mu_l^{(0)})}\int_0^a d\rho\,\rho F(\rho)J_0\left(\mu_l^{(0)}\frac{\rho}{a}\right)\delta_{n,0}.$$

Masalaning yechimi:

$$u(\rho, t) = \sum_{l=1}^{\infty} \left(a_{0l}\cos(\omega_{0l}t) + b_{0l}\sin(\omega_{0l}t)\right) J_0\left(\mu_l^{(0)}\frac{\rho}{a}\right).$$

§5. Helmholtz tenglamasi – tortburchak uchun chegaraviy masala.

Helmholtz tenglamasini yechishga oid bir misolni ko'rib chiqaylik:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u, \quad u|_L = 0,$$
$$0 \le x \le l, \qquad 0 \le y \le m. \tag{32}$$

Bu (x, y) tekislikda to'rtburchak ichida berilgan Dirichlet masalasi. Tortburchak chegarasi L da noma'lum funksiya nolga teng. Shu vaqtning o'zida bu masala xususiy qiymatlar masalasidir, chunki bu masalaning yechimi har qanday λ uchun ham mavjud bo'lavermaydi. Masalaning yechimini

$$u(x, y) = X(x)Y(y)$$

ko'rinishda qidiramiz. Bu bizga quyidagini beradi:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda. \tag{33}$$

Bu tenglamaning mumkin bo'lgan ko'rinishlaridan biri:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \lambda = -\mu. \tag{34}$$

Bu yerda biz yangi noma'lum doimiy μ ni kiritdik. Natijada, biz ikkita xususiy qiymatlar masalasini olamiz:

$$X''(x) + \mu X(x) = 0, \quad X(0) = X(l) = 0;$$
$$Y''(y) + \nu Y(y) = 0, \quad Y(0) = Y(m) = 0, \ \ \nu = \lambda - \mu. \tag{35}$$

Bunday masalalarni yechishda tajribamiz bor. Birinchi satrdagi maslaning yechimi

$$\mu = \left(\frac{k\pi}{l}\right)^2, \quad k = 1, 2, 3, ...$$

qiymatlar uchungina, ikkinchi satrqadi masalaning yechimi esa

$$\nu = \left(\frac{j\pi}{m}\right)^2, \quad j = 1, 2, 3, ...$$

qiymatlar uchungina mavjuddir. Demak, λ ham ixtiyoriy emas:

$$\lambda_{kj} = \pi^2\left(\frac{k^2}{l^2} + \frac{j^2}{m^2}\right).$$

Yechimlarni ham keltiraylik:

$$X_k(x) = \sqrt{\frac{2}{l}}\sin\frac{k\pi x}{l}, \quad Y_j(y) = \sqrt{\frac{2}{m}}\sin\frac{j\pi y}{m}.$$

Umumiy yechim:

$$u(x, y) = \sum_{k,j=1}^{\infty} X_k(x)Y_j(y) = \sum_{k,j=1}^{\infty} \frac{2}{\sqrt{lm}}\sin\frac{k\pi x}{l}\sin\frac{j\pi y}{m}.$$

$\{X_k\}$ va $\{Y_j\}$ to'plamlar to'liq sistemalarni hosil qiladi, demak, masalaning boshqa xususiy qiymatlari va yechimlari yo'q. λ_{kj} xususiy qiymatlar karrali bo'lishi mumkin, uning karraliliga

$$\frac{k_1^2}{l^2} + \frac{j_1^2}{m^2} = \frac{k_2^2}{l^2} + \frac{j_2^2}{m^2}$$

tenglamaning butun sonlardagi yechimlarining soniga bog'liq bo'ladi. Masalan, $l = m = 1$ hol uchun λ_{55} ning karraligi uchga teng: $\lambda_{55} = \lambda_{71} = \lambda_{17}$.

7.10-mashq. Laplace tenglamasi $\Delta u = 0$ ning $0 \le x \le a$, $0 \le y \le b$ to'rburchak ichidagi yechimini toping. $u(x, y)$ shu to'rtburchak chegaralarida quyidagi qiymatlarni qabul qiladi:

$$u(0, y) = A\sin\frac{\pi y}{b}, \quad u(a, y) = 0, \quad u(x, 0) = B\sin\frac{\pi x}{a}, \quad u(x, b) = 0.$$

§6. Shar uchun Dirichlet va Neumann masalalari.

Radiusi a bo'lgan shar uchun ichki va tashqi Dirichlet va Neumann masalalarini yechaylik. Masalalarning qo'yilishi quyidagicha. Ichki masala:

$$\Delta u = 0, \quad r < a; \quad r = a \quad \text{da} \quad \text{D}) : u = f; \quad \text{N}) : \frac{\partial u}{\partial r} = f.$$

Tashqi masala:

$$\Delta u = 0, \quad r > a; \quad r = a \quad \text{da} \quad \text{D}) : u = f; \quad \text{N}) : \frac{\partial u}{\partial r} = f; \ \lim_{r\to\infty} u = 0.$$

Ko'rilyapgan sohada $u \in C^2$, chegarada esa $u \in C$ - Dirichlet masalasi uchun va $u \in C^1$ - Neumann masalasi uchun. Neumann masalasi uchun

$$\int dSf = a^2 \int d\Omega f(\theta, \varphi) = 0$$

bo'lishi ham kerak.

Haqiqatda §2.7.-paragrafdagi (72)-formula Laplace tenglamasining sferik sistemadagi yechimlarini beradi, ammo u yerda chegaraviy shartlar muhokama qilinmagan edi. Tushunarliki, ichki masala haqida gap ketayotgan bo'lsa, (72)-ning birinchi (r^n ga proporsional bo'lgan) qismini olishimiz kerak, tashqi masala haqida gap ketayotgan bo'lsa (72)-ning ikkinchi (r^{-n-1} ga proporsional bo'lgan) qismini olishimiz lozim.

Chegaraviy shartni ifodalaydigan funksiya $f = f(\theta, \varphi)$ ni sferik funksiyalar bo'yicha qatorga yoyamiz:

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} A_{nm} Y_n^m(\theta, \varphi), \qquad A_{nm} = \int d\Omega \, Y_n^{m*}(\theta, \varphi) f(\theta, \varphi).$$

Aytilganlarni (shu jumladan, §3.-paragrafdagi chegaraviy shartlarning muhokamasini) hisobga olib, quyidagi natijalarga kelish mumkin:

Dirichlet ichki masalasining yechimi:

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \sum_{m=-n}^{m=n} A_{nm} Y_n^m(\theta, \varphi), \quad r < a;$$

Neumann ichki masalasining yechimi:

$$u(r, \theta, \varphi) = \sum_{n=1}^{\infty} \frac{a}{n} \left(\frac{r}{a}\right)^n \sum_{m=-n}^{m=n} A_{nm} Y_n^m(\theta, \varphi) + C, \quad r < a;$$

Dirichlet tashqi masalasining yechimi:

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=-n}^{m=n} A_{nm} Y_n^m(\theta, \varphi), \quad r > a;$$

Neumann tashqi masalasining yechimi:

$$u(r, \theta, \varphi) = -\sum_{n=0}^{\infty} \frac{a}{n+1} \left(\frac{a}{r}\right)^{n+1} \sum_{m=-n}^{m=n} A_{nm} Y_n^m(\theta, \varphi), \quad r > a.$$

Paydo bo'lgan hamma qatorlar yaqinlashuvchi bo'ladi va o'zining yaqinlashuv sohasida tekis yaqinlashuvchi bo'ladi, buning isbotini [3] kitobning §25 dan topish mumkin.

Ikkinchi masala: Bu gal ikkinchi chegaraviy shart quyidagicha:

$$u_r(a,z)=0, \quad \Rightarrow \quad J_0'(\lambda a)=-J_1(\lambda a)=0.$$

Demak, $\lambda a=\mu_n^{(1)}$ - J_1 ning nollari. Yechim:

$$u(r,z)=\sum_{n=1}^{\infty} c_n J_0\left(\mu_n^{(1)}\frac{r}{a}\right)\operatorname{sh}\left(\mu_n^{(1)}\frac{z}{a}\right).$$

Yuqori asosdagi chegaraviy shart:

$$u(r,h)=f(r)=\sum_{n=1}^{\infty} c_n J_0\left(\mu_n^{(1)}\frac{r}{a}\right)\operatorname{sh}\left(\mu_n^{(1)}\frac{h}{a}\right)$$

dan c_n larni topamiz:

$$c_n=\frac{2}{a^2 J_0^2(\mu_n^{(1)})\operatorname{sh}\left(\mu_n^{(1)}\frac{h}{a}\right)}\int_0^a dr r f(r) J_0\left(\mu_n^{(1)}\frac{r}{a}\right).$$

To'liq yechimni topdik:

$$u(r,z)=\sum_{n=1}^{\infty}\frac{2J_0\left(\mu_n^{(1)}r/a\right)\operatorname{sh}\left(\mu_n^{(1)}z/a\right)}{a^2 J_0^2(\mu_n^{(1)})\operatorname{sh}\left(\mu_n^{(1)}h/a\right)}\int_0^a dr r f(r) J_0\left(\mu_n^{(1)}\frac{r}{a}\right).$$

Uchinchi masala:

Bu galda chegaraviy shart murakkabroq:

$$u_r(a,z)=-\alpha u(a,z), \quad \alpha>0.$$

Buni quyidagicha yozishimiz kerak:

$$\left.\frac{d}{dr}J_0(\lambda r)\right|_{r=a}+\alpha J_0(\lambda a)=0.$$

Agarda J_0' deb, uning argumenti bo'yicha hosilani belgilasak, yuqoridagi tenglama quyidagicha yoziladi:

$$J_0'(\lambda a)+\frac{\alpha}{\lambda}J_0(\lambda a)=0.$$

Hosil bo'lgan tenglamaning yechimlarini λ_n deb belgilaylik. Yana yuqoridagi amallarni bajarsak, c_n koeffisienti uchun quyidagi ifodani topamiz:

$$c_n=\frac{2}{a^2}\frac{1}{(1+\alpha^2 a^2/\lambda_n^2)J_0^2(\lambda_n)}\int_0^a dr r f(r) J_0\left(\lambda_n\frac{r}{a}\right).$$

VIII BOB. GREEN FUNKSIYASI METODI

§1. δ-funksiya

Matematik fizikada moddiy nuqta, nuqtaviy zaryad, zaryadlangan sirt va h.k. shunga o'xshash ideallashtirilgan tushunchalar ko'p uchrab turadi. Bunday kattaliklarning zichligini oddiy funksiyalar yordamida ta'riflab bo'lmaydi.

Masalan, massasi m ga teng bo'lgan moddiy nuqta tushunchasini ko'raylik. Agar chekli massa bir nuqtada joylashgan bo'lsa, (shu nuqta koordinat boshi bo'lsin) uning zichligi uchun

$$\rho(\mathbf{r}) = \begin{cases} \infty, & r = 0; \\ 0, & r >\neq 0. \end{cases} \tag{1}$$

deb olishimiz kerak. Tabiiyki, bunday funksiyaga oddiy funksiyaga qaragandek qaray olmaymiz - uni na differensiallash mumkin, na integrallash. Vaholangki, jismning zichligidan olingan integral shu jismning massasini berishi kerak:

$$\int d^3x\rho(\mathbf{r}) = m. \tag{2}$$

Demak, nuqtaviy kattaliklarning zichligiga boshqacha yondoshishimiz lozim. Buning uchun biz tajribada biror jismning nuqtadagi zichligini o'lchaganda haqiqatda shu nuqtaning kichik atrofida bo'lgan o'rtacha zichliknigina o'lchay olishimizni eslashimiz kerak. Odatda shu o'rtacha zichlik nuqtadagi zichlik deb e'lon qilinadi. Shunga asosan nuqtaviy zarra zichligi haqida gapirganimizda uning massasini yetarli darajada kichik bo'lgan ε radisuli shar bo'yicha bir tekisda taqsimlangan deb qarab, uning o'rtacha zichligi uchun (zarraning massasi birga teng $m = 1$ va u koordinat boshida joylashgan deb olaylik)

$$\rho_\varepsilon(\mathbf{r}) = \begin{cases} \dfrac{3}{4\pi\varepsilon^3}, & r \leq \varepsilon; \\ 0, & r > \varepsilon. \end{cases} \tag{3}$$

ifodani ko'rish tabiiyroqdir. Albatta, $\varepsilon \longrightarrow 0$ limitda biz yana o'sha(1)-formulani olamiz, ammo (1)-dan farqli o'laroq bu gal ρ_ε dan hajm bo'yicha olingan integral shu hajmdagi massani beradi:

$$\int_V \rho_\varepsilon(\mathbf{r})d^3x = \begin{cases} 1, & 0 \in V; \\ 0, & 0 \overline{\in} V. \end{cases} \tag{4}$$

Ana endi $\varepsilon \to 0$ limitga o'tishimiz mumkin! Bundan ko'rinib turibdiki, $\varepsilon \longrightarrow 0$ limitni nuqtadagi limit ma'nosida tushunish, ya'ni, (3)-formulada bevosita $\varepsilon \longrightarrow 0$ limitga o'tish to'g'ri natijaga olib kelmaydi. Bu limitga (3)-funksiyani integrallaganimizdan keyingina o'tishimiz mumkin.

Shu mulohazadan xulosa qilib $\{\rho_\varepsilon(\mathbf{r}), \varepsilon \longrightarrow 0\}$ limitni ***sust limit*** ma'nosida tushunaylik. Bu degani, $\varepsilon \longrightarrow 0$ limit faqatgina integral ostidagina ma'noga ega: ya'ni, bu limitga integral ostida $\rho_\varepsilon(\mathbf{r})$ funksiyamiz boshqa ixtiyoriy bir uzluksiz funksiya $\varphi(\mathbf{r})$ bilan kirgandagina o'tamiz: $\left\{\int \rho_\varepsilon \varphi d^3x, \varepsilon \longrightarrow 0\right\}$. (3)-ta'rifdan keltirib chiqarish qiyin emaski,

$$\lim_{\varepsilon \longrightarrow 0} \int \rho_\varepsilon(\mathbf{r})\varphi(\mathbf{r})dx = \varphi(0). \tag{5}$$

Darhaqiqat, $\varphi(\mathbf{r})$ funsiyaning uzluksizligi shuni bildiradiki, ixtiyoriy $\eta > 0$ uchun shunday $\varepsilon > 0$ topiladiki, $|\mathbf{r}| < \varepsilon$ bo'lganda $|[\varphi(\mathbf{r}) - \varphi(0)]| < \eta$. Natijada,

$$\begin{aligned} \left|\int \rho_\varepsilon(\mathbf{r})\varphi(\mathbf{r})d^3x - \varphi(0)\right| &= \frac{3}{4\pi\varepsilon^3}\left|\int_{|\mathbf{r}|\leq\varepsilon} [\varphi(\mathbf{r}) - \varphi(0)]\, d^3x\right| \leq \\ &\leq \frac{3}{4\pi\varepsilon^3}\int_{|\mathbf{x}|\leq\varepsilon} |\varphi(\mathbf{r}) - \varphi(0)|\, d^3x < \eta \end{aligned} \tag{6}$$

ni olamiz. Demak, $\varphi(0)$ son $\int d^3r\varphi(\mathbf{r})\rho_\varepsilon(\mathbf{r}) = (\varphi, \rho_\varepsilon)$, $\varepsilon \longrightarrow 0$ ketma-ketlikning sust limiti bo'lar ekan. Ixtiyoriy uzluksiz $\varphi(\mathbf{r})$ funksiya uchun bunday limit uning noldagi qiymati $\varphi(0)$ ni mos qo'yar ekan.

Xuddi shunday, massasi birga teng $m = 1$ nuqtaviy zarracha $\mathbf{r} = \mathbf{r}_0$ nuqtada joylashgan bo'lsa, uning zichligi sifatida quyidagini qabul qilamiz:

$$\rho_\varepsilon(\mathbf{r} - \mathbf{r}_0) = \begin{cases} \dfrac{3}{4\pi\varepsilon^3}, & |\mathbf{r} - \mathbf{r}_0| \leq \varepsilon; \\ 0, & |\mathbf{r} - \mathbf{r}_0| > \varepsilon. \end{cases} \tag{7}$$

(6)-ni keltirib chiqargandek

$$\lim_{\varepsilon\longrightarrow 0}\int \rho_\varepsilon(\mathbf{r}-\mathbf{r}_0)\varphi(\mathbf{r})d^3x = \varphi(\mathbf{r}_0) \tag{8}$$

ekanligini ham tekshirib ko'rishimiz mumkin.

Bunday hol matematikada quyidagicha ta'riflanadi: $\{\rho_\varepsilon(\mathbf{r}), \varepsilon \longrightarrow 0\}$ ketma-ketlikning sust limiti yetarli darajada silliq bo'lgan $\varphi(\mathbf{r})$ funksiyalar ustida aniqlangan chiziqli funksional $\varphi(\mathbf{r}_0)$ ni beradi [1].

Har gal $\{\rho_\varepsilon(\mathbf{r}), \varepsilon \longrightarrow 0\}$ deb yozib o'tirmaslik uchun

$$\lim_{\varepsilon\longrightarrow 0}\rho_\varepsilon(\mathbf{r}) = \delta(\mathbf{r})$$

belgi kiritaylik. Bunda biz (birlik massali) nuqtaviy zarrachaning zichligini

$$\rho(\mathbf{r}) = \delta(\mathbf{r})$$

deb belgilagan bo'lamiz. Massasi m bo'lgan zarracha uchun esa

$$\rho(\mathbf{r}) = m\delta(\mathbf{r})$$

deb yozishimiz kerak. Yangi kiritilgan $\delta-$ funksiyaning asosiy xossasi quyidagichadir:

$$\int \delta(\mathbf{r})\varphi(\mathbf{r})d^3x = \varphi(0). \tag{9}$$

Buni ko'pincha

$$(\delta, \varphi) = \varphi(0) \tag{10}$$

ko'rinishda ham yoziladi. Yangi kiritilgan funksiya ***Dirakning delta-funksiyasi*** deyiladi.

Agar $\mathbf{r}_k, k = 1, 2, 3, ...$ nuqtalarda joylashgan m_k diskret massalar sistemasi berilgan bo'lsa, bu sistemaning zichligi

$$\rho(\mathbf{r}) = \sum_{k=1}^{n} m_k\delta(\mathbf{r}-\mathbf{r}_k)$$

formula orqali ifodalanadi. Ko'rinib turibdiki,

$$\int_V d^3x\rho(\mathbf{r}) = \sum_{k=1}^{n} m_k$$

[1] Funksional deganda biror bir $f(x)$ funksiyadan olingan va shu funksiyaning ko'rinishiga bog'liq bo'lgan aniq integral - songa aytiladi.

bo'ladi.

Xuddi shunday, nuqtaviy zaryadlar sistemasi haqida gap ketsa,

$$\rho_e(\mathbf{r}) = \sum_{k=1}^{n} e_k \delta(\mathbf{r} - \mathbf{r}_k)$$

deb yozishimiz kerak.

Biz ko'rdikki, nuqtaviy massa va shunga o'xshash tushunchalarni kiritish uchun sust limit tushunchasidan foydalanishimiz kerak. Bunday kattaliklarning nuqtadagi qiymati haqida gapirishning ma'nosi yo'q - bu shu kattaliklarning fizik ma'nosiga ham to'g'ri keladi. Yuqorida aytganimizdek, tajribada faqat o'rtacha, ya'ni, integral kattaliklar o'lchanadi - kiritilgan $\delta-$funksiya ham xuddi shunday ma'noga ega.

Biz $\delta-$funksiyani uch o'lchamli holda kiritdik, uning xossalarini bir o'lchamli holda o'rganish osonroq. Shuning uchun shu paytgacha aniqlagan xossalarni bir joyga yig'aylik:

1. $\delta(x - x_0) = \begin{cases} \infty, & x = x_0; \\ 0, & x \neq x_0. \end{cases}$.

2. $\int\limits_{-\infty}^{\infty} dx \delta(x - x_0) = 1;$

3. $\int\limits_{-\infty}^{\infty} dx f(x) \delta(x - x_0) = f(x_0).$

Birinchi xossa shartli ma'noga ega.

Uch o'lchamli $\delta-$funksiya quyidagicha aniqlanadi:

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0).$$

Ba'zi -bir hollarda u $\delta^{(3)}(\mathbf{r} - \mathbf{r}_0)$ ko'rinishda ham belgilanadi.

Ko'pincha $\delta-$funksiya oddiy funksiyalarning limiti sifatida ham ko'riladi. Fizik masalalarni yechganda ba'zi hollarda shunday qilish maqsadga muvofiq bo'ladi. δ-ni oddiy funksiyalarning limiti sifatida berish uchun yuqoridagi uch xossaning bajarilishini tekshirib ko'rishimiz kerak. Shu maqsadda eng keng tarqalgan to'rta misolni ko'raylik va yuqoridagi uchta xossalarning $\varepsilon \to 0$ da bajarilishini tekshiraylik ($f(x)$ funksiya uzluksiz va cheksiz tartibdagi uzluksiz hosilalarga ega deb qaraymiz):

I. Birinchi misol: $\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}, & |x| < \varepsilon; \\ 0, & |x| > \varepsilon. \end{cases}$

1. $\lim\limits_{\varepsilon\to 0} \delta_\varepsilon(x) = \begin{cases} \infty, & x = 0; \\ 0, & |x| \neq 0. \end{cases}$

2. $\int\limits_{-\infty}^{\infty} \delta_\varepsilon(x)dx = \int\limits_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}dx = 1.$

3. $\left|\lim\limits_{\varepsilon\to 0} \int\limits_{-\infty}^{\infty} f(x)\delta_\varepsilon(x)dx - f(0)\right| \le \lim\limits_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int\limits_{-\varepsilon}^{\varepsilon} \left|f(x) - f(0)\right|dx \to 0.$

II. Ikkinchi misol: $\delta_\varepsilon(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\varepsilon} \exp\left(-\frac{x^2}{\varepsilon^2}\right).$

1. $\lim\limits_{\varepsilon\to 0} \delta_\varepsilon(x) = \begin{cases} \infty, & x = 0; \\ 0, & |x| \neq 0. \end{cases}$

2. $\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{1}{\varepsilon} \exp\left(-\frac{x^2}{\varepsilon^2}\right) dx = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp\left(-y^2\right) dy = 1.$

3.

$$\left|\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{1}{\varepsilon} \exp\left(-\frac{x^2}{\varepsilon^2}\right) f(x)dx - f(0)\right| \le$$

$$\le \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp\left(-y^2\right) |f(\varepsilon y) - f(0)|\, dy \to 0, \quad \varepsilon \to 0.$$

Uchinchi va tortinchi misollar sifatida quyidagilarni olamiz:

$$\text{III.}\quad \delta_\varepsilon(x) = \frac{\sin(\varepsilon x)}{\pi x}, \quad \varepsilon \to \infty; \tag{11}$$

$$\text{IV.}\quad \delta_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}, \quad \varepsilon \to 0.$$

Bu funksiyalar uchun ham uchala xossalarning bajarilishini bevosita tekshirib chiqish mumkin.

Keltirilgan misollardan oydinki, δ-funksiya juft funksiyadir:

$$\delta(-x) = \delta(x). \tag{12}$$

δ-funksiyaning hosilasini ham sust limit ko'rinishida kiritish mumkin. Buning uchun yana asosiy funksiya $f(x)$ larning yetarli darajada silliqligi va cheksizlikda yetarli darajada tez nolga intilishini ishlatamiz:

$$(\delta', f) = \int_{-\infty}^{\infty} dx\delta'(x)f(x) = \delta(x)f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\delta(x)f'(x) = -f'(0). \quad (13)$$

Va shu mulohazani davom ettirib, umuman olganda,

$$\left(\delta^{(n)}, f\right) = (-1)^n f^{(n)}(0) \quad (14)$$

bo'lishini topish mumkin. Bu xossani

$$\delta^{(n)}(x)f(x) = (-1)^n f^{(n)}(0)\delta(x) \quad (15)$$

ma'noda ham tushunish mumkin. Xususan, $n = 0$ holda:

$$\delta(x)f(x) = \delta(x)f(0). \quad (16)$$

Umumlashgan funksiyalar ichida eng ko'p tarqalganlaridan biri teta-funksiyadir (pog'onacha):

$$\theta(x) = \begin{cases} 1, & x > 0; \\ 0, & x < 0. \end{cases}$$

Bu funksiya uchun

$$\frac{d}{dx}\theta(x) = \delta(x) \quad (17)$$

ekanligini isbot qilaylik:

$$(\theta', f) = \int_{-\infty}^{\infty} dx\theta'(x)(x)f(x) = \theta(x)f(x)|_{\infty}^{\infty} - \int_{0}^{\infty} df(x) = f(0).$$

δ-funksiyaning Fourier tasvirini topaylik. Buning uchun

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\tilde{\delta}(k)\exp(-ikx)$$

formuladan uning teskarisiga o'tamiz:

$$\tilde{\delta}(k) = \int_{-\infty}^{\infty} dx\delta(x)\exp(ikx) = 1. \quad (18)$$

Ya'ni, δ - funksiyaning Fourier-tasviri birga teng ekan, bu esa bizga δ - funksiyaning eng mashhur tasavvurini beradi:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(-ikx). \tag{19}$$

Bu tasavvurdan foydalanib $\delta(x)$ funksiyaning yuqoridagi uchinchi limit formasini keltirib chiqarishimiz mumkin:

$$\delta(x) = \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L} dk \exp(-ikx) = \lim_{L\to\infty}\frac{\sin(Lx)}{\pi x}.$$

Uch o'lchamli $\delta-$funksiyaning integral tasavvurini topaylik:

$$\delta^{(3)}(\mathbf{r}) = \delta(x)\delta(y)\delta(z) =$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk_x \exp(-ik_x x)\int_{-\infty}^{\infty} dk_y \exp(-ik_y y)\int_{-\infty}^{\infty} dk_z \exp(-ik_z z) = \tag{20}$$

$$= \int \frac{d^3k}{(2\pi)^3}\exp(-i\mathbf{k}\cdot\mathbf{r}).$$

Bu formula matematik fizikada ko'p ishlatiladigan formulalar qatoriga kiradi.

8.1-mashq. Quyidagini isbot qiling:

$$\delta[a(x-x_0)] = \frac{1}{|a|}\delta(x-x_0).$$

8.2-mashq. Quyidagini ko'rsating:

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0), \qquad f(x_0) = 0.$$

Avvalgi misol shu misolning xususiy holidir. Agar $f(x) = 0$ tenglamaning yechimlari bir nechta bo'lsa, $\{x_i,\ i = 1, 2, ...\}$

$$\delta(f(x)) = \sum_{i=1,2,...}\frac{1}{|f'(x_i)|}\delta(x-x_i)$$

bo'ladi.

8.3-mashq.

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n}; \\ n, & -\frac{1}{2n} < x < \frac{1}{2n}; \\ 0, & \frac{1}{2n} < x. \end{cases}$$

funksiya uchun $\lim_{n\to\infty}\delta_n(x)=\delta(x)$ ekanligini ko'rsating.

8.4-mashq. Sferik (r,θ,φ) sistemada $\delta(\mathbf{r}-\mathbf{r}_0)$ funksiya

$$\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$$

bo'lishini ko'rsating.

8.5-mashq.

$$\delta(\varphi_1-\varphi_2)=\frac{1}{2\pi}\sum_{m=-\infty}^{m=\infty}\exp\left[im(\varphi_1-\varphi_2)\right]$$

ekanligini isbot qiling.

8.6-mashq. θ-funksiya uchun quyidagi tasavvurni isbot qiling:

$$\theta(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\tau\frac{e^{ix\tau}}{\tau-i\varepsilon}=\begin{cases}1, & x>0;\\ 0, & x<0.\end{cases}$$

§2. Chiziqli differensial operatorning fundamental yechimi (Green funksiyasi)

Matematik fizika tenglamalariuni yechishning yana bir muhim metodi - Green funksiyasi metodidir. Bu metodni kiritish uchun ixtiyoriy chiziqli differensial tenglamani olaylik:

$$Lu=f, \tag{21}$$

bu yerda L - biror differensial operator, masalan,

$$L=\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\quad \text{yoki}\quad L=\frac{\partial^2}{\partial t^2}-\Delta$$

va h.k. Noma'lum funksiya $u(x)$ shu tenglamaning yechimi qidirilayotgan sohada va uning chegarasida yetarli darajada silliq deb hisoblaymiz.

L operatorning fundamental yechimi, yoki ***Green[2] funksiyasi*** $G(x)$ quyidagicha kiritiladi:

$$L_{x_1}G(x_1,x_2)=\delta(x_1-x_2). \tag{22}$$

x deganda n-o'lchamli fazo vektorini ko'zda tutamiz: $x=\{x^1,x^2,x^3,...,x^n\}$. $n=3$ bo'lganda $x\to\mathbf{r}$ bo'ladi. Shunga yarasha $\delta(x_1-x_2)$ -funksiya ham n-o'lchamli funksiyadir. L_{x_1} deganimizda operator x_1 argumentga ta'sir qilayapti demoqchimiz. Fundamental yechim mavjud bo'lsa (22)-ning yechimini bir zumda yozib olishimiz mumkin:

$$u(x)=u_0(x)+\int dx'\,G(x,x')\,f(x'), \tag{23}$$

[2]George Green (1793-1841) - ingliz matematigi. Rus tilida - Грин.

bu yerda $u_0(x)$ bir jinsli tenglamaning yechimi:

$$Lu_0 = 0.$$

Isbot:

$$Lu(x) = \int dx' \, LG(x, x') \, f(x') = \int dx' \, \delta(x - x') \, f(x') = f(x). \qquad (24)$$

$u_0(x)$ haddan chegaraviy shartlarni qanoatlantirish uchun foydalanish mumkin.

Quyidagi teorema differensial operatorlarning fundamental yechimlarini topishda muhim rol o'ynaydi:

Teorema:

$$\theta(t)e^{-at} \qquad \text{va} \qquad \theta(t)\frac{\sin at}{a}$$

funksiyalar

$$\frac{d}{dt} + a \qquad \text{va} \qquad \frac{d^2}{dt^2} + a^2$$

operatorlarning fundamental yechimlari bo'ladi.

Isbot:

Bevosita hisoblab topamiz:

$$\left(\frac{d}{dt} + a\right)\theta(t)e^{-at} = \delta(t)e^{-at} - a\theta(t)e^{-at} + a\theta(t)e^{-at} = \delta(t)e^{-at} = \delta(t).$$

Bu yerda 16- va 17- formulalar ishlatildi.

Teoremaning ikkinchi qismini isbot qilish uchun quyidagidan boshlaylik ($Z(t)$ - ixtiyoriy ikki marta differensiallanuvchi funksiya):

$$\frac{d^2}{dt^2}(\theta(t)Z(t)) = \delta'(t)Z(t) + 2\delta(t)Z'(t) + \theta(t)Z''(t) = -\delta(t)Z'(t) +$$

$$+ 2\delta(t)Z'(t) + \theta(t)Z''(t) = \delta(t)Z'(t) + \theta(t)Z''(t) = \delta(t)Z'(0) + \theta(t)Z''(t),$$

bu yerda 15-formula $n = 1$ va $n = 0$ hollarda ishlatildi. Oxirgi natijadan kelib chiqqan holda, quyidagini olamiz:

$$\left(\frac{d^2}{dt^2} + a^2\right)\theta(t)\frac{\sin at}{a} = \delta(t). \qquad (25)$$

Teorema isbot qilindi.

§3. Laplace operatorining fundamental yechimi

Laplace operatorining fundamental yechimini topish uchun

$$\Delta G(\mathbf{r}) = \delta^{(3)}(\mathbf{r}) \tag{26}$$

tenglamani yechish kerak. Buning uchun Fourier almashtirish metodidan foydalanamiz:

$$G(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\tilde{G}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r}), \qquad \delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{27}$$

Birinchisini (26)-ga olib borib qo'yaylik:

$$\Delta G(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\tilde{G}(\mathbf{k})\Delta\exp(-i\mathbf{k}\cdot\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}(-\mathbf{k}^2)\tilde{G}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{28}$$

Demak,

$$\int \frac{d^3k}{(2\pi)^3}(-\mathbf{k}^2)\tilde{G}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\exp(-i\mathbf{k}\cdot\mathbf{r}),$$

yoki,

$$\tilde{G}(\mathbf{k}) = -\frac{1}{k^2} \tag{29}$$

formulaga kelamiz. Shu bilan Laplace operatorining Green funksiyasining integral tasavvuri topildi:

$$G(\mathbf{r}) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{k^2}\exp(-i\mathbf{k}\cdot\mathbf{r}). \tag{30}$$

Ammo bu integralni bevosita hisoblash cheksizlikka olib keladi. Shuning uchun uning o'rniga

$$G_\lambda(\mathbf{r}) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{k^2+\lambda^2}\exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{31}$$

ni hisoblaymiz va oxirida $\lambda \to 0$ limitga o'tamiz. $\mathbf{k}$ fazoda sferik koordinat sistemasini kiritib, $\mathbf{k}$ va $\mathbf{r}$ vektorlar orasidagi burchakni θ, deb belgilaylik:

$$G_\lambda(r) = -\frac{1}{(2\pi)^3}\int_0^\infty \frac{k^2dk}{k^2+\lambda^2}\int_0^{2\pi} d\varphi \int_0^\pi d\theta\exp(-ikr\cos\theta) =$$

$$= -\frac{1}{(2\pi)^2 ir}\int_0^\infty \frac{kdk}{k^2+\lambda^2}\left(\exp(ikr) - \exp(-ikr)\right) =$$

$$= -\frac{1}{2(2\pi)^2 ir}\int\limits_{-\infty}^{\infty}\frac{kdk}{k^2+\lambda^2}\left(\exp(ikr)-\exp(-ikr)\right) = -\frac{1}{4\pi r}.$$

Oxirgi integralni hisoblashda chegirmalar nazariyasi va Jordan lemmasidan foydalandik. Integral ostidagi funksiya $k = \pm i\lambda$ nuqtalarda birinchi tartibli qutblarga ega[3], birinchi integralni hisoblashda $k = +i\lambda$ qutb olinadi va Jordan lemmasi bo'yicha kontur yuqori yarim tekislikda joylashgan yarim aylanma bilan to'ldiriladi. Ikkinchi integralni hisoblaganda $k = -i\lambda$ qutb olinadi va Jordan lemmasi bo'yicha kontur quyi yarim tekislikda joylashgan yarim aylanma bilan to'ldiriladi. Oxirida λ nolga intiltiriladi. Demak,

$$G(\mathbf{r}) = -\frac{1}{4\pi r} \tag{32}$$

yoki, umumiyroq formada

$$G(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} \tag{33}$$

ekan.

Shu natijadan foydalanib, Poisson tenglamasi

$$\Delta\varphi = -f \tag{34}$$

ning yechimini topish mumkin. (23)-formula bo'yicha bu tenglamaning yechimi

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}\int\frac{f(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \tag{35}$$

ga teng. Elektrostatikada nuqtaviy zaryad uchun $f(\mathbf{r}) = \rho(\mathbf{r}) = e\delta(\mathbf{r})$ ekanligini hisobga olsak,

$$\varphi(\mathbf{r}) = \frac{e}{4\pi}\int\frac{\delta(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' = \frac{e}{4\pi r} \tag{36}$$

ifodani olamiz. Bu - Coulomb[4] qonuni.

Hisoblarimizdan kelib chiqadiki,

$$\Delta\frac{1}{r} = -4\pi\delta(\mathbf{r}). \tag{37}$$

[3]Nima uchun boshidan $\lambda = 0$ deb olish mumkinmasligi endi tushunarli - bu holda integrallash konturi ikki maxsus nuqta orasida siqilib qolgan bo'lar ekan, *pinch* deyiladigan bunday maxsus nuqta integrallanuvchi maxsus nuqtalarga kirmaydi.

[4]Rus tilida - Кулон. Charles-Augustin de Coulomb (14.06.1736 - 23.08.1806) (Шарль-Огюстен де Кулон)

Bu formulani bevosita tekshirib ko'raylik. Uning uchun sferik koordinat sistemasiga o'taylik. Agar $r \neq 0$ bo'lsa.

$$\Delta\frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right) = 0$$

ekanligiga ishonch hosil qilamiz. $r = 0$ nuqta maxsus nuqtadir, bu nuqtada hosila olib bo'lmaydi, ammo Gauss teoremasidan foydalanib (37)-ning o'ng tomonida delta-funksiya borligini ko'rsatishimiz mumkin:

$$\int d^3r\Delta\frac{1}{r} = \int d^3r \mathrm{div}\,\mathrm{grad}\frac{1}{r} = -\int d\mathbf{S}\cdot\mathbf{r}/r^3 = -4\pi.$$

Oxirgi ikki formula (37)-ning yana bir isbotini beradi.

§4. Ikki o'lchamli Laplace operatorining Green funksiyasi

Ikki o'lchamli Laplace operatorining Green funksiyasi G_2

$$\Delta_2 G_2(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G_2(x,y) = \delta(x)\delta(y) = \delta(\mathbf{r})$$

formula orqali aniqlanadi.

$$G_2(\mathbf{r}) = \frac{1}{2\pi}\ln r$$

ekanligini isbot qilaylik. Bu esa

$$\Delta_2 \ln r = 2\pi\delta(\mathbf{r}) \tag{38}$$

ekanligini ko'rsatishga teng. Isbotni uch qismga bo'lamiz. Birinchidan,

$$\Delta_2 \ln r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\ln r\right) = 0, \qquad r \neq 0.$$

Ikkinchidan, $\Delta_2 \ln r$ dan ixtiyoriy R radiusli doira bo'yicha integralga Gauss teoremasini qo'llaymiz:

$$\int_V d^2r\Delta_2 \ln r = \oint_S d\mathbf{S}\cdot\nabla\ln r = R\int_0^{2\pi} d\varphi\frac{1}{R} = 2\pi.$$

Ikkita oxirgi natijadan delta-funksiyaning birinchi va ikkinchi xossalari bajarilganligi kelib chiqadi. Ixtiyoriy cheksiz silliq $f(\mathbf{r})$ funksiya uchun uchinchi xossasini tekshiraylik:

$$\int d^2r\Delta_2 \ln r f(\mathbf{r}) = \int d^2r\,\Delta_2 \ln r\,[f(\mathbf{r}) - f(0) + f(0)] = 2\pi f(0) +$$

$$+\int d^2r\,\Delta_2 \ln r\,[f(\mathbf{r})-f(0)].$$

$r\neq 0$ da $\Delta_2 \ln r = 0$ bo‘lgani uchun integralga faqat $r = 0$ nuqtaning atrofi hissa qo‘shadi. Shuning uchun integralni kichik ε radiusli doira bo‘yicha integralga almashtiramiz va integrallash o‘zgaruvchisi ustida $\mathbf{r} = \varepsilon\tilde{\mathbf{r}}$ almashtirish bajaramiz. Bunda integral $0 \leq \tilde{\mathbf{r}} \leq 1$ bo‘yicha olingan bo‘ladi:

$$\int d^2r\,\Delta_2 \ln r\,[f(\mathbf{r})-f(0)] = \int d^2\tilde{r}\,\widetilde{\Delta_2} \ln \tilde{r}\,[f(\varepsilon\tilde{\mathbf{r}})-f(0)] \xrightarrow{\varepsilon\to 0} 0.$$

Shu bilan (38)-formula isbot qilindi.

§5. Multipol yoyilma

Bizga VIII.1-rasmda ko‘rsatilgan zaryadlar taqsimoti berilgan bo‘lsin. Shu zaryadlar sistemasining P nuqtada hosil qilgan elektrostatik potensialini topaylik. Elektrostatik potensial uchun tenglama

$$\Delta\varphi = -\rho$$

ga Green funksiyasi (33)-ni qo‘llasak, potensial uchun

$$\varphi(\mathbf{R}) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|}d^3r$$

yechim kelib chiqadi. Agar sistema diskret zaryadlardan tashkil topgan bo‘lsa,

$$\varphi(\mathbf{R}) = \frac{1}{4\pi}\sum_a \frac{e_a}{|\mathbf{R}-\mathbf{r}_a|}$$

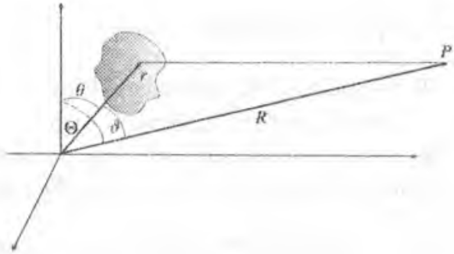
bo‘ladi.

$\rho(\mathbf{r})$ - cheklangan zaryadlar taqsimotiga mos kelsin, ularni o‘z ichiga olgan sohaning o‘lchami a kuzatish nuqtasigacha bo‘lgan masofa R ga nisbatan kichik bo‘lsin: $a \ll R$. $r \leq a$ bo‘lgani uchun $r \ll R$ bo‘ladi. Undan tashqari

$$|\mathbf{R}-\mathbf{r}| = \sqrt{(\mathbf{R}-\mathbf{r})^2} = \sqrt{\mathbf{R}^2 - 2\mathbf{R}\cdot\mathbf{r}+\mathbf{r}^2} = R\sqrt{1-2\frac{r}{R}\cos\vartheta + \left(\frac{r}{R}\right)^2}$$

ϑ burchak $\mathbf{R}$ va $\mathbf{r}$ vektorlar orasidagi burchak. Bundan kelib chiqadiki, integral ostidagi $1/|\mathbf{R}-\mathbf{r}|$ ifodaga Legendre yoyilmasi (35)-ni qo‘llash mumkin:

$$\frac{1}{|\mathbf{R}-\mathbf{r}|} = \frac{1}{R}\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left(\frac{r}{R}\right)^n.$$



VIII.1-rasm: Zaryadlar taqsimoti

Shu bilan potensial quyidagi ko'rinishga keltirildi:

$$\varphi(\mathbf{R}) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|}d^3r = \frac{1}{4\pi R}\sum_{n=0}^{\infty}\int d^3r\rho(\mathbf{r})\left(\frac{r}{R}\right)^n P_n(\cos\vartheta).$$

(VIII.1)-rasmdagi burchaklarga diqqat bilan qaraylik. Sferik sistemada

$$d^3r = drr^2d\theta\sin\theta d\varphi,$$

demak,

$$\varphi(\mathbf{R}) = \frac{1}{4\pi R}\sum_{n=0}^{\infty}\int\limits_0^{\infty} drr^2\rho(\mathbf{r})\int\limits_0^{\pi} d\theta\sin\theta\int\limits_0^{2\pi} d\varphi\left(\frac{r}{R}\right)^n P_n(\cos\vartheta).$$

Qo'shish teoremasi (78) bo'yicha (I.3- va VIII.1-rasmlarni solishtiring)

$$P_n(\cos\vartheta) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\theta,\varphi)Y_n^{m*}(\Theta,\Phi),$$

bu yerda (θ,φ) burchaklar $\mathbf{r}$ vektor (θ - $\mathbf{r}$ va z-o'qi orasidagi burchak) bilan bog'liq, (Θ,Φ) burchaklar $\mathbf{R}$ vektor (Θ - $\mathbf{R}$ va z-o'qi orasidagi burchak) bilan bog'liq. Qo'shish formulasini integralga qo'yamiz:

$$\varphi(\mathbf{R}) = \frac{1}{4\pi R}\sum_{n=0}^{\infty}\frac{1}{R^n}\sqrt{\frac{4\pi}{2n+1}}\sum_{m=-n}^{n} Q_n^m Y_n^{m*}(\Theta,\Phi) =$$
$$= \sum_{n=0}^{\infty}\varphi^{(n)}(\mathbf{R}) = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \cdots, \qquad (39)$$

bu yerda

$$Q_n^m = \sqrt{\frac{4\pi}{2n+1}} \int\limits_0^\infty dr r^{2+n} \int\limits_0^\pi d\theta \sin\theta \int\limits_0^{2\pi} d\varphi \rho(\mathbf{r}) Y_n^m(\theta, \varphi) \tag{40}$$

va

$$\varphi^{(n)}(\mathbf{R}) = \frac{1}{4\pi R^{n+1}} \sqrt{\frac{4\pi}{2n+1}} \sum_{m=-n}^{n} Q_n^m Y_n^{m*}(\Theta, \Phi).$$

Olingan yoyilma (39) ***multipol yoyilma*** deyiladi. Ma'lum bir n uchun m soni $m = -n$ dan $m = +n$ gacha bo'lgan $2n+1$ ta qiymat qabul qilgani uchun Q_n^m lar $2n+1$ ta komponentaga ega bo'lgan kattalikni tashkil qiladi, ular sistemaning 2^n-pol momentlari deyiladi - $n = 1$ da dipol, $n = 2$ da kvadrupol, $n = 3$ da oktupol va h.k. Agar sistemaning to'liq zaryadi noldan farqli bo'lsa yoyilma

$$\varphi^{(0)} = \frac{1}{4\pi R} \int d^3 r \rho(\mathbf{r}) = \frac{q}{4\pi R}$$

haddan boshlanadi. Undan keyingi had

$$\varphi^{(1)} = \frac{1}{R^2} \frac{1}{\sqrt{12\pi}} \left(Q_1^{-1} Y_1^{-1*} + Q_1^0 Y_1^{0*} + Q_1^1 Y_1^{1*} \right),$$

sistemaning dipol momentiga mos keladi. §2.11.-paragrafda yechilgan misollar mana shu multipol momentlarga tegishli edi.

8.7-mashq. I.4-misolning natijasidan foydalanib dipol momenti $\mathbf{d}$ va (Q_1^m, $m = -1, 0, 1$) lar orasidagi bog'lanishlar quyidagicha bo'lishini ko'rsating:

$$Q_1^0 = d_z, \quad Q_1^1 = \frac{1}{\sqrt{2}}(d_x + i d_y), \quad Q_1^{-1} = \frac{-1}{\sqrt{2}}(d_x - i d_y).$$

8.1-misol. VIII.2-rasmda ko'rsatilgan zaryadlangan halqaning elektr maydon potensialini toping. Halqaning to'liq zaryadi - q, halqaning radiusi - a, A - kuzatish nuqtasi.

Zaryad taqsimoti quyidagicha ifodalanadi (delta-funksiyaning sferik sistemadagi ifodasi VIII.4-mashqdan olingan):

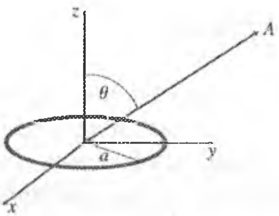
$$\rho(\mathbf{r}) = \frac{q}{r^2} \delta(r-a) \delta(\cos\theta).$$

Bu ifodani (40)-ga qo'yamiz:

$$Q_n^m = q \sqrt{\frac{4\pi}{2n+1}} \int\limits_0^\infty dr r^n \delta(r-a) \int\limits_{-1}^{1} d(\cos\theta) \delta(\cos\theta) \int\limits_0^{2\pi} d\varphi Y_n^m(\theta, \varphi).$$

Sferik funksiyaning (69)-ifodasidan foydalanib φ bo'yicha integralni birinchi hisoblaylik:

$$\int_0^{2\pi} d\varphi e^{im\varphi} = \frac{1}{im}\left(e^{2\pi im} - 1\right) = 2\pi\delta_{m,0}.$$



VIII.2-rasm: Zaryadlangan halqa

Demak, multipol momentlarning faqat $m = 0$ bo'lgan hadigina noldan farqli bo'lar ekan:

$$Q_n^0 = qa^n P_n(0).$$

Ammo $P_{2n+1}(0) = 0$ (I-bobdagi 45-formulaga qarang), shuning uchun multipol momentlarning faqat juftlari qoladi:

$$\varphi(\mathbf{R}) = \frac{1}{4\pi R}\sum_{n=0}^{\infty}\frac{1}{R^{2n}}\sqrt{\frac{4\pi}{4n+1}}Q_{2n}^0 Y_{2n}^{0*}(\theta,\varphi) = \frac{q}{4\pi R}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n} P_{2n}(0)P_{2n}(\cos\theta).$$

$P_{2n}(0)$ uchun (I-bobdagi 46)-formuladan foydalanamiz, natijada a-radiusli zaryadi q bo'lgan halqaning potensiali

$$\varphi(\mathbf{R}) = \frac{q}{4\pi R}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}P_{2n}(\cos\theta) =$$

$$= \frac{q}{4\pi R}\left[1 - \left(\frac{a}{R}\right)^2\frac{3\cos^2\theta - 1}{4} + \cdots\right], \qquad R > a,$$

ko'rinishda ifodalanishini topamiz. Albatta, $R \gg a$ masofalarda

$$\varphi(\mathbf{R}) \simeq \frac{q}{4\pi R}$$

bo'ladi.

§6. Xususiy funksiyalar, xususiy qiymatlar, δ-funksiya va Green funksiyasi

Chiziqli L operator berilgan bo'lsin. $\varphi_n(x)$ funksiyalar uning (λ_n xususiy qiymatlarga mos keluvchi) xususiy funksiyalarining ortonormal sistemasini bersin:

$$L\varphi_n(x) = \lambda_n\varphi_n(x).$$

x deganda, k-o'lchamli fazo vektorini ko'zda tutamiz: $x = \{x_1, x_2, x_3, ..., x_k\}$. $k = 3$ bo'lganda $x \to \mathrm{r}$ bo'ladi.

δ-funksiyani quyidagi yoyilma ko'rinishida qidiramiz:

$$\delta(x_1 - x_2) = \sum_n a_n(x_2)\varphi_n(x_1),$$

$a_n(x_2)$ – noma'lum koeffisientlar. Bu formulaning ikkala tomonini $\varphi^*_m(x_1)$ ga ko'paytirib x_1 bo'yicha integrallaymiz. Chap tomondan

$$\int_{-\infty}^{\infty} dx_1\, \varphi^*_m(x_1)\delta(x_1 - x_2) = \varphi^*_m(x_2)$$

kelib chiqadi. O'ng tomondan (φ_n larning ortonormalligidan)

$$\sum_n a_n(x_2) \int d^n x_1\, \varphi^*_m(x_1)\varphi_n(x_1) = a_m(x_2)$$

kelib chiqadi. Demak,

$$\delta(x_1 - x_2) = \sum_n \varphi^*_n(x_2)\varphi_n(x_1) \tag{41}$$

ekan.

Quyidagi birjinslimas tenglama berilgan bo'lsin:

$$L\psi + \lambda\psi = -\rho. \tag{42}$$

Bu tenglama uchun Green funksiyasini quyidagicha ta'riflaymiz:

$$(L + \lambda)G(x_1 - x_2) = \delta(x_1 - x_2).$$

Green funksiyasini L operatorining xususiy funksiyalar orqali ifodalaylik:

$$G(x_1 - x_2) = \sum_n b_n(x_2)\varphi_n(x_1).$$

Ko'rinib turibdiki,

$$(L + \lambda)G(x_1 - x_2) = \sum_n b_n(x_2)(\lambda_n + \lambda)\varphi_n(x_1) = \sum_n \varphi^*_n(x_2)\varphi_n(x_1).$$

Demak,

$$G(x_1 - x_2) = \sum_n \frac{\varphi^*_n(x_2)\varphi_n(x_1)}{\lambda_n + \lambda}.$$

(42)-tenglamaning yechimi uchun quyidagiga egamiz:

$$\psi(x) = -\int d^n x'\, G(x - x')\rho(x') = -\sum_n \frac{\varphi_n(x)}{\lambda_n + \lambda} \int d^n x'\, \varphi^*_n(x')\rho(x').$$

§7. Helmholtz tenglamasining Green funksiyasi

Quyidagi tenglama Helmholtz[5] tenglamasi deyiladi:

$$\Delta\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = -4\pi\rho(\mathbf{r}). \tag{43}$$

Laplace operatorining xususiy funksiyalari va xususiy qiymatlarini $\{\varphi_n(\mathbf{r}),\ -k_n^2\}$ deb belgilaymiz:

$$\Delta\varphi_n(\mathbf{r}) = -k_n^2\varphi_n(\mathbf{r}), \qquad n = 0, 1, 2, ... \tag{44}$$

Helmholtz tenglamasining Green funksiyasini

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \sum_n b_n(\mathbf{r}_2)\varphi_n(\mathbf{r}_1).$$

ko'rinishda qidiramiz. Helmholtz operatorining unga ta'siri δ-funksiyani beradi:

$$\sum_n b_n(\mathbf{r}_2)(-k_n^2 + k^2)\varphi_n(\mathbf{r}_1) = \delta(\mathbf{r}_1 - \mathbf{r}_2) = \sum_n \varphi_n^*(\mathbf{r}_2)\varphi_n(\mathbf{r}_1).$$

Helmholtz operatori uchun Green funksiyasini topdik:

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \sum_n \frac{\varphi_n^*(\mathbf{r}_2)\varphi_n(\mathbf{r}_1)}{-k_n^2 + k^2}. \tag{45}$$

Agar bu formulada $k^2 = 0$ deb olsak, Laplace operatorining Green funksiyasini topgan bo'lamiz. Rostdan ham, quyidagi funksiyalar

$$\varphi_n(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{k}_n\cdot\mathbf{r}}$$

(44)-ning yechimlarining ortonormal sistemani hosil qiladi. Bu funksiyalar (44)-ning yechimi ekanligini va ularning ortogonalligini ham tekshirish qiyin emas:

$$(\varphi_n(\mathbf{r}),\ \varphi_m(\mathbf{r})) = \frac{1}{(2\pi)^3}\int d^3r e^{i(\mathbf{k_n}-\mathbf{k_m})\cdot\mathbf{r}} = \delta(\mathbf{k_n} - \mathbf{k_m}).$$

Demak, (45)-dan $k^2 = 0$ hol uchun quyidagini olamiz:

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{(2\pi)^3}\sum_n \frac{1}{-k_n^2} e^{-i\mathbf{k}_n(\mathbf{r}_1-\mathbf{r}_2)}.$$

Agar to'lqin vektorlari uzluksiz sistemani hosil qilsa, yig'indining o'rniga integralga o'tish kerak:

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{k^2} e^{-i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}.$$

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[5]Rus tilida - Гельмгольц

Laplace operatorining fundamental yechimi uchun (30)-formulani qaytatdan oldik.

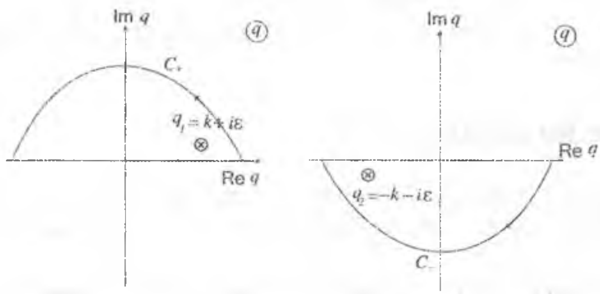
Helmholtz operatorining Green funksiyasiga o'taylik. Uni ham Laplace operatorining xususiy funksiyalari orqali topamiz. Boshidan uzluksiz impulslarga o'tib ($\mathbf{k_n} \to \mathbf{q}$, Helmholtz tenglamasidagi $\mathbf{k}$ bilan adashtirmaslik uchun), quyidagini olamiz:

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{k^2 - q^2} e^{-i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)}.$$

Integral ostidagi funksiya $q^2 = k^2$ nuqtada qutbga ega, bu maxsus nuqta integrallash konturining ustida yotibdi. Shuning uchun uni aylanib o'tish yo'lini ko'rsatishimiz kerak. Haqiqiy o'q ustidagi maxsus nuqta - qutbni aylanib o'tish yo'lini quyidagicha tanlab olamiz:

$$G(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{(2\pi)^3} \lim_{\varepsilon \to 0} \int \frac{d^3q}{k^2 + i\varepsilon - q^2} e^{-i\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)}.$$

Bu holda qutblar $q^2 = k^2 + i\varepsilon$ tenglama orqali aniqlanadigan yangi nuqtalarga ko'chib o'tadi, ε - cheksiz kichik bo'lgani uchun ularni $q_1 = k + i\varepsilon$ va $q_2 = -k - i\varepsilon$ deb belgilaymiz. Yangi qutblar haqiqiy o'qdan q kompleks tekisligining yuqori va quyi yarimtekisliklariga siljiydi. Ularning yangi holati VIII.3-rasmda ko'rsatilgan. Shu rasmning o'zida Jordan lemmasidan foydalanib, q_1



VIII.3-rasm: Helmholtz operatorining Green funksiyasiga oid

va q_2 qutblar uchun konturlarni qanday yopish kerakligi ko'rsatilgan: Jordan lemmasidan kelib chiqadiki radiusi cheksizga intiltirilganda C_+ va C_- konturlar bo'yicha integrallar nolga teng.

Integralni hisoblaylik:

$$G(\mathbf{r}_1-\mathbf{r}_2)=\frac{1}{(2\pi)^3}\lim_{\varepsilon\to0}\int\limits_0^\infty\frac{q^2dq}{k^2+i\varepsilon-q^2}\int\limits_0^\pi\sin\theta d\theta\exp\left(-iq|\mathbf{r}_1-\mathbf{r}_2|\cos\theta\right)\int\limits_0^{2\pi}d\varphi=$$

$$=\frac{1}{i(2\pi)^2|\mathbf{r}_1-\mathbf{r}_2|}\lim_{\varepsilon\to0}\int\limits_0^\infty\frac{qdq}{q^2-k^2-i\varepsilon}\left(\exp\left(-iq|\mathbf{r}_1-\mathbf{r}_2|\right)-\exp\left(iq|\mathbf{r}_1-\mathbf{r}_2|\right)\right)=$$

$$=\frac{1}{2i(2\pi)^2|\mathbf{r}_1-\mathbf{r}_2|}\lim_{\varepsilon\to0}\int\limits_{-\infty}^\infty\frac{qdq}{q^2-k^2-i\varepsilon}\left(\exp\left(-iq|\mathbf{r}_1-\mathbf{r}_2|\right)-\exp\left(iq|\mathbf{r}_1-\mathbf{r}_2|\right)\right).$$

Integralni q_1 qutbni inobatga olib hisoblasak,

$$G(\mathbf{r}_1-\mathbf{r}_2)=-\frac{1}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\exp\left(ik|\mathbf{r}_1-\mathbf{r}_2|\right) \tag{46}$$

formulani olamiz, q_2 qutbni inobatga olsak,

$$\bar{G}(\mathbf{r}_1-\mathbf{r}_2)=-\frac{1}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\exp\left(-ik|\mathbf{r}_1-\mathbf{r}_2|\right) \tag{47}$$

javob topiladi. Bu ikki funksiyalar o'zaro kompleks qo'shmadir.

8.8-mashq.

$$\Delta G(\mathbf{r})+k^2G(\mathbf{r})=\delta(\mathbf{r})$$

tenglamaga (46)- va (47)-formulalarni qo'yib bevosita hisoblash orqali bu tenglamaning bajarilishini ko'rsating.

§8. Green formulalari

Elliptik tenglamani chegaralangan sohada yechganimizda chegaraviy shartlarni hisobga olishimiz kerak. Green funksiyasi metodida bu ish quyidagicha bajariladi.

Bizga ikkita $u(\mathbf{r})$ va $v(\mathbf{r})$ funksiyalar berilgan bo'lsin. Masalaning yechimi qidirilayotgan sohani G deb va uning chegarasini S deb belgilaylik. Unda

$$u,v\in C^2(G),\qquad u,v\in C(S)$$

deb talab qilamiz. Gauss teoremasidan foydalanib,

$$\int\limits_V dV u\Delta v = \int\limits_V dV\nabla(u\nabla v) - \int\limits_V dv\nabla u\cdot\nabla v = \int\limits_S u\nabla v\cdot d\mathbf{S} - \int\limits_V dV\nabla u\cdot\nabla v =$$

$$= \int\limits_S u\frac{\partial v}{\partial n}dS - \int\limits_V dV\nabla u\cdot\nabla v \tag{48}$$

formulani keltirib chiqarishimiz mumkin. Ba'zi-bir hollarda bu formula ***Greenning birinchi formulasi*** deyiladi. Agar bu formulada u va v funksiyalarning o'rinlarini almashtirib olinganini yuqoridagidan ayirsak,

$$\int\limits_V dV(u\Delta v - v\Delta u) = \int\limits_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \tag{49}$$

formulaga kelamiz. Buning nomi ***Greenning ikkinchi formulasi.***

§9. Chegaraviy masalaga Green formulalarini qo'llash

Quyidagi chegaraviy masalani ko'raylik:

$$\Delta u = f(\mathbf{r}) \quad \in G, \qquad \left(\alpha\frac{\partial u}{\partial n} + \beta u\right)_S = \varphi(r). \tag{50}$$

- Agar $\alpha = 0$, $\beta \neq 0$ bo'lsa bu Dirichlet masalasi;
- Agar $\alpha \neq 0$, $\beta = 0$ bo'lsa bu Neumann masalasi;
- Agar $\alpha \neq 0$, $\beta \neq 0$ bo'lsa bu uchinchi chegaraviy masala.

Albatta, har bir masala ichki yoki tashqi bo'lishi mumkin. Bu haqidagi informatsiya G sohaning ta'rifida berilgan bo'ladi, deb qaraymiz. Masalan, G soha R radiusli sharning ichi desak, ichki masala haqida gap ketayotgan bo'ladi. G soha sharning yoki ellipsoidning tashqarisi deyilgan bo'lsa, tashqi chegaraviy masala haqida gap ketayotgan bo'ladi.

Green funksiyasidan esa quyidagilar talab qilinadi:

$$\Delta G = \delta \quad \in G, \qquad \left(\alpha\frac{\partial G}{\partial n} + \beta G\right)_S = 0. \tag{51}$$

Avvalgi paragrafdagi (49)-formulada $v(\mathbf{r}) = G(\mathbf{r} - \mathbf{r}_0)$ deb olaylik. $\Delta G(\mathbf{r} - \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$ ni hisobga olsak, quyidagiga kelamiz:

$$u(\mathbf{r}_0) = \int\limits_V dV G(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r}) + \int\limits_S \left(u(\mathbf{r}) \frac{\partial G(\mathbf{r} - \mathbf{r}_0)}{\partial n} - G(\mathbf{r} - \mathbf{r}_0) \frac{\partial u(\mathbf{r})}{\partial n} \right) dS. \tag{52}$$

$\mathbf{r}_0$ nuqta G sohada yotibdi. Ikkinchi integral chegara S bo'yicha olinadi, integral ostidagi $\mathbf{r}$ o'zgaruvchi mana shu S sirtning ustida yotadi. n normal S sirtning har bir nuqtasida unga perpendikular bo'lgan yo'nalishga ega.

Agar ***birinchi chegaraviy masala*** ko'rilyapgan bo'lsa,

$$G\Big|_S = 0, \quad u\Big|_S = \varphi,$$

bo'ladi va yechim quyidagi holda aniqlanadi:

$$u(\mathbf{r}_0) = \int\limits_V dV G(\mathbf{r}_0 - \mathbf{r}) f(\mathbf{r}) + \int\limits_S dS \varphi(\mathbf{r}) \frac{\partial G(\mathbf{r}_0 - \mathbf{r})}{\partial n}. \tag{53}$$

Ikkinchi chegaraviy masala uchun

$$\frac{\partial G}{\partial n}\Big|_S = 0, \qquad \frac{\partial u}{\partial n}\Big|_S = \varphi.$$

Ammo Green funksiyasiga qo'yilgan bunday shart Green funksiyasining ta'rifi bo'lgan

$$\Delta_{\mathbf{r}} G(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

formulaga ziddir, chunki bu formulaga Gauss teoremasini qo'llasak,

$$\int\limits_V dV \Delta_{\mathbf{r}} G(\mathbf{r} - \mathbf{r}') = \int\limits_S \frac{\partial G}{\partial n} dS = 1$$

shart kelib chiqadi. Shuning uchun Green funksiyasi uchun

$$\frac{\partial G}{\partial n} = \frac{1}{S}$$

shart kiritish kerak, bu yerda S - V hajmni o'z ichiga olgan sirtning yuzasi. Agar tashqi Neumann masalasi haqida gap ketayotgan bo'lsa, unda $S = \infty$ va Green funksiyasi uchun ikkala shart bir xil bo'ladi. Agar ichki Neumann masalasi haqida gap ketayotgan bo'lsa, uning yechimi

$$u(\mathbf{r}_0) = \int\limits_V dV G(\mathbf{r}_0 - \mathbf{r}) f(\mathbf{r}) + \langle u \rangle_S - \int\limits_S dS \varphi(\mathbf{r}) G(\mathbf{r}_0 - \mathbf{r}). \tag{54}$$

bo'ladi, bu yerda $\langle u\rangle_S$ - u funksiyaning S sirt bo'yicha o'rtacha qiymati. Bu - konstanta, ichki Neauman masalasi ixtiyoriy konstantagacha aniqlanganligini muhokama qilgan edik.

Uchinchi chegaraviy masala uchun esa

$$\left.\frac{\partial u}{\partial n}\right|_S = -\frac{\beta}{\alpha}u|_S + \frac{\varphi}{\alpha}, \quad \left.\frac{\partial G}{\partial n}\right|_S = -\frac{\beta}{\alpha}G,$$

va

$$u(\mathbf{r}') = \int\limits_V dVG(\mathbf{r}' - \mathbf{r})f(\mathbf{r}) - \frac{1}{\alpha}\int\limits_S dSG\varphi. \tag{55}$$

(53)-, (54)- va (55)- formulalar birinchi, ikkinchi va uchinchi chegaraviy masalalarning yechimlarini beradi, ulardagi ikkinchi hadlar chegaraviy shartlarni o'z ichiga olgan. Ularga kirgan Green funksiyasi G Laplace operatorining cheksiz fazodagi Green funksiyasi $1/(-4\pi|\mathbf{r} - \mathbf{r}'|)$ ning o'zi emas, balki (51)-masalaning yechimidir. Bu yechimni biz

$$G = G_0 + v(\mathbf{r}), \qquad G_0(\mathbf{r}' - \mathbf{r}) = -\frac{1}{4\pi|\mathbf{r}' - \mathbf{r}|}, \qquad \Delta v = 0 \in G$$

ko'rinishda olishimiz kerak. Birinchi chegaraviy masala uchun $\left.G\right|_S = 0$ bo'lgani uchun

$$\left.v\right|_S = \left.\frac{1}{4\pi|\mathbf{r}' - \mathbf{r}|}\right|_S$$

deb olishimiz kerak.

Misollar

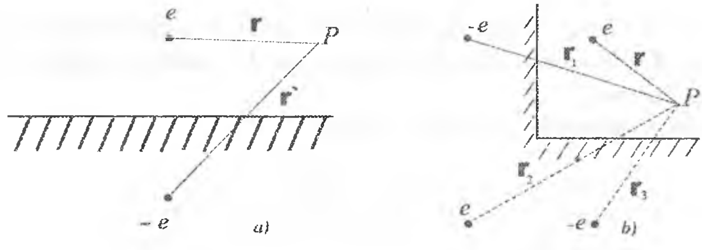
Ideal o'tkazgichdan tashqarida joylashgan zaryad

(VIII.4)-rasmning a) qismida o'tkazgich sirtidan tepada turgan e zaryadning P kuzatish nuqtasida hosil qilgan potensialini toping. Masalaning qo'yilishi:

$$\Delta\varphi = -4\pi e\delta(\mathbf{r}), \qquad \varphi|_S = 0.$$

Yechim:

$$\varphi(r) = \frac{e}{r} - \frac{e}{r'}.$$



VIII.4-rasm: e zaryad hosil qilgan potensial

Masalani yechish uchun biz mavhum zaryad $-e$ kiritdik, u tekislikdan pastda joylashgan. Elektromagnetizmda bunday metod akslantirish metodi deyiladi. Oydinki,

$$\Delta\frac{1}{r'} = -4\pi\delta(\mathbf{r}') = 0,$$

chunki masala berilgan soha - sirtdan yuqorida yotadi, P nuqtani yuqori yarim sirtning xohlagan joyida olsak ham hamma vaqt $r' \neq 0$ bo'ladi. Demak, $\varphi(r)$ tenglamani ham, chegaraviy shartni ham qanoatlantiradi.

To'g'ri burchakli ideal o'tkazgich uchun masala

Cheksiz to'g'ri burchakli sohada joylashgan e zaryad hosil qilgan maydonni toping, to'g'ri burchakning sirtida $\varphi = 0$. Bu masalaning yechimi

$$\varphi = e\left(\frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3}\right)$$

ko'rinishga ega bo'ladi ((VIII.4)-rasmning b) qismiga qarang). Fiktiv zaryadlar r_1, r_2, r_3 nuqtalarga qo'yilgan.

§10. Issiqlik tarqalishi masalasi

§10.1. Issiqlik tarqalishi operatorining fundamental yechimi

Issiqlik tarqalishi (diffuziya) tenglamasining

$$\left(\frac{\partial}{\partial t} - a^2\Delta\right) u(x, y, z, t) = f(x, y, z, t)$$

umumiy yechimini topish uchun issiqlik tarqalishi operatorining fundamental yechimini topamiz:

$$\left(\frac{\partial}{\partial t} - a^2\Delta\right) G(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t).$$

Bu tenglamaning ustida uch o'lchamli Fourier almashtirishi bajaramiz:

$$\int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}} \left(\frac{\partial}{\partial t} - a^2\Delta\right) G(\mathbf{r}, t) = \int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}}\delta(\mathbf{r})\delta(t) = \delta(t).$$

Green funksiyasi uchun quyidagi formulani ishlatib:

$$G(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\tilde{G}(\mathbf{k}, t)\exp(-i\mathbf{k}\cdot\mathbf{r})$$

(28)-formulani hisobga olib, yuqoridagi tenglamani

$$\left(\frac{\partial}{\partial t} + a^2k^2\right) \tilde{G}(\mathbf{k}, t) = \delta(t)$$

ko'rinishga keltiramiz. Bu tenglamaga §2.-paragrafning oxiridagi teoremaning birinchi qismini ishlatsak,

$$\tilde{G}(\mathbf{k}, t) = \theta(t)e^{-a^2k^2t}$$

ekanligini darhol topamiz. Demak, izlanayotgan fundamental yechimning integral tasavvuri quyidagicha ko'rinishga ega ekan:

$$G(\mathbf{r}, t) = \theta(t)\int \frac{d^3k}{(2\pi)^3}\, e^{-a^2k^2t - i\mathbf{k}\cdot\mathbf{r}}.$$

$\mathbf{k}$ vektor bo'yicha integralni hisoblash qiyin emas. Buning uchun eksponentadagi ifodani to'liq kvadratga keltirish yetarli:

$$G(\mathbf{r}, t) = \theta(t)\int \frac{d^3k}{(2\pi)^3}\, e^{-a^2(\mathbf{k} + i\mathbf{r}/(2a^2t))^2t - r^2/(4a^2t)} = \frac{\theta(t)}{(2a\sqrt{\pi t})^3}\, e^{-\frac{r^2}{4a^2t}}.$$

§10.2. Cauchy masalasining yechimi

Cheksiz fazoda $u_t - a^2\Delta u = f$ tenglama uchun Cauchy masalasining yechimini Green funksiyasi yordamida ifodalab olishga hamma narsa tayyor. Cheksiz fazoda bu tenglamaga faqat boshlang'ich shart $u(\mathbf{r}, 0) = \varphi(\mathbf{r})$ beriladi, δ-funksiya yordamida bu shartni tenglamaning o'ng tomonidagi manbaga qo'shib qo'yishimiz mumkin:

$$u_t(\mathbf{r}, t) - a^2\Delta u(\mathbf{r}, t) = f'(\mathbf{r}, t), \qquad f'(\mathbf{r}, t) = f(\mathbf{r}, t) + \delta(t)\varphi(\mathbf{r}). \qquad (56)$$

Ko'rinib turibdiki, bu formula boshlang'ich shartni $t = 0$ vaqt momentida ta'sir qiluvchi manba sifatida talqin qilishni taklif etadi. Fizik mulohazalar nuqtai-nazaridan bu talqin manbaning ham, boshlang'ich shartning ham ma'nosiga mos keladi. (56)-formulalar issiqlik tarqalishi (diffuziya) masalasining ma'lum bir qo'yilishiga mos keladi. Bu masalaning yechimi (cheksiz fazoda, chegaralar yo'q!) Green funksiyasi metodida quyidagicha ifodalanadi:

$$u(\mathbf{r},t) = \int d\tau \int d^3r' G(\mathbf{r}-\mathbf{r}', t-\tau) f'(\mathbf{r}', \tau) =$$

$$= \frac{1}{(2a\sqrt{\pi})^3} \int_0^t \frac{d\tau}{(t-\tau)^{3/2}} \int d^3r' f'(\mathbf{r}', \tau) \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^2}{4a^2(t-\tau)}\right). \tag{57}$$

Bu formulaga f' ning ta'rifini qo'yamiz:

$$u(\mathbf{r},t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^3} \int d^3r' \varphi(\mathbf{r}') \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^2}{4a^2 t}\right) +$$

$$+ \frac{1}{(2a\sqrt{\pi})^3} \int_0^t \frac{d\tau}{(t-\tau)^{3/2}} \int d^3r' f(\mathbf{r}', \tau) \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^2}{4a^2(t-\tau)}\right). \tag{58}$$

(57)-formulada integral ostida $\delta(\tau)$ ni ishlatganimizda τ ni nolga yuqoridan intiltiramiz deb hisoblash kerak, buni odatda quyidagicha belgilanadi: $\tau \to 0^+$. Olingan formula uch o'lchamli fazoga tegishli, uni bir o'lchamli fazo uchun yozib olish qiyin emas:

$$u(x,t) = \frac{\theta(t)}{2a\sqrt{\pi t}} \int dx' \varphi(x') \exp\left(-\frac{(x-x')^2}{4a^2 t}\right) +$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{3/2}} \int dx' f(x', \tau) \exp\left(-\frac{(x-x')^2}{4a^2(t-\tau)}\right). \tag{59}$$

§10.3. Chegaraviy shartlar

Chegaraviy shartlarni muhokama qilish uchun issiqlik tarqalishi tenglamasini (3)-formada yozib olaylik ($k =$ const deb olamiz):

$$c\rho \frac{\partial u}{\partial t} = k\Delta u + F.$$

Eslatib o'tamiz, c - muhitning issiqlik sig'imi (muhitni muvozanatda turibdi deb qaraganimiz uchun $c = c_P$ bo'lishi kerak), ρ—muhit zichligi, k - issiqlik tarqalish koeffisienti, F- manba zichligi.

Muvozanatda turgan va umumiy chegaraga ega bo'lgan ikkita muhit berilgan bo'lsin. Ikkita muhitning chegarsida temperaturalar teng bo'lishi kerak (muvozanat sharti):

$$u_1 = u_2.$$

Undan tashqari, bir muhitdan (k_1) chiqayotgan issiqlik oqimi ikkinchi muhitga (k_2) kirib kelayotgan issiqlik oqimiga teng bo'lishi kerak. Ixtiyoriy sirt elementi $d\mathbf{S}$ uchun buni

$$k_1 \nabla u_1 d\mathbf{S} = k_2 \nabla u_2 d\mathbf{S}$$

ko'rinishda yozib olish mumkin. Gradientning sirt elementiga proyeksiyasi shu sirtga normal hosila bo'ladi, shuning uchun bu chegaraviy shartni

$$k_1 \frac{\partial u_1}{\partial n} = k_2 \frac{\partial u_2}{\partial n}$$

ko'rinishda olish qulaydir.

Agar muhitlar chegarasida tashqi issiqlik manbalari mavjud bo'lsa

$$k_1 \frac{\partial u_1}{\partial n} - k_2 \frac{\partial u_2}{\partial n} = q^{(s)}$$

bo'ladi, bu yerda $q^{(s)}$ - manbaning sirt zichligi.

§10.4. Xususiy hollar

Boshlang'ich temperatura faqat bitta koordinataga bog'liq

Faraz qilaylik, tashqi manba bo'lmasin va

$$\varphi(\mathbf{r}) = \varphi(x)$$

bo'lsin. Bu holda $u(x, y, z, t)$ temperatura ham koordinatlardan faqat x ning funksiyasi bo'lib chiqadi:

$$u(x,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^3} \int d^3r' \varphi(x') \exp\left(-\frac{(\mathbf{r} - \mathbf{r}')^2}{4a^2 t}\right) =$$

$$= \frac{\theta(t)}{2a\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} dx' \varphi(x') \exp\left(-\frac{(x - x')^2}{4a^2 t}\right).$$

Biz bu natijani olishda yaxshi ma'lum bo'lgan

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{x^2}{a}\right) = \sqrt{\frac{\pi}{a}}$$

formulani ikki marta ishlatdik.

Endi faraz qilaylik, butun boshlang'ich issiqlik $x = 0$ nuqta atrofidagi cheksiz kichik qatlamda mujassamlangan bol'sin, buni $\varphi(x) = \varphi_0\delta(x)$ orqali ifodalash mumkin. Bu holda

$$u(x,t) = \frac{\theta(t)}{2a\sqrt{\pi t}}\varphi_0 \exp\left(-\frac{x^2}{4a^2t}\right).$$

Issiqlik (modda) $x = 0$ nuqtadan $x = l$ nuqtaga yetib borgan bo'lsin, l nuqtadagi temperatura (konsentratsiya) boshlang'ich temperatura (konsentratsiya) φ_0 bilan solishtirilganda sezilarli bo'lishi uchun eksponentadagi faktorning tartibi bir atrofida bo'lishi kerak: $l^2/a^2t \sim 1$. Demak,

$$l \sim \sqrt{t}a = \sqrt{\frac{kt}{c\rho}}. \tag{60}$$

Diffuziya masalalari uchun

$$l \sim \sqrt{tD}. \tag{61}$$

Ya'ni, issiqlikning (modda konsentratsiyasining) tarqalish sohasi o'lchamining tartibi vaqtdan olingan ildizga proporsional ekan.

Agar boshlang'ich temperatura (konsentratsiya) bitta nuqtadagina noldan farqli bo'lsa (issiqlikning ma'lum bir miqdori $\mathbf{r} = 0$ nuqtada mujassamlashgan bo'lsa),

$$\varphi(\mathbf{r}) = \varphi_0\delta(\mathbf{r})$$

bo'ladi va (58)-formula bu holda quyidagi natijaga olib keladi:

$$u(r,t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^3}\varphi_0 \exp\left(-\frac{r^2}{4a^2t}\right).$$

t vaqt ichida issiqlik (modda) tarqalishi sohasi o'lchami uchun (60)- va (61)-formulalar bu holda ham o'rinli, faqat endi l markazgacha masofani bildiradi.

Olingan formulaning fizik ma'nosini talqin qilaylik. Bu formuladagi φ_0 had o'zining kelib chiqishi bo'yicha issiqlik manbai zichligi intensivligi F bilan quyidagicha bog'langan:

$$F(\mathbf{r},t) = c\rho\,\varphi_0\delta(\mathbf{r})\delta(t).$$

$\delta-$funksiyalarning o'lchamliklarini hisobga olsak, ($[\delta(x)] = [x]^{-1}$) $c\rho\varphi_0$ ning o'lchamligi issiqlik miqdori Q ning o'lchamligi bilan bir xil bo'lib chiqadi (SI sistemasida $[Q]$ = Joule, CGS sistemasida $[Q]$ = erg). $Q = 1$ deb olaylik, bu holda $r = 0$ nuqtaga $t = 0$ vaqt momentida (oniy) birlik issiqlik miqdori kiritilsa, u hosil qilgan temperatura $t > 0$ da fazoda quyidagicha taqsimlangan bo'ladi:

$$u(r,t) = \frac{1}{c\rho}\frac{1}{(2a\sqrt{\pi t})^3}\exp\left(-\frac{r^2}{4a^2t}\right).$$

$c\rho u$ dan olingan integral butun issiqlik miqdori Q ni berishi kerak, rostdan ham bu formuladan butun fazo bo'yicha integral hisoblasak, $Q = 1$ ni olamiz.

Yarim-fazodagi issiqlik taqsimoti: birinchi tur chegaraviy shart

$x \geq 0$ yarim-fazo uchun birinchi chegaraviy masala berilgan bo'lsin: $x = 0$ sirtda ma'lum temperatura berilgan, uni nolga teng deb olamiz - $u(0,y,z;t) = 0$, $x > 0$ sohada temperaturaning boshlang'ich taqsimoti berilgan - $u(x,y,z;0) = \varphi(x,y,z)$.

Umumiy formulalarni qo'llash maqsadida, masalaga $x < 0$ soha ham quyidagi yo'l bilan kiritiladi: faraz qilaylik, $t = 0$ vaqt momentida $x < 0$ sohada ham temperatura taqsimoti berilgan bo'lsin, va u taqsimot

$$\varphi(-x,y,z;0) = -\varphi(x,y,z;0) \tag{62}$$

shartga bo'ysunsin. Bundan chegaraviy shartning $t = 0$ da avtomatik ravishda bajarilishi kelib chiqadi:

$$u(0,y,z;0) = \varphi(0,y,z) = -\varphi(0,y,z) = 0.$$

Boshlang'ich taqsimotning simmetriyasidan kelib chiqadiki, bu chegaraviy shart ixtiyotiy $t > 0$ da ham bajariladi. (58)-yechimda $f = 0$ deb olamiz va dx' bo'yicha integralni ikki qismga bo'lamiz: $-\infty$ dan 0 gacha va 0 dan ∞ gacha va (62)-shartni ishlatamiz:

$$u(x,y,z;t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^3}\int\limits_{-\infty}^{\infty} dy' \int\limits_{-\infty}^{\infty} dz' \exp\left(-\frac{(y-y')^2 + (z-z')^2}{4a^2t}\right)\cdot$$

$$\int\limits_0^{\infty} dx'\varphi(x',y',z')\left[\exp\left(-\frac{(x-x')^2}{4a^2t}\right) - \exp\left(-\frac{(x+x')^2}{4a^2t}\right)\right].$$

Agar temperaturaning boshlang'ich taqsimoti faqat x ga bo'gliq bo'lsa bu formulada y' va z' bo'yicha integrallarni hisoblab tashlash mumkin:

$$u(x,y,z;t) = \frac{\theta(t)}{2a\sqrt{\pi t}} \int_0^\infty dx'\varphi(x') \left[\exp\left(-\frac{(x-x')^2}{4a^2t}\right) - \exp\left(-\frac{(x+x')^2}{4a^2t}\right)\right].$$

Yarim-fazodagi issiqlik taqsimoti: ikkinchi tur chegaraviy shart

Fazo $x = 0$ tekislik bilan ikki qismga bo'lingan bo'lsin. Masalani quyidagicha qo'yamiz: $x > 0$ yarim tekislikda boshlang'ich temperatura $\varphi(x,y,z)$ berilgan, $x = 0$ tekislik issiqlik o'tkazmaydigan bo'lganda $x > 0$ yarim tekislikda temperatura taqsimotini toping. Bu shartlarni matematik ko'rinishga keltiraylik:

$$c\rho\frac{\partial u}{\partial t} = k\Delta u; \quad u_0(x,y,z) = u(x,y,z;0) = \varphi(x,y,z), \;\; x > 0; \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0. \tag{63}$$

Chegaraviy shartni qanoatlantirish uchun masalani $x < 0$ sohaga simmetrik ravishda davom ettiramiz, buning uchun boshlang'ich shartni

$$\varphi(-x,y,z) = \varphi(x,y,z)$$

deb olsak, yetarlidir. Bu holda

$$\left.\frac{\partial u_0}{\partial x}\right|_{x=0} = \frac{\partial\varphi(0,y,z)}{\partial x} = -\frac{\partial\varphi(0,y,z)}{\partial x} = 0$$

bo'ladi. Masalaning simmetriyasidan ixtiyoriy vaqt momentida ham $(\partial u/\partial x)\Big|_{x=0} = 0$ bo'lishi kelib chiqadi. Demak, boshlang'ich shartni simmetrik ravishda butun fazoga kengaytirsak, chegarviy shart bajarilgan bo'lib chiqadi. Shuning uchun (63)-masalaning yechimi (58)-formuladan osongina olinadi:

$$u(x,y,z;t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^3} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \exp\left(-\frac{(y-y')^2 + (z-z')^2}{4a^2t}\right)\cdot$$

$$\cdot \int_0^\infty dx'\varphi(x',y',z') \left[\exp\left(-\frac{(x-x')^2}{4a^2t}\right) + \exp\left(-\frac{(x+x')^2}{4a^2t}\right)\right].$$

Chegaraviy shart $(\partial u/\partial x)\Big|_{x=0} = 0$ ning bajarilishi ko'rinib turibdi.

Ikkinchi chegraviy masalaning umumiy ko'rinishiga o'tish mumkin: $x = 0$ chegarada vaqtning ma'lum funksiyasi bo'lgan issiqlik oqimi berilgan bo'lsin:

$$-k\frac{\partial u}{\partial x}\bigg|_{x=0} = q(t).$$

Boshlang'ich shart:

$$u_0(x, y, z) = u(x, y, z; 0) = 0.$$

Agar boshlang'ich temperatura noldan farqli bo'lsa, uni yechimga qo'shib qo'yish qiyin emas.

$x = 0$ tekisligida berilgan issiqlik oqimini shu tekislik bo'yicha taqsimlangan manba sifatida qaraymiz: $f \sim q(t)\delta(x)$. O'lchamliklarning tahlilidan

$$f \sim \frac{1}{c\rho}q(t)\delta(x) \sim \frac{a^2}{k}q(t)\delta(x)$$

bo'lishi kerakligini topish mumkin. Ammo bu manbadan chiqayotgan issiqlik oqimi $x > 0$ sohaga ham $x < 0$ sohaga ham ketayapti, shuning uchun uni 2 ga ko'paytirishimiz kerak:

$$f = \frac{2}{c\rho}q(t)\delta(x) = \frac{2a^2}{k}q(t)\delta(x)$$

Manba uchun bu formulani (58)-ga qo'ysak berilgan masalaning yechimi topiladi (integral ostida y, z larga bog'liqlik yo'q bo'lgani uchun ular b'oyicha integrallab tashlandi):

$$u(x, y, z; t) = \frac{a}{k\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}}q(\tau)\exp\left(-\frac{x^2}{4a^2(t-\tau)}\right).$$

Masalan, $x = 0$ nuqtada ((y, z) tekislikda), temperatura

$$u(0, y, z; t) = \frac{a}{k\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}}q(\tau)$$

bo'ladi. Agar $q = \text{const}$ bo'lsa,

$$u(0, y, z; t) = \frac{aq}{k}\sqrt{\frac{t}{\pi}}$$

bo'ladi.

§11. Uch o'lchamli fazoda to'lqin tarqalishi masalasi

§11.1. To'lqin operatorining fundamental yechimi

Uch o'lchamli fazoda to'lqin tarqalishini (11)-tenglama ifodalashini keltirib chiqargan edik. U tenglamada a parametr to'lqin tarqalishi tezligi edi. Endi uni c harfi bilan belgilaylik. Quyidagi operator

$$\frac{\partial^2}{c^2\partial t^2} - \Delta = \frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

D'Alémbert, yoki ***to'lqin operatori*** deyiladi. Shu operatorning fundamental yechimi $G(t, \mathbf{r})$ ni topaylik. Uni quyidagicha ta'riflaymiz:

$$\left(\frac{\partial^2}{c^2\partial t^2} - \Delta\right) G(t, \mathbf{r}) = \delta^{(4)}(t, \mathbf{r}).$$

Bu yerda to'rt o'lchamli delta-funksiya o'zining odatdagi ma'nosiga ega: $\delta^{(4)}(t, \mathbf{r}) = \delta(t)\delta(\mathbf{r})$. Tenglama ustida yana uch o'lchamli Fourier-almashtirish bajaramiz:

$$\int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}} \left(\frac{\partial^2}{c^2\partial t^2} - a^2\Delta\right) G(\mathbf{r}, t) = \int d^3r\, e^{i\mathbf{k}\cdot\mathbf{r}} \delta(\mathbf{r})\delta(t) = \delta(t).$$

Green funksiyasini ham uch o'lchamli Fourier-integraliga yoyamiz:

$$G(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3} \tilde{G}(\mathbf{k}, t) \exp(-i\mathbf{k} \cdot \mathbf{r}).$$

Natijada,

$$\left(\frac{\partial^2}{c^2\partial t^2} + \mathbf{k}^2\right) \tilde{G}(\mathbf{k}, t) = \delta(t)$$

tenglamani olamiz. 133-betdagi teorema bo'yicha bu tenglamaning yechimi

$$\tilde{G}(\mathbf{k}, t) = \theta(t)\frac{c\sin(ckt)}{k}, \qquad k = \sqrt{\mathbf{k}^2}. \tag{64}$$

Quyidagi integralni hisoblash qoldi:

$$G(\mathbf{r}, t) = \theta(t) \int \frac{d^3k}{(2\pi)^3} \frac{c\sin(ckt)}{k} \exp(-i\mathbf{k} \cdot \mathbf{r}).$$

Sferik sistemaga o'tib avval burchaklar bo'yicha integrallarni hisoblaymiz:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \exp(-ikr\cos\theta) = \frac{2\pi}{kri}\left(e^{ikr} - e^{-ikr}\right).$$

Qolgan integral:

$$G(\mathbf{r},t) = \frac{c\,\theta(t)}{4\pi^2 ir}\int\limits_0^\infty dk\sin(ckt)\left(e^{ikr} - e^{-ikr}\right) =$$

$$= \frac{c\,\theta(t)}{8\pi^2 r}\int\limits_0^\infty dk\left(e^{-ikct} - e^{ikct}\right)\left(e^{ikr} - e^{-ikr}\right) =$$

$$= \frac{c\,\theta(t)}{16\pi^2 r}\int\limits_{-\infty}^\infty dk\left(e^{-ikct} - e^{ikct}\right)\left(e^{ikr} - e^{-ikr}\right) = \tag{65}$$

$$= \frac{c\,\theta(t)}{16\pi^2 r}\int\limits_{-\infty}^\infty dk\left(e^{ik(r-ct)} + e^{-ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)}\right) =$$

$$= \frac{c\,\theta(t)}{4\pi r}\left(\delta(ct - r) - \delta(ct + r)\right) = \frac{c^2\theta(t)}{2\pi}\delta(c^2t^2 - r^2).$$

Haqiqatda oxirgi tenglik simvolik ahamiaytga ega, chunki $\theta(t)$ borligi uchun $t > 0$ va ikkinchi delta-funksiyaning argumenti hech qachon nolga teng bo'lishi mumkin emas. Shuning uchun haqiqatda

$$G(\mathbf{r},t) = \frac{c\,\theta(t)}{4\pi r}\delta(ct - r) \tag{66}$$

deb olishimiz kerak. Ammo yuqoridagi (65)-formula o'zining Lorentz-invariantligi sababli ko'p hollarda qulay bo'lishi mumkin.

Formulaga $\theta(t)$ kirgani uchun $t < 0$ da $G(\mathbf{r},t) = 0$ bo'ladi. Bunday Green funksiyalari ***kechikuvchi*** deyiladi.

§11.2. Ixtiyoriy harakatdagi zaryadlar hosil qilgan maydon

$\mathbf{j}(\mathbf{r},t)$ tok zichligi (harakatdagi zaryad) hosil qilgan kechikuvchi vektor-potensialni topaylik. Elektromagnit maydon vektor-potensiali quyidagi tenglamaga bo'ysunadi:

$$\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{c^2\partial t^2} - \Delta\mathbf{A}(\mathbf{r},t) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t). \tag{67}$$

Green funksiyasi metodi bo'yicha bu tenglamaning yechimi

$$\mathbf{A}(\mathbf{r},t) = \frac{4\pi}{c}\int G(\mathbf{r}-\mathbf{r}', t-t')\,\mathbf{j}(\mathbf{r}',t')\,d^3r'dt'$$

ga teng. Green funksiyasi uchun (66)-formulani ishlatamiz:

$$\mathbf{A}(\mathbf{r},t) = \int\limits_{-\infty}^{t} dt' \int \frac{\delta\left[c(t-t') - |\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}',t')\,d^3r'.$$

VIII.1-mashq natijasidan foydalanib yorug'lik tezligi c ni delta-funksiya argumentidan chiqarib tashlaymiz:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int\limits_{-\infty}^{t} dt' \int \frac{\delta\left[(t-|\mathbf{r}-\mathbf{r}'|/c) - t'\right]}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}',t')\,d^3r'.$$

t' bo'yicha integralni delta-funksiya yordamida hisoblash qiyin emas, natijada,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int d^3r'\,\frac{\mathbf{j}\left(\mathbf{r}', t - \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} \tag{68}$$

formulani olamiz. Topilgan ifoda ***kechikuvchi potensial*** deyiladi. Bunday nomning sababi integral ostidagi tokning vaqt argumentida - t vaqtda $\mathbf{r}$ nuqtadagi potensialni $t - |\mathbf{r}-\mathbf{r}'|/c$ vaqtda $\mathbf{r}'$ nuqtada turgan zaryadlar hosil qiladi. Bu formulada elektromagnit maydonning yorug'lik tezligi bilan harakat qilishi hisobga olingan bo'lib chiqayapti: $\mathbf{r}'$ nuqtadan $\mathbf{r}$ nuqtaga yetib kelish uchun maydon $|\mathbf{r}-\mathbf{r}'|/c$ vaqt sarf qilishi kerak, u kechikib keladi.

Olingan formula (68) da tok zichligi vaqtga bog'liq bo'lmasin, deb faraz qilaylik: $\mathbf{j} = \mathbf{j}(\mathbf{r})$. Bu holda

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{69}$$

formulaga kelamiz. Olingan natija Laplace operatorining Green funksiyasi orqali ham olinishi mumkin edi. Bunga ishonch hosil qilish qiyin emas - vaqtga bog'liqlik yo'qligida (67)-tenglama Poisson tenglamasiga aylanadi:

$$\Delta\mathbf{A}(\mathbf{r}) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r}).$$

(33)-, (34)- va (35)-formulalarni eslash qoldi, ularni qo'llasak, yana (69)-yechimga kelinadi.

Yana bir xususiy holni ko'rib chiqaylik - harakatdagi zaryadlar hosil qilgan maydonni monoxromatik to'lqinlarga yo'yish masalasini. Monoxromatik to'lqin vaqtga sodda bo'lgan $e^{-i\omega t}$ ko'rinishda bog'liq bo'ladi, bu yerda ω - mana shu to'lqinning chastotasi:

$$\mathbf{A}(\mathbf{r},t) = e^{-i\omega t}\mathbf{A}(\mathbf{r}).$$

Albatta tok zichligi ham vaqtga shunday ko'rinishda bog'liq bo'lishi kerak:

$$\mathbf{j}(\mathbf{r},t) = e^{-i\omega t}\mathbf{j}(\mathbf{r}).$$

(68)-yechimda mana shu almashtirishlarni bajarsak va $\omega = ck$ formula orqali to'lqin vektori kiritsak kechikuvchi maydonning monoxromatik komponentasi uchun

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}') \tag{70}$$

formulani olamiz.

Xuddi shu formulaga boshqa nuqtai nazardan kelish mumkin. **A** va **j** lar uchun monoxromatik holdagi vaqtga bog'liqlikni (67)-tenglamaga qo'ysak, u Helmholtz tenglamasiga aylanadi:

$$(\Delta + k^2)\mathbf{A}(\mathbf{r}) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{r}).$$

Bu tenglamaga Helmholtz operatorining Green funksiyasi (46)-ni qo'llasak yana (70)-yechimni olamiz.

§11.3. Kirchhoff formulasi

Yuqorida topilgan hamma yechimlar birjinsli bo'lmagan ((67)- va undan olingan) tenglamalarning xususiy integrallaridir. Birjinsli bo'lmagan tenglamaning umumiy yechimi mana shu xususiy yechim + birjinsli tenglamaning umumiy yechimi bo'lishi kerak. Shu yechimni topaylik.

Masalaning qo'yilishi:

$$\frac{\partial^2 u(\mathbf{r},t)}{c^2\partial t^2} - \Delta u(\mathbf{r},t) = f(\mathbf{r},t), \qquad u\Big|_{t=0} = u_0(\mathbf{r}), \quad u_t\Big|_{t=0} = u_1(\mathbf{r}). \tag{71}$$

Bu - Cauchy masalasi. Green funksiyasi metodidan unumli foydalanish maqsadida boshlang'ich shartlarni ***oniy ta'sir qiluvchi*** manbalar sifatida qaraymiz va (71)-tenglamadagi f manbani quyidagi umumlashgan $\tilde{f}$ manbaga almashtiramiz:

$$\tilde{f}(\mathbf{r},t) = f(\mathbf{r},t) + \frac{1}{c^2}\delta(t)u_1(\mathbf{r}) + \frac{1}{c^2}\delta'(t)u_0(\mathbf{r}).$$

Bunday yondashishga asos quyidagicha. Tenglamani yangi o'ng tomon bilan yozib olaylik:

$$\frac{\partial^2 \tilde{u}(\mathbf{r},t)}{c^2 \partial t^2} - \Delta \tilde{u}(\mathbf{r},t) = f(\mathbf{r},t) + \frac{1}{c^2}\delta(t)u_1(\mathbf{r}) + \frac{1}{c^2}\delta'(t)u_0(\mathbf{r}). \qquad (72)$$

Tenglamaning o'ng tomoni o'zgargani uchun uning yechimini ham boshqa harf bilan belgiladik. Bu tenglamani t bo'yicha $-\varepsilon$ dan ε gacha integrallab $\varepsilon \to 0$ limitga o'tamiz. $t = -\varepsilon$ da $u(\mathbf{r},t)$ va uning hamma hosilalari nolga teng, natijada, faqat quyidagi hadlar qoladi:

$$\left.\frac{\partial \tilde{u}(\mathbf{r},t)}{\partial t}\right|_{t=0} = u_1(\mathbf{r}). \qquad (73)$$

Delta-funksiyaning hosilasi kirgan oxirgi hadning nolga tengligi ham oydindir:

$$\int_{-\varepsilon}^{\varepsilon} dt\delta'(t)u_0(\mathbf{r}) = -\int_{-\varepsilon}^{\varepsilon} dt\delta(t)\frac{d}{dt}u_0(\mathbf{r}) = 0.$$

Endi oxirgi hadni chap tomonga o'tkazamiz:

$$\frac{\partial}{c^2\partial t}\left(\frac{\partial \tilde{u}(\mathbf{r},t)}{\partial t} - \delta(t)u_0(\mathbf{r})\right) = \Delta\tilde{u}(\mathbf{r},t) + f(\mathbf{r},t) + \frac{1}{c^2}\delta(t)u_1(\mathbf{r})$$

Chap tomondagi ifodaning vaqt bo'yicha hosilasi delta-funksiya kirmagan bitta funksiya va delta-funksiyali haddan iborat, (17)-formulani eslasak, oxirgi tenglamaning integrali

$$\frac{\partial \tilde{u}(\mathbf{r},t)}{\partial t} - \delta(t)u_0(\mathbf{r}) = g_1(\mathbf{r},t)) + \theta(t)u_1(\mathbf{r})$$

ko'rinishga ega bo'lishi kerakligini tushunish qiyin emas, bu yerda g_1 - bir uzluksiz funksiya. Shu yerda yuqoridagi amalni yana bir marta bajaramiz: bu tenglikni t bo'yicha $-\varepsilon$ dan ε gacha integrallab, $\varepsilon \to 0$ limitga o'tamiz. Natijada, quyidagi hosil bo'ladi:

$$\tilde{u}(\mathbf{r},0) = u_0(\mathbf{r}). \qquad (74)$$

(73)- va (74)-formulalar $u(\mathbf{r},t)$ funksiyasiga qo'yilgan boshlang'ich shartlarning o'zi, shuning uchun (72)-tenglamaning yechimi (71)-tenglamaning yechimining o'zi. Bu mulohazalar boshlang'ich shartlarni (72)-tenglamaga o'tish yo'li bilan hisobga olishning to'la-to'kis isboti emas, aniq isbotni [3] kitobning §13 da topish mumkin.

(72)-tenglamaning qulayligi unga boshlang'ich shartlar manbaning qismi sifatida bevosita kiritilgan, bu esa bu tenglamaga Green funksiyasi metodini bevosita qo'llash imkoniyatini beradi:

$$u(\mathbf{r}, t) = \int dt' d^3r' G(\mathbf{r} - \mathbf{r}', t - t') \bar{f}(\mathbf{r}', t').$$

Bu yerda Green funksiyasi sifatida (66)-formula olinadi. Integral osti uchta haddan iborat, ularning birinchisi elektromagnit potensial misolida keltirib chiqarilgan (68)-formula ko'rinishga ega (faqat koeffisient o'zgaradi), ikkinchi va uchinchi hadlarning ustida esa quyidagi amallarni bajaramiz. Ikkinchi had:

$$\frac{1}{4\pi c} \int \frac{\theta(t - t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left[c(t - t') - |\mathbf{r} - \mathbf{r}'|\right] \delta(t') u_1(\mathbf{r}') dt' d^3r'.$$

Birinchi delta-funksiyaning argumentida $|\mathbf{r} - \mathbf{r}'|$ ga bog'liq bo'lgan hadning mavjudligi d^3r' bo'yicha integraldan radiusi $|\mathbf{r} - \mathbf{r}'| = c(t - t')$ bo'lgan sirt $dS_{r'}$ bo'yicha integralga o'tishga imkon beradi:

$$\frac{1}{4\pi c^2} \int \frac{\theta(t - t')}{t - t'} \delta(t') u_1(\mathbf{r}') dt' dS_{r'}.$$

$u_1(\mathbf{r}')$ ning argumenti mana shu sirtning ustida yotibdi. Delta-funksiyadan foydalanib vaqt bo'yicha integralni oson hisoblaymiz:

$$\frac{1}{4\pi c^2 t} \int dS_{\{r=ct\}} u_1(\mathbf{r}).$$

Vaqt bo'yicha delta-funksiyani ishlatgandan keyin integrallash sirti $r = ct$ radiusli sferaga aylanadi. Uchinchi had:

$$\frac{1}{4\pi c} \int \frac{\theta(t - t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left[c(t - t') - |\mathbf{r} - \mathbf{r}'|\right] \delta'(t') u_0(\mathbf{r}') dt' d^3r'.$$

Yana birinchi delta-funksiyadan foydalanib d^3r' bo'yicha integraldan radiusi $|\mathbf{r} - \mathbf{r}'| = c(t - t')$ bo'lgan sirt $dS_{r'}$ bo'yicha integralga o'tamiz:

$$\frac{1}{4\pi c^2} \int \frac{\theta(t - t')}{t - t'} \delta'(t') u_0(\mathbf{r}') dt' dS_{r'}.$$

Vaqt bo'yicha integralga delta-funksiya o'zining hosilasi bilan kirgan, (14)-formulani $n = 1$ holda qo'llasak yuqoridagi ifoda quyidagi holga keladi:

$$-\frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left[\frac{1}{t} \int u_0(\mathbf{r}) dS_{\{r=ct\}}\right].$$

Olingan hamma formulalarni bir joyga yig'ib quyidagi ***Kirchhoff***[6] ***formulasi*** deyiladigan yechimga kelamiz:

$$u(\mathbf{r},t)=\int dt'd^3r'G(\mathbf{r}-\mathbf{r}',t-t')\tilde{f}(\mathbf{r}',t')=\frac{1}{4\pi}\int d^3r'\frac{f\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|}+$$

$$+\frac{1}{4\pi c^2t}\int dS_{\{r=ct\}}u_1(\mathbf{r})-\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\int dS_{\{r=ct\}}u_0(\mathbf{r})\right].\tag{75}$$

§12. Ikki o'lchamli fazo uchun to'lqin tenglamasining yechimi

Ikki o'lchamli fazoda to'lqin tenglamasi uchun Cauchy masalasi quyidagicha qo'yiladi:

$$\frac{\partial^2u(x,y,t)}{c^2\partial t^2}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y,t)=f(x,y,t),$$

$$u\Big|_{t=0}=u_0(x,y),\qquad u_t\Big|_{t=0}=u_1(x,y);\qquad u\in C^2R\times T.$$

Tenglamaga kirgan operatorning fundamental yechiminining ta'rifi:

$$\left(\frac{\partial^2}{c^2\partial t^2}-\Delta_2\right)G_2(\mathbf{r},t)=\delta(\mathbf{r})\delta(t),\quad \mathbf{r}=\{x,y\},\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

$\mathbf{r}=\{x,y\}$ bo'yicha Fourier almashtirish bajaramiz:

$$\left(\frac{\partial^2}{c^2\partial t^2}+k^2\right)\tilde{G}_2(\mathbf{k},t)=\delta(t).$$

(25)-formuladan foydalanib quyidagini olamiz:

$$\tilde{G}_2(k,t)=c\theta(t)\frac{\sin(ckt)}{k}.$$

8.9-mashq.

$$\int\frac{\theta(R-r)}{\sqrt{R^2-r^2}}e^{i\mathbf{k}\cdot\mathbf{r}}d^2r=2\pi R\int\limits_0^{\pi/2}d\theta\cos\theta J_0(kR\cos\theta)$$

ekanligini ko'rsating.

[6] Gustav Robert Kirchhoff (1824-1887) - buyuk nemis fizigi. Rus tilida - Кирхгоф.

8.10-mashq.

$$\int_0^{\pi/2} d\theta \cos\theta J_0(x\cos\theta) = \frac{\sin x}{x}$$

ekanligini ko'rsating.

Shu ikkala mashqning natijalaridan foydalanib Green funksiyasini olamiz:

$$G_2(\mathbf{r},t) = c\theta(t)\int \frac{\sin(kct)}{k} e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{d^2k}{(2\pi)^2} = \frac{c}{2\pi}\frac{\theta(ct-r)}{\sqrt{c^2t^2-r^2}}.$$

Formulada $\theta(ct-r)$ bor bo'lgani uchun $\theta(t)$ ni tashlab yubordik.

Yuqoridagi muhokama asosida Cauchy masalasini quyidagi ko'rinishga keltirib olamiz:

$$\frac{\partial^2\tilde{u}(\mathbf{r},t)}{c^2\partial t^2} - \Delta_2\tilde{u}(\mathbf{r},t) = f(\mathbf{r},t) + \frac{1}{c^2}\delta(t)u_1(\mathbf{r}) + \frac{1}{c^2}\delta'(t)u_0(\mathbf{r}).$$

Topilgan Green funksiyasi $G_2(\mathbf{r},t)$ bu masalaning yechimini darhol yozib olishga imkon beradi:

$$u(\mathbf{r},t) = \frac{c}{2\pi}\int d^2\mathbf{r}'\int_0^\infty dt' \frac{\theta\left(c(t-t')-|\mathbf{r}-\mathbf{r}'|\right)}{\sqrt{c^2(t-t')^2-|\mathbf{r}-\mathbf{r}'|^2}}\cdot$$

$$\cdot\left(f(\mathbf{r}',t') + \frac{1}{c^2}\delta(t')u_1(\mathbf{r}') + \frac{1}{c^2}\delta'(t')u_0(\mathbf{r}')\right).$$

Birinchi had quyidagi ko'rinishga keltiriladi (qulaylik uchun $t'=\tau$ va $\mathbf{r}'=\boldsymbol{\rho}$ belgilashlar kiritaylik):

$$\frac{c}{2\pi}\int d^2\mathbf{r}'\int_0^\infty dt' \frac{\theta\left(c(t-t')-|\mathbf{r}-\mathbf{r}'|\right)}{\sqrt{c^2(t-t')^2-|\mathbf{r}-\mathbf{r}'|^2}} f(\mathbf{r}',t') =$$

$$= \frac{c}{2\pi}\int_0^t d\tau \int_{U_2} \frac{d^2\boldsymbol{\rho} f(\boldsymbol{\rho},\tau)}{\sqrt{c^2(t-\tau)^2-|\mathbf{r}-\boldsymbol{\rho}|^2}},$$

bu yerda U_2 - markazi $\mathbf{r}$ nuqtada va radiusi $c(t-\tau)$ bo'lgan doiraning ichi. Agar ikki o'lchamli masalani z o'qiga bog'liqligi yo'q uch o'lchamli masala deb qarasak, bu doira ixtiyoriy $z=\text{const}$ tekislikning ustida yotadi. Integral ostidagi θ-funksiya argumentining ko'rinishidan shu xulosaga kelamiz.

Ikkinchi had:

$$\frac{1}{2\pi c}\int d^2\mathbf{r}'\int_0^\infty dt' \frac{\theta\left(c(t-t')-|\mathbf{r}-\mathbf{r}'|\right)}{\sqrt{c^2(t-t')^2-|\mathbf{r}-\mathbf{r}'|^2}}\delta(t')u_1(\mathbf{r}') = \frac{1}{2\pi c}\int_{U_{ct}} \frac{d^2\boldsymbol{\rho}\, u_1(\boldsymbol{\rho})}{\sqrt{c^2t^2-|\mathbf{r}-\boldsymbol{\rho}|^2}},$$

bu yerda U_{ct} - radiusi ct bo'lgan doiraning ichi.

Uchinchi had:

$$\frac{1}{2\pi c}\int d^2\mathbf{r}'\int\limits_0^\infty dt'\frac{\theta\left(c(t-t')-|\mathbf{r}-\mathbf{r}'|\right)}{\sqrt{c^2(t-t')^2-|\mathbf{r}-\mathbf{r}'|^2}}\delta'(t')u_0(\mathbf{r}')=$$

$$=\frac{-1}{2\pi c}\int d^2\boldsymbol{\rho}\int\limits_0^\infty d\tau\,\delta(\tau)\frac{\partial}{\partial\tau}\frac{\theta\left(c(t-\tau)-|\mathbf{r}-\boldsymbol{\rho}|\right)}{\sqrt{c^2(t-\tau)^2-|\mathbf{r}-\boldsymbol{\rho}|^2}}=$$

$$=\frac{1}{2\pi c}\frac{\partial}{\partial t}\int\limits_{U_{ct}}\frac{d^2\boldsymbol{\rho}\,u_0(\boldsymbol{\rho})}{\sqrt{c^2t^2-|\mathbf{r}-\boldsymbol{\rho}|^2}}.$$

Topilgan uchala hadlarni bir joyga yig'amiz:

$$u(\mathbf{r},t)=\frac{c}{2\pi}\int\limits_0^t d\tau\int\limits_{U_2}\frac{d^2\boldsymbol{\rho} f(\boldsymbol{\rho},\tau)}{\sqrt{c^2(t-\tau)^2-|\mathbf{r}-\boldsymbol{\rho}|^2}}+$$

$$+\frac{1}{2\pi c}\int\limits_{U_{ct}}\frac{d^2\boldsymbol{\rho}\,u_1(\boldsymbol{\rho})}{\sqrt{c^2t^2-|\mathbf{r}-\boldsymbol{\rho}|^2}}+\frac{1}{2\pi c}\frac{\partial}{\partial t}\int\limits_{U_{ct}}\frac{d^2\boldsymbol{\rho}\,u_0(\boldsymbol{\rho})}{\sqrt{c^2t^2-|\mathbf{r}-\boldsymbol{\rho}|^2}}.$$

Olingan formula Poisson formulasi deyiladi.

§13. Bir o'lchamli fazo uchun to'lqin tenglamasining yechimi

Bir o'lchamli fazoda to'lqin tenglamasi Cauchy shartlari bilan berilgan bo'lsin:

$$\frac{\partial^2 u(x,t)}{c^2\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}=f(x,t),\qquad u\Big|_{t=0}=u_0(x),\quad u_t\Big|_{t=0}=u_1(x).\qquad(76)$$

Bir o'lchamli fazo uchun to'lqin operatorining Green funksiyasi quyidagicha aniqlanadi:

$$\left(\frac{\partial^2}{c^2\partial t^2}-\frac{\partial^2}{\partial x^2}\right)G_1(x,t)=\delta(x)\delta(t).$$

Ushbu Green funksiyasini hisoblab topib va undan foydalanib, D'Alembert formulasi (12)-ni keltirib chiqaraylik.

Green funksiyasi uchun tenglamada x o'zgaruvchi bo'yicha Fourier-almashtirish bajaramiz:

$$\left(\frac{\partial^2}{c^2\partial t^2}+k^2\right)\tilde{G}_1(k,t)=\delta(t).$$

(25)-formuladan foydalanib quyidagini olamiz:

$$\tilde{G}_1(k,t) = c\,\theta(t)\frac{\sin(ckt)}{k}.$$

Fourier almashtirish bajarib Green funksiyasiga qaytib kelish mumkin:

$$G_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikx}\tilde{G}_1(k,t) = \frac{c\,\theta(t)}{4\pi i}\int_{-\infty}^{\infty}\frac{dk}{k}e^{-ikx}\left(e^{ickt} - e^{-ickt}\right).$$

Integral ostidagi $1/k$ funksiyani $1/(k-i\varepsilon)$ ga almashtiramiz va integralni $\varepsilon \to 0$ ma'noda tushunamiz:

$$G_1(x,t) = \frac{c\,\theta(t)}{4\pi i}\int_{-\infty}^{\infty}\frac{dk}{k-i\varepsilon}e^{-ikx}\left(e^{ik(ct-x)} - e^{-ik(ct+x)}\right).$$

Jordan lemmasidan foydalanib quyidagilarga kelinadi:

$x > 0$ holda: ikkinchi had nolni beradi, birinchi haddan

$$G_1(x,t) = \frac{1}{2}c\,\theta(t)\theta(ct-x)$$

kelib chiqadi.

$x < 0$ holda: ikkala had ham hissa qo'shadi -

$$G_1(x,t) = \frac{1}{2}c\,\theta(t)(1-\theta(|x|-ct)) = \frac{1}{2}c\,\theta(t)\theta(ct-|x|).$$

Ko'rinib turibdiki, ikkala formulani birlashtirish mumkin:

$$G_1(x,t) = \frac{c}{2}\theta(ct-|x|).$$

$\theta(t)$ ni tashlab yuborildi, chunki ikkinchi θ-funksiyaning argumenti kuchliroq shartni o'z ichiga olgan.

Topilgan Green funksiyasi bir o'lchamli to'lqin tenglamasining yechimi darhol beradi:

$$u(x,t) = \frac{c}{2}\int_{-\infty}^{\infty} dx'\int_{-\infty}^{\infty} dt'\,\theta\left[c(t-t') - |x-x'|\right]\cdot$$

$$\cdot\left(f(x',t') + \frac{1}{c^2}\delta(t')u_1(x') + \frac{1}{c^2}\delta'(t')u_0(x')\right).$$

Birinchi hadni quyidagi ko‘rinishga keltiramiz (avvalgi formulalar bilan solishtirish qulayligi uchun integrallash o‘zgaruvchilari ustida $(x', t') \to (y, \tau)$ almashtirish bajardik):

$$\frac{c}{2}\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} d\tau\,\theta\,[c(t-\tau) - |x-y|]\,f(y,\tau) = \frac{c}{2}\int\limits_{0}^{t} d\tau \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} dy f(y,\tau).$$

Chegaralar quyidagicha aniqlanadi:

$$x - y > 0 \quad \text{bo‘lsin, bunda} \quad c(t-\tau) - x + y > 0 \quad \to \quad y > x - c(t-\tau);$$

$$x - y < 0 \quad \text{bo‘lsin, bunda} \quad c(t-\tau) - y + x > 0 \quad \to \quad y < x + c(t-\tau).$$

Vaqt bo‘yicha integraldagi chegaralarning aniqlanishini tushunish qiyin emas.

Ikkinchi had:

$$\frac{1}{2c}\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} d\tau\,\theta\,[c(t-\tau) - |x-y|]\,\delta(\tau)u_1(y) = \frac{1}{2c}\int\limits_{x-ct}^{x+ct} dy\, u_1(y).$$

Chegaralar avvalgi holdagidek aniqlanadi, $\delta(\tau)$ ning mavjudligi ularni yanada soddalashtiradi. Uchinchi had:

$$\frac{1}{2c}\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} d\tau\,\theta\,[c(t-\tau) - |x-y|]\,\delta'(\tau)u_0(y) =$$

$$= -\frac{1}{2c}\int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} d\tau\,\delta(\tau)u_0(y)\frac{d}{d\tau}\theta\,[c(t-\tau) - |x-y|] =$$

$$= \frac{1}{2}\int\limits_{-\infty}^{\infty} dy\,\delta(ct - |x-y|)\,u_0(y) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right).$$

Hamma qismlarni bir joyga to‘plab yana, albatta, o‘zimizga ma’lum bo’lgan D’Alembert formulasi (12)-ni olamiz:

$$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} dy\, u_1(y) + \frac{c}{2}\int\limits_{0}^{t} d\tau \int\limits_{x-c(t-\tau)}^{x+c(t-\tau)} dy f(y,\tau).$$

Oxirgi had oldidagi koeffisientning V-bobdagi (12)-formulaning oxirgi hadi oldidagi koeffisient bilan farqi V-bobdagi (9)-tenglamaga f ning (71)-tenglamadagi f ga nisbatan boshqa koeffisient bilan kirganligi bilan tushuntiriladi.

Mashqlarga ko‘rsatmalar va ularning yechimlari

1.1.

$$J_\nu(-x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!(k+\nu)!}\left(-\frac{x}{2}\right)^{2k+\nu} = (-1)^\nu J_\nu(x).$$

1.2. $n \geq 1$ butun son bo‘lganda $(-n)! = \infty$, demak,

$$\frac{1}{(k-n)!} = 0, \qquad k < n.$$

Shuning uchun Bessel funksiyasi qatori (10) $k = 0$ emas, $k = n$ haddan boshlanadi:

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n} = \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2} + \cdots =$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} = (-1)^n J_n(x).$$

1.3. l’Hôpital qoidasini qo‘llang.

1.4. (18)-formulaga olib kelgan amalni m-marta qo‘llang.

1.5. (14)-formulada $t = e^{i\theta}$ almashtirish bajaramiz va

$$t - \frac{1}{t} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

ekanligidan foydalanamiz.

1.6. $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$ dan kelib chiqadi.

1.7. Bevosita hisoblanadi.

1.8. Bevosita hisoblanadi.

1.9. (31)-formulani $\cos(n\theta)$ ga ko‘paytirib 0 dan π gacha integrallaymiz:

$$\frac{1}{\pi}\int_0^\pi d\theta\cos(x\sin\theta) = \sum_{m=-\infty}^{\infty} J_m(x)\int_0^\pi d\theta\cos(m\theta)\cos(n\theta) = \frac{1}{2}\sum_{m=-\infty}^{\infty} J_m(x)\delta_{mn}.$$

Shu joyda birinchi mashqdan foydalanilsa, talab qilingan javob kelib chiqadi. Ikkinchi formula ham xuddi shu yo'l bilan olinadi.

1.10.

$$t - \frac{1}{t} = ie^{i\theta} + ie^{-i\theta} = 2i\cos\theta$$

formuladan kelib chiqadi.

1.11. (18)-formulaning ikkinchisida $n = 0$ deb olinsa, quyidagi kelib chiqadi:

$$\left(\frac{d}{x\,dx}\right) J_0(x) = -\frac{J_1(x)}{x}.$$

Bu ifodadan $d/(xdx)$ bo'yicha yana bir marta hosila olamiz, uning o'ng tomoniga (18)-formulani yana bir marta qo'llaymiz:

$$\left(\frac{d}{x\,dx}\right)^2 J_0(x) = -\left(\frac{d}{x\,dx}\right)\frac{J_1(x)}{x} = (-1)^2\frac{J_2(x)}{x^2}$$

va h.k.

1.12. (21)-integraldagi C kontur nol nuqtani o'z ichiga olgan birlik aylana deb olinsa $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ va

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{ix\sin\theta - in\theta} = \frac{1}{2\pi}\int_0^{\pi} d\theta e^{ix\sin\theta - in\theta} + \frac{1}{2\pi}\int_{\pi}^{2\pi} d\theta e^{ix\sin\theta - in\theta}$$

bo'ladi. Ikkinchi integralda $\theta \to \theta - \pi$ almashtirish bajarilsa quyidagi olinadi:

$$J_n(x) = \frac{1}{2\pi}\int_0^{\pi} d\theta e^{-in\theta}\left(e^{ix\sin\theta} + (-1)^n e^{-ix\sin\theta}\right).$$

n juft bo'lganda:

$$J_n(x) = \frac{1}{2\pi}\int_0^{\pi} d\theta e^{-in\theta}\left(e^{ix\sin\theta} + e^{-ix\sin\theta}\right) = \frac{1}{\pi}\int_0^{\pi} d\theta\cos(x\sin\theta)\cos(n\theta);$$

n toq bo'lganda:

$$J_n(x) = \frac{1}{2\pi}\int_0^{\pi} d\theta e^{-in\theta}\left(e^{ix\sin\theta} - e^{-ix\sin\theta}\right) = \frac{1}{\pi}\int_0^{\pi} d\theta\sin(x\sin\theta)\sin(n\theta).$$

Tashlab yuborilgan integrallarning nolga tengligi ularning integral osti ifodalarining integrallash sohasida toqligidan kelib chiqadi. 8-mashqning natijalari bu ikki formulani birga ko'rishga imkoniyat beradi:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi d\theta\,[\cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta)] =$$

$$= \frac{1}{\pi}\int_0^\pi d\theta\cos[n\theta - x\sin\theta].$$

1.13.

$$J_5(x) = \left(1 - \frac{72}{x^2} + \frac{384}{x^4}\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x).$$

1.14. Bevosita hisoblanadi.

1.15. Bir tomondan

$$J_0'(x) = -\frac{x}{\pi}\int_0^\pi d\theta\sin(x\sin\theta)\sin\theta,$$

ikkinchi tomondan

$$J_1(x) = \frac{1}{\pi}\int_0^\pi d\theta\cos(\theta - x\sin\theta) = \frac{x}{\pi}\int_0^\pi d\theta\sin(x\sin\theta)\sin\theta,$$

chunki

$$\int_0^\pi d\theta\cos(x\sin\theta)\cos\theta = \frac{1}{x}\sin(x\sin\theta)\Big|_0^\pi = 0.$$

1.16.

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right).$$

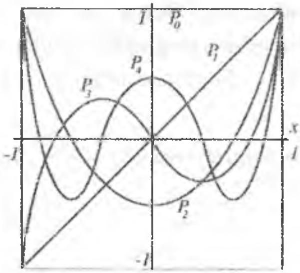
1.17. Javob VIII.5-rasmda ko'rsatilgan.

1.18. Bevosita hisoblanadi.

1.19. Bevosita hisoblanadi.

1.20.

$$P_n^n(x) = \frac{1}{2^n n!}\left(1 - x^2\right)^{n/2}\frac{d^{2n}}{dx^{2n}}\left(x^2 - 1\right)^n = \frac{1}{2^n n!}\left(1 - x^2\right)^{n/2}\frac{d^{2n}}{dx^{2n}}x^{2n} =$$



VIII.5-rasm: $P_0(x), P_1(x), P_2(x), P_3(x)$ va $P_4(x)$ larning grafiklari

$$= \frac{1}{2^n n!}\left(1-x^2\right)^{n/2}(2n)! = (2n-1)!!\left(1-x^2\right)^{n/2} = (2n-1)!!\sin^n\theta,$$

chunki

$$(2n)! = 2n(2n-1)(2n-2)(2n-3)(2n-4)\cdots 1 =$$

$$= 2n\cdot(2n-2)\cdot(2n-4)\cdots(2n-1)\cdot(2n-3)\cdot(2n-5)\cdots 1 = 2^n n!(2n-1)!!$$

1.21. Potensial uchun

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}(r^2+a^2-2ra\cos\theta)^{-1/2}$$

ifodani kichik parametr bo'yicha qatorga yoyish kerak, kichik parametr esa bu holda $r/a < 1$, shuning uchun

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a}\frac{1}{\sqrt{1+\frac{r^2}{a^2}-2\frac{r}{a}\cos\theta}} = \frac{q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty}P_n(cos\theta)\left(\frac{r}{a}\right)^n.$$

1.22. Zaryadlar sistemasining potensiali:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty}P_n(\cos\theta)\left[\left(\frac{2a}{r}\right)^n - 2\left(\frac{a}{r}\right)^n + 2\left(-\frac{a}{r}\right)^n - \left(-\frac{2a}{r}\right)^n\right].$$

$n = 0, 1, 2$ hadlar nolni beradi, noldan farqli birinchi had $n = 3$ bo'lgan had:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}P_3(\cos\theta)12\frac{a^3}{r^3} + \cdots = \frac{3qa^3}{2\pi\varepsilon_0}\frac{5\cos^3\theta - 3\cos\theta}{r^4} + \cdots$$

2.1-mashq. Bevosita hisoblanadi.

2.2-mashq. $D=-1<0$, tenglama - elliptik. $\zeta=y+3x,\ \eta=x$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\zeta}+u_{\eta\eta}+u_{\eta}=0.$$

2.3-mashq. $D=0$, tenglama - parabolik. $\zeta=y-\frac{1}{2}x,\ \eta=y+\frac{1}{2}x$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\eta\eta}-\frac{1}{2}u_{\zeta}-\frac{1}{2}u_{\eta}=0.$$

2.4-mashq. $D=x$. $x>0$ sohada $D>0$, tenglama giperbolik tipga tegishli. $\zeta=y+\frac{2}{3}x^{3/2},\ \eta=y-\frac{2}{3}x^{3/2}$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\eta}=\frac{1}{6(\zeta-\eta)}(u_{\zeta}-u_{\eta}).$$

$x<0$ sohada $D<0$, tenglama elliptik. $\zeta=y,\ \eta=\frac{2}{3}(-x)^{3/2}$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\zeta}+u_{\eta\eta}=\frac{1}{3\eta}u_{\eta}.$$

2.5-mashq. $D=y$. $y>0$ sohada tenglama giperbolik tipga oid $D>0$. $\zeta=2\sqrt{y}+x,\ \eta=2\sqrt{y}-x$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\eta}=\frac{1}{2(\zeta+\eta)}(u_{\zeta}+u_{\eta}).$$

$y<0$ sohada tenglama elliptik tipga oid $D<0$. $\zeta=2\sqrt{-y},\ \eta=x$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\zeta}+u_{\eta\eta}=\frac{1}{\zeta}u_{\zeta}.$$

2.6-mashq. Bu tenglama uchun $D=-xy$. Birinchi va uchinchi choraklarda $D<0$, ikkinchi va to'rtinchi choraklarda $D>0$. Birinchi chorakda $\zeta=2\sqrt{y},\ \eta=2\sqrt{x}$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\zeta}+u_{\eta\eta}-\frac{1}{\zeta}u_{\zeta}-\frac{1}{\eta}u_{\eta}=0.$$

Uchinchi chorakda $\zeta=2\sqrt{-y},\ \eta=2\sqrt{-x}$ almashtirish bajarsak, xuddi shu tenglamani yana olamiz.

Ikkinchi chorakda $(y>0,\ x<0)$ $\zeta=\sqrt{y}+\sqrt{-x}$, $\eta=\sqrt{y}-\sqrt{-x}$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\eta}+\frac{\eta}{\zeta^2-\eta^2}u_\eta-\frac{\zeta}{\zeta^2-\eta^2}u_\zeta=0.$$

To'rtinchi chorakda $(y<0,\ x>0)$ $\zeta=\sqrt{-y}+\sqrt{x}$, $\eta=\sqrt{-y}-\sqrt{x}$ almashtirish yordamida yana xuddi shu kanonik ko'rinishning o'zini olaqmiz.

2.7-mashq. $D=-x^2y^2<0$, tenglama elliptik tipga oid. $\zeta=\frac{1}{2}y^2$, $\eta=\frac{1}{2}x^2$ almashtirish bu tenglamani quyidagi kanonik ko'rinishga keltiradi:

$$u_{\zeta\zeta}+u_{\eta\eta}+\frac{1}{2\zeta}u_\zeta+\frac{1}{2\eta}u_\eta=0.$$

2.8-mashq. $D=-x^2y^2<0$, tenglama elliptik tipga oid. $\zeta=\ln y$, $\eta=\ln x$ almashtirish bu tenglamani quyidagi kanonik ko'rinishga keltiradi:

$$u_{\zeta\zeta}+u_{\eta\eta}-u_\zeta-u_\eta=0.$$

2.9-mashq. Giperbolik tenglama: $D=x^2y^2>0$. $\zeta=y/x$, $\eta=xy$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\eta}-\frac{1}{2\eta}u_\zeta=0.$$

2.10-mashq. $D=x^2y^2>0$, tenglama giperbolik tipga oid. $\zeta=\frac{1}{2}(y^2-x^2)$, $\eta=\frac{1}{2}(x^2+y^2)$ almashtirish bu tenglamani quyidagi kanonik ko'rinishga keltiradi:

$$u_{\zeta\eta}-\frac{\eta}{2(\zeta^2-\eta^2)}u_\zeta+\frac{\zeta}{2(\zeta^2-\eta^2)}u_\eta=0.$$

2.11-mashq. $D=-(1+x^2)(1+y^2)<0$, tenglama - elliptik. $\zeta=\ln(x+\sqrt{1+x^2})$, $\eta=\ln(y+\sqrt{1+y^2})$ almashtirish yordamida quyidagi kanonik ko'rinishga keltiriladi:

$$u_{\zeta\zeta}+u_{\eta\eta}-2u=0.$$

2.12-mashq. $D=0$, tenglama parabolik tipga oid. $\zeta=y+\ln x$, $\eta=y-\ln x$ almashtirish bu tenglamani quyidagi kanonik ko'rinishga keltiradi:

$$u_{\eta\eta}+\frac{1}{4}(u_\eta-u_\zeta)=0.$$

2.13-mashq. $D=0$, tenglama parabolik tipga oid. $\zeta=\frac{1}{2}y^2-x$, $\eta=\frac{1}{2}y^2+x$ almashtirish bu tenglamani quyidagi kanonik ko'rinishga keltiradi:

$$u_{\eta\eta}+\frac{1}{4(\zeta+\eta)}(u_\zeta+u_\eta)=0.$$

2.14-mashq. Parabolik tenglama: $D = 0$. $\zeta = \frac{1}{2}(y^2 - x^2)$, $\eta = \frac{1}{2}(x^2 + y^2)$ almashtirish yordamida quyidagi kanonik koʻrinishga keltiriladi:

$$u_{\eta\eta} + \frac{\zeta}{\eta^2 - \zeta^2}u_\zeta + \frac{\eta}{\eta^2 - \zeta^2}u_\eta = 0.$$

2.15-mashq: $\zeta = x$, $\eta = \frac{1}{2}(x + y + z)$, $\xi = \frac{1}{2}(3x + y - z)$ almashtirish yordamida tenglama quyidagi kanonik koʻrinishga keladi:

$$u_{\zeta\zeta} + u_{\eta\eta} - u_{\xi\xi} + 3u_\zeta + \frac{3}{2}u_\eta + \frac{9}{2}u_\xi = 0.$$

2.16-mashq: $\zeta = x$, $\eta = \frac{1}{\sqrt{2}}(x+y)$, $\xi = 2x+y+z$ almashtirish yordamida tenglama quyidagi kanonik koʻrinishga keladi:

$$u_{\zeta\zeta} + u_{\eta\eta} = 0.$$

3.1-mashq. Masalaning qoʻyilishi:

$$u_{tt} - a^2u_{xx} = 0, \quad u(0,t) = 0,\ u_x(l,t) = 0,\ u(x,0) = 0,\ u_t(x,0) = v;$$

$$0 \leq x \leq l, \quad t > 0.$$

Bu yerda $a = \sqrt{\gamma p_0/\rho_0}$ tovush tezligi, $\gamma = c_P/c_V$.

3.2-mashq.

$$u_{tt} - a^2u_{xx} = -\alpha u_t, \quad u(0,t) = u(l,t) = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x);$$

$$0 \leq x \leq l, \quad t > 0.$$

3.3-mashq.

$$u_{tt} - a^2u_{xx} = 0, \quad u(0,t) = 0,\ u_x(l,t) = \frac{\alpha u_l(l,t)}{ES},$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ 0 \leq x \leq l, \quad t > 0.$$

α - sterjen oʻng uchining elastiklik koeffisienti.

3.4-mashq.

$$u_{tt} - a^2u_{xx} = g, \quad u(0,t) = 0, \quad u_x(l,t) = 0,$$

$$u(x,0) = u_t(x,0) = 0; \quad 0 \leq x \leq l, \quad t > 0.$$

3.5-mashq.

$$u_{tt} - a^2u_{xx} = 0, \quad u(0,t) = \mu(t), \quad u_x(l,t) = \frac{\Phi(t)}{ES},$$

$$u(x,0) = u_t(x,0) = 0; \quad 0 \le x \le l, \quad t > 0.$$

3.6-mashq.

$$u_{tt} - a^2 u_{xx} = g, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = 0,$$

$$Pu_{tt}(l,t) = -ESu_x(l,t) + P, \quad 0 \le x \le l, \quad t > 0.$$

Oxirgi chegaraviy shartning kelib chiqishi quyidagicha: $Pu_{tt}(l,t)$–sterjenning $x = l$ nuqtasiga ta'sir qilayotgan kuch, u ikki qismdan iborat - birinchisi qaytaruvchi elastik kuch $-ESu_x(l,t)$, ikkinchisi - yukning og'irlik kuchi P.

4.1-mashq.

$$u_t - a^2 u_{xx} = 0, \quad u(0,t) = u_1, \; u(l,t) = u_2, \quad u(x,0) = \varphi(x), \; u(x,0) = \psi(x).$$

4.2-mashq.

$$u_t - a^2 u_{xx} = \frac{q}{c}\delta(x - vt), \quad u(x,0) = \varphi(x), \; a^2 = k/(c\rho), \; -\infty < x < \infty, t > 0.$$

4.3-mashq.

$$u_t - a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u(r,t)}{\partial r}\right) = \frac{Q}{c\rho}, \quad u(r,0) = f(r), \; 0 \le r < R.$$

Issiqlik tarqalishi tenglamasi sferik simmetriyani hisobga olib yozilgan, masalaning shartlarida θ va φ burchaklarga bog'lanish yo'q. Chegaraviy shartlar: a) $u(R,t) = 0$; b) $u_r(R,t) + hu(R,t) = 0$. Ushbu va keyingi masalalarda $|u(0,t)| < \infty$ bo'lishi kerak.

4.4-mashq.

$$u_t - a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u(r,t)}{\partial r}\right) = 0, \quad u(r,0) = 0, \; 0 \le r < R, \; u_r(R,t) = \frac{q}{k}.$$

4.5-mashq. Tenglama:

$$\Delta u = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Masalada φ ga bog'liq bo'lgan shartlar yo'q, shu sababdan tenglamadagi ikkinchi had ham yo'q:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0.$$

Chegaraviy shartlar:

1. $u(r,0)=0, \quad u(a,z)=0, \quad u(r,h)=f(r);$

2. $u(r,0)=0, \quad u_r(a,z)=0, \quad u(r,h)=f(r);$

3. $u(r,0)=0, \quad u_r(a,z)=-\alpha u(a,z),\ \alpha>0, \quad u(r,h)=f(r).$

Temperaturaning barqaror taqsimoti haqida gap ketayotgani uchun boshlang'ich shartlar yo'q.

5.2-mashq. Bu tenglama giperbolik tipga oid, $D=4>0$, uning ikkita xarakteristikasi $\zeta=y-x$ va $\eta=y+3x$. Demak, uning umumiy yechimi

$$u(x,y)=f(y-x)+g(y+3x).$$

5.3-mashq. Bu giperbolik tenglama, uning xarakteristikalari $\zeta=y-x/3,\ \eta=y+2x$. Tenglamaning kanonik ko'rinishi:

$$u_{\zeta\eta}-\frac{3}{7}u_\eta=-\frac{6}{49}.$$

Bu tenglamani

$$\frac{\partial}{\partial\eta}\left(u_\zeta-\frac{3}{7}u\right)=-\frac{6}{49}$$

ko'rinishda yozib olsak,

$$u_\zeta-\frac{3}{7}u=-\frac{6}{49}\eta+f_1(\zeta)$$

ekanligini topish mumkin, bu yerda $f_1(\zeta)$ - o'z o'zgaruvchisining ixtiyoriy funksiyasi (argumentining o'zgarish sohasida C^2 sinfga tegishli, albatta).

$$u=ve^{3\zeta/7}$$

almashtirish bajarib

$$v=\frac{2}{7}\eta e^{-3\zeta/7}+f_2(\zeta)+g(\eta)$$

ekanligini topamiz, bu yerda f_2 va g funksiyalar yana C^2 sinfiga tegishli ixtiyoriy funksiyalar. Yechim:

$$u(x,y)=\frac{2}{7}(y+2x)+f(3y-x)+g(y+2x)e^{(3y-x)/7}.$$

5.4-mashq. $u(x,y)=v(x,y)e^{-bx-ay}$ almashtirish bajarsak, $v_{xy}=0$ tenglamaga kelamiz. Demak, berilgan tenglamaning yechimi

$$u(x,y)=(f(x)+g(y))\,e^{-bx-ay}.$$

5.5-mashq. $u(x,y) = v(x,y)e^{3x+2y}$ almashtirish bajarsak,

$$v_{xy} = 2e^{-2x-y}$$

tenglamaga kelamiz. Uning yechimi:

$$v(x,y) = e^{-2x-y} + f(x) + g(y).$$

Demak, berilgan tenglamaning yechimi:

$$u(x,y) = e^{x+y} + (f(x) + g(y))\, e^{3x+2y}.$$

5.6-mashq. Berilgan tenglama $\zeta = y/x$ va $\eta = xy$ almashtirish orqali quyidagi kanonik ko'rinishga keltiriladi (II.9-mashqning yechimiga qarang):

$$u_{\zeta\eta} - \frac{1}{2\eta}u_\zeta = 0.$$

Demak,

$$u_\eta - \frac{1}{2\eta}u = g_1(\eta), \quad g_1(\eta) \in C^2 - \text{noma'lum funksiya.}$$

$u = \sqrt{\eta}v$ almashtirish bajarsak, bu tenglamaning yechimi darhol topiladi, undan esa

$$u(x,y) = g(xy) + \sqrt{xy} f\left(\frac{y}{x}\right)$$

yechimni topamiz. Bu - birinchi kvadrantda. Umumiy holda,

$$u(x,y) = g(xy) + \sqrt{|xy|} f\left(\frac{y}{x}\right)$$

deb yozamiz, $|xy|$ - har bir kvadrantda musbat qilib tanlab olinishi kerak.

5.7-mashq. Boshlang'ich shartlar: $u(x,0) = \varphi(x) = f(x)$, $u_t(x,0) = \psi(x) = -af'(x)$, bularni (8)-formulaga qo'ysak, $u(x,t) = f(x - at)$ ekanligini topamiz.

6.1-mashq. Yechimni

$$u(x,t) = \frac{xt}{l} + v(x,t)$$

ko'rinishda qidiramiz. $v(x,t)$ uchun masala:

$$v_{tt} - v_{xx} = 0, \quad v(0,t) = v(l,t) = 0, \; v(x,0) = 0, \; v_t(x,0) = -\frac{x}{l}.$$

Bu masalani yechib $u(x,t)$ ni topamiz:

$$u(x,t) = \frac{xt}{l} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi t}{l}.$$

6.1-mashq. Yechimni

$$u(x,t) = t + 1 + x(t^2 - t + 1) + v(x,t)$$

ko'rinishda qidiramiz. Bunda

$$v_{tt} - v_{xx} = -2x, \quad v(0,t) = v(1,t) = 0, \quad v(x,0) = 0,\ v_t(x,0) = x - 1.$$

Ikkinchi bosqichda

$$v = \tilde{v} + w(x)$$

almashtirish bajaramiz, bunda $w(x)$ uchun quyidagi masalani olamiz:

$$w''(x) = 2x, \qquad w(0) = w(1) = 0.$$

Uning yechimi:

$$w(x) = \frac{x}{3}(x^2 - 1).$$

$\tilde{v}$ uchun masala:

$$\tilde{v}_{tt} - \tilde{v}_{xx} = 0, \quad \tilde{v}(0,t) = \tilde{v}(1,t) = 0, \quad \tilde{v}(x,0) = -\frac{x}{3}(x^2 - 1),\ \tilde{v}_t(x,0) = x - 1.$$

Bu masalani yechish qiyin emas, boshlang'ich masalaning yechimi:

$$u(x,t) = t + 1 + x(t^2 - t + 1) + \frac{x}{3}(x^2 - 1) -$$

$$-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(n\pi x)\left(\frac{(-1)^n}{n}\cos(n\pi t) + \sin(n\pi t)\right).$$

6.3-mashq. Tenglama $u(x,t) = X(x)T(t)$ almashtirish yordamida

$$\frac{T''(t)}{T(t)} + 4 = \frac{X''(x)}{X(x)} = -\lambda$$

ko'rinishga keltiriladi. Natijada,

$$X''(x) + \lambda X(x) = 0,\ X(0) = X(1) = 0 \quad \text{va} \quad T''(t) + (4 + \lambda)T(t) = 0$$

masalalarni olamiz. Demak, yechim

$$u(x,t) = \sum_{n=1}^{\infty}\sin(n\pi x)\left(a_n\cos\left(t\sqrt{4 + n^2\pi^2}\right) + b_n\sin\left(t\sqrt{4 + n^2\pi^2}\right)\right)$$

ko'rinishga ega. Boshlang'ich shartlarni ishlatish quyidagiga olib keladi:

$$u(x,t) = -\frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\pi x]}{(2k+1)^3}\cos\left[t\sqrt{4+(2k+1)^2\pi^2}\right].$$

6.4-mashq. Yechimni

$$u(x,t) = t(2-x) + v(x,t)$$

ko'rinishda qidiramiz. $v(x,t)$ uchun quyidagi tenglamaga kelamiz:

$$v_{tt} - v_{xx} - v = t(2-x).$$

Uning yechimini

$$v(x,t) = \sum_{n=1} v_n(t)\sin\frac{n\pi x}{2}$$

ko'rinishda qidirish kerak, chunki $v(0,t) = v(2,t) = 0$. O'ng tomondagi $t(2-x)$ funksiyani $\sin\frac{n\pi x}{2}$ bo'yicha Fourier-qatorga, yoysak quyidagi tenglamaga kelamiz:

$$\ddot{v}_n(t) + \lambda_n^2 v_n(t) = \frac{4}{n\pi}t, \qquad \lambda_n^2 = \left(\frac{n\pi}{2}\right)^2 - 1.$$

Uning yechimi:

$$v_n(t) = \frac{4t}{n\pi\lambda_n^2} + a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t).$$

Demak,

$$u(x,t) = t(2-x) + \frac{4t}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\lambda_n^2}\sin\frac{n\pi x}{2} + \sum_{n=1}^{\infty}\sin\frac{n\pi x}{2}\left(a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t)\right).$$

Boshlang'ich shartlardan foydalangandan keyin quyidagini olamiz:

$$u(x,t) = t(2-x) + \sum_{n=1}^{\infty}\left(\frac{4t}{n\pi\lambda_n^2} - \frac{n\pi}{\lambda_n^3}\sin(\lambda_n t)\right)\sin\frac{n\pi x}{2}.$$

6.5-mashq. Yechish bosqichlari avvalgi masaladan farq qilmaydi. Yechim:

$$u(x,t) = \frac{xt}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\lambda_n^2}\left(t - \frac{1}{\lambda_n}\sin(\lambda_n t)\right)\sin\frac{n\pi x}{2}, \quad \lambda_n^2 = \left(\frac{n\pi}{l}\right)^2 - 1.$$

6.6-mashq. Yechim bitta garmonikadan iborat:

$$u(x,t) = \frac{l}{2\pi a}\sin\frac{2\pi x}{l}\sin\frac{2\pi at}{l}.$$

6.7-mashq. Yechim ikkita garmonikadan iborat:

$$u(x,t) = \sin\frac{5\pi x}{2l}\cos\frac{5\pi a t}{2l} + \frac{2l}{a\pi}\sin\frac{\pi x}{2l}\sin\frac{\pi a t}{2l}.$$

6.8-mashq.

$$u(x,t) = \frac{8l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi a t}{2l}+$$

$$+\frac{2l}{\pi a}\sin\frac{\pi x}{2l}\sin\frac{\pi a t}{2l} + \frac{2l}{3\pi a}\sin\frac{3\pi x}{2l}\sin\frac{3\pi a t}{2l}.$$

6.9-mashq. Yechim uchta garmonikadan iborat:

$$u(x,t) = \cos\frac{\pi x}{2l}\cos\frac{\pi a t}{2l} + \frac{2l}{3a\pi}\cos\frac{3\pi x}{2l}\sin\frac{3a\pi t}{2l} + \frac{2l}{5a\pi}\cos\frac{5\pi x}{2l}\sin\frac{5a\pi t}{2l}.$$

6.10-mashq. Umumiy yechim:

$$u(x,t) = \sum_{n=0}^{\infty}\cos\frac{n\pi x}{l}\left(a_n\cos\frac{n\pi a t}{l} + b_n\sin\frac{n\pi a t}{l}\right),$$

boshlang'ich shartlardan

$$a_n = \frac{2l}{n^2\pi^2}\left((-1)^n - 1\right) \qquad \text{va} \qquad b_n = 0, \qquad n \neq 0$$

va

$$a_0 = \frac{l}{4}, \qquad b_n = \left.\frac{\sin(n\pi a)}{2n^2\pi^2}\right|_{n=0}$$

ekanligi kelib chiqadi. Umumiy yechimdagi qavs ichidagi ikkinchi hadda $n \to 0$ limitga ehtiyotkorlik bilan o'tish kerak:

$$u(x,t) = \frac{1}{2}l + \frac{l}{4} - \frac{4l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)a\pi t}{2l}.$$

6.11-mashq. $u(x,t) = A$.

6.12-mashq.

$$u(x,t) = u_1 + \frac{x}{l}(u_2 - u_1)+$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{u_0 - u_1 + (u_2 - u_1)(-1)^n}{n}\sin\frac{n\pi x}{l}\exp\left(-\frac{n^2\pi^2 a^2}{l^2}t\right).$$

6.13-mashq.

$$u(x,t) = u_1 + \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}\exp\left(-\frac{(2n+1)^2\pi^2a^2}{l^2}t\right).$$

6.14-mashq.

$$u(x,t) = u_2 + \frac{4(A-u_2)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\frac{(2n+1)\pi x}{2l}\exp\left(-\frac{(2n+1)^2\pi^2a^2}{4l^2}t\right) +$$

$$-\frac{8A}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2l}\exp\left(-\frac{(2n+1)^2\pi^2a^2}{4l^2}t\right).$$

6.15-mashq. Chegaraviy shartlar birjinslimas bo'lgani uchun - $u(0,t) = u_1$, $u_x(l,t) = q/k$ - yechim $u(x,t) = u_1 + qx/k + v(x,t)$ ko'rinishda qidiriladi, bunda $v(0,t) = v_x(l,t) = 0$ bo'lib chiqadi:

$$u(x,t) = u_1 + \frac{q}{k}x - \frac{8ql}{k\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\exp\left(-\frac{(2n+1)^2\pi^2a^2}{4l^2}t\right) -$$

$$+\frac{4(u_0-u_1)}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2l}\exp\left(-\frac{(2n+1)^2\pi^2a^2}{4l^2}t\right).$$

6.16-mashq.

$$u(x,t) = \sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{l}e^{-\frac{n^2\pi^2a^2t}{l^2}}$$

$$a_n = \frac{2}{l}\int_0^l (x^2-l^2)\cos\frac{n\pi x}{l}dx = \frac{4l^2}{n^2\pi^2}(-1)^n, \qquad n\neq 0, \quad a_0 = -\frac{2}{3}l^2.$$

Yechim:

$$u(x,t) = -\frac{2}{3}l^2 + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}\exp\left(-\frac{n^2\pi^2a^2t}{l^2}\right).$$

6.17-mashq.

$$u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{l}\exp\left[-t\left(1+\frac{(2n+1)^2\pi^2}{l^2}\right)\right].$$

6.18-mashq.

$$u(x,t) = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin[(2n+1)x]\exp\left(-4t-(2n+1)^2t\right).$$

6.19-mashq:

$$u(x,t) = -\frac{2Al}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\exp\left(-\frac{n^2\pi^2a^2t}{l^2}\right).$$

6.20-mashq:

$$u(x,t)=\frac{4Al}{\pi}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)\pi}-2(-1)^n\right]\sin\frac{(2n+1)\pi x}{2l}\exp\left[-\frac{(2n+1)^2a^2\pi^2}{4l^2}t\right].$$

6.21-mashq:

$$u(x,t) = \frac{8Al}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi x}{2l}\exp\left[-\frac{(2n+1)^2a^2\pi^2}{4l^2}t\right].$$

6.22-mashq: $u(x,t) = u_0$.

6.23-mashq:

$$u(x,t) = -\frac{2Al}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\exp\left(-\beta t-\frac{n^2\pi^2a^2t}{l^2}\right).$$

6.24-mashq:

$$u(x,t) = u_1+\frac{x}{l}(u_2-u_1)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-u_1+(-1)^nu_2}{n}\sin\frac{n\pi x}{l}\exp\left(-\frac{n^2\pi^2a^2t}{l^2}\right).$$

$\lim t\to\infty$ da $u \simeq u_1+\frac{x}{l}(u_2-u_1)$ bo‘ladi.

6.25-mashq: Yechimni $u(x,t) = v(x,t)+w(x)$ ko1rinishda qidiramiz, bunda $w(x)$ funksiya uchun tenglama, chegaraviy shartlar va yechim quyidagicha:

$$a^2w''(x)+\sin\frac{\pi x}{l}=0, \qquad w(0)=w(l)=0; \qquad w(x)=\frac{l^2}{a^2\pi^2}\sin\frac{\pi x}{l}.$$

To‘liq yechim:

$$u(x,t) = \frac{l^2}{a^2\pi^2}\left(1-\exp\left(-\frac{\pi^2a^2t}{l^2}\right)\right)\sin\frac{\pi x}{l}.$$

6.26-mashq: Yechimni quyidagicha qidiramiz: $u(x,t) = u_0+v(x,t)$. $v(x,t)$ uchun quyidagi masala paydo bo‘ladi (qulaylik uchun $h=\alpha/(c\rho)$ deb belgiladik):

$$v_t-a^2v_{xx}=-hv, \quad v(0,t)=u_1-u_0, \; v(l,t)=u_2-u_0, \; v(x,0)=\varphi(x)-u_0.$$

Chegaraviy shartlarni bir jinsliga aylantirish maqsadida

$$v(x,t) = u_1 - u_0 + \frac{x}{l}(u_2 - u_1) + \bar{v}(x,t)$$

almashtirish bajaramiz. $\bar{v}$ uchun tenglama:

$$\bar{v}_t - a^2\bar{v}_{xx} = -h\bar{v} - h\left(u_1 - u_0 + \frac{x}{l}(u_2 - u_1)\right).$$

$\bar{v}$ uchun chegaraviy sharlar bir jinslidir. Noma'lum $\bar{v}$ ni yana ikkiga bo'lamiz: $\bar{v}(x,t) = z(x,t) + w(x)$ va $w(x)$ ni quyidagi tenglama va shartlarga bo'ysundiramiz:

$$a^2w''(x) = hw(x) + h\left(u_1 - u_0 + \frac{x}{l}(u_2 - u_1)\right), \qquad w(0) = w(l) = 0.$$

Bu tenglamaning yechimi

$$w(x) = (u_1 - u_0)\operatorname{ch}\frac{\sqrt{h}x}{a} + \frac{u_2 - u_0 - (u_1 - u_0)\operatorname{ch}\frac{\sqrt{h}l}{a}}{\operatorname{sh}\frac{\sqrt{h}l}{a}}\operatorname{sh}\frac{\sqrt{h}x}{a} - u_1 + u_0 - \frac{x}{l}(u_2 - u_1).$$

Shu bilan quyidagiga keldik:

$$u(x,t) = u_0 + (u_1 - u_0)\operatorname{ch}\frac{\sqrt{h}x}{a} + \frac{u_2 - u_0 - (u_1 - u_0)\operatorname{ch}\frac{\sqrt{h}l}{a}}{\operatorname{sh}\frac{\sqrt{h}l}{a}}\operatorname{sh}\frac{\sqrt{h}x}{a} + z(x,t) =$$

$$= p(x) + z(x,t).$$

$z(x,t)$ uchun masala:

$$z_t - a^2z_{xx} = -hz, \quad z(0,t) = z(l,t) = 0, \quad z(x,0) = \varphi(x) - p(x).$$

Bu masalaning yechimi:

$$z(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\exp\left(-ht - \frac{n^2\pi^2a^2}{l^2}t\right),$$

$$a_n = \frac{2}{l}\int_0^l (\varphi(x) - p(x))\sin\frac{n\pi x}{l}dx.$$

6.27-mashq: Masalaning fazoviy qismi uchun quyidagiga egamiz:

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x, \quad X'(0) - hX(0) = 0, \; X'(l) = 0.$$

Uning yechimi:

$$X_k(x) = c_1\left(\cos\lambda_k x + \frac{h}{\lambda_k}\sin\lambda_k x\right), \qquad \mathrm{tg}(\lambda_k l) = \frac{h}{\lambda_k}, \quad k = 0, 1, 2, ...$$

Quyidagini hisoblab topish mumkin (bu bir muncha hisobni talab qiladi):

$$(X_n, X_m) = c_1^2 \int\limits_0^l dx X_n(x) X_m(x) = 0, \qquad m \neq n.$$

Agar c_1 ni quyidagicha tanlab olsak:

$$c_1 = \frac{1}{\sqrt{\int_0^l dx\left(\cos\lambda_k x + \frac{h}{\lambda_k}\sin\lambda_k x\right)^2}} = \frac{\sqrt{2}\lambda_k}{\sqrt{h + l(\lambda_k^2 + h^2)}}$$

$X_k(x)$ funksiyaning normasi birga teng bo'ladi: $||X_k|| = 1$. Natijada, $\{X_k(x),\ k = 0, 1, 3, ...\}$ funksiyalar to'plami ortonormal sistemani hosil qiladi: $(X_n, X_m) = \delta_{mn}$. Demak,

$$u(x,t) = \sum_{k=0}^{\infty} a_k X_k(x) e^{-\lambda_k^2 t}, \qquad a_k = \frac{2}{l}\int\limits_0^l dx\varphi(x) X_k(x).$$

6.28-mashq: Avvalgi mashqdan oz farq qiladi. Masalaning fazoviy qismi uchun quyidagiga egamiz:

$$X(x) = c_1\cos\lambda x + c_2\sin\lambda x, \quad X'(0) - hX(0) = 0,\ X'(l) + hX(l) = 0.$$

Uning yechimi:

$$X_k(x) = c_1\left(\cos\lambda_k x + \frac{h}{\lambda_k}\sin\lambda_k x\right), \qquad \mathrm{tg}(\lambda_k l) = \frac{2\lambda_k h}{\lambda_k^2 + h^2}, \quad k = 0, 1, 2, ...$$

Agar c_1 ni quyidagicha tanlab olsak:

$$c_1 = \frac{1}{\sqrt{\int_0^l dx\left(\cos\lambda_k x + \frac{h}{\lambda_k}\sin\lambda_k x\right)^2}} = \frac{2\sqrt{2}\lambda_k^2 h}{(\lambda_k^2 + h^2)\sin(\lambda_k l)\sqrt{h + l(\lambda_k^2 + h^2)}}$$

$X_k(x)$ funksiyaning normasi birga teng bo'ladi: $||X_k|| = 1$. Natijada, $\{X_k(x),\ k = 0, 1, 3, ...\}$ funksiyalar to'plami ortonormal sistemani hosil qiladi:

$(X_n, X_m) = \delta_{mn}$. Demak,

$$u(x,t) = \sum_{k=0}^{\infty} a_k X_k(x) e^{-\lambda_k^2 t}, \qquad a_k = \frac{2}{l}\int_0^l dx \varphi(x) X_k(x).$$

7.1-mashq: VIII.8-mashqning natijasiga asosan

$$\Delta \frac{\sin kr}{r} = -k^2 \frac{\sin kr}{r}.$$

7.2-mashq: Poisson formulalarining isboti quyidagi sodda hisobga asoslangan:

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha = -\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} t^n (e^{i\alpha} + e^{-i\alpha}) = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{1 - te^{i\alpha}} + \frac{1}{1 - te^{-i\alpha}}\right) =$$

$$= -\frac{1}{2} + \frac{1 - t\cos\alpha}{1 + t^2 - 2t\cos\alpha} = \frac{1}{2}\frac{1 - t^2}{1 + t^2 - 2t\cos\alpha}.$$

7.3-mashq:

$$u = \frac{1}{2} + \frac{1}{2}\rho^2 \cos(2\varphi) = \frac{1}{2} + \frac{1}{2}(x^2 - y^2).$$

7.4-mashq:

$$u = \frac{3}{8} + \frac{\rho^2}{2}\cos(2\varphi) + \frac{\rho^4}{8}\cos(4\varphi) = \frac{3}{8} + \frac{1}{2}(x^2 - y^2) - \frac{3}{4}x^2y^2 + \frac{1}{8}(x^4 + y^4).$$

7.5-mashq:

$$u = \frac{3\rho}{4}\sin\varphi - \frac{\rho^3}{4}\sin(3\varphi) = \frac{3}{4}y - \frac{3}{4}x^2 y + \frac{1}{4}y^3.$$

7.6-mashq:

$$u = \frac{1}{2} - \frac{\rho^2}{32}\cos(2\varphi) + \frac{5\rho^4}{16}\cos(4\varphi) + \frac{\rho^6}{32}\cos(6\varphi) =$$

$$= \frac{1}{2} + \frac{1}{32}(y^2 - x^2) + \frac{5}{16}(x^4 + y^4 - 6x^2y^2) + \frac{1}{32}\left(x^6 - y^6 + 15x^2y^2(y^2 - x^2)\right).$$

7.7-mashq: (26)-shart bajarilgan.

$$u = A\rho\cos\varphi + C = Ax + C.$$

7.8-mashq: (26)-shart bajarilgan.

$$u = \frac{A\rho^2}{2R}\cos(2\varphi) + C = A\frac{x^2 - y^2}{2R} + C.$$

7.9-mashq: (26)-shart bajarilgan.

$$u = -\frac{\rho}{12}\sin\varphi + \frac{3}{4}\frac{\rho^3}{R^2}\sin(3\varphi) + C = -\frac{y}{12} + \frac{9x^2y}{4R^2} - \frac{3y^3}{4R^2} + C.$$

7.10-mashq: Yechimni $u(x,y) = v(x,y) + w(x,y)$ ko'rinishda qidiramiz, v va w funksiyalar uchun quyidagi chegaraviy shartlarni olamiz:

$$v(0,y) = A\sin\frac{\pi y}{b}, \quad v(a,y) = v(x,0) = v(x,b) = 0;$$

$$w(0,y) = w(a,y) = w(x,b) = 0, \quad w(x,0) = B\sin\frac{\pi x}{a}.$$

v uchun masala quyidagicha yechiladi:

$$v_{xx} + v_{yy} = 0, \quad v(x,y) = X(x)Y(y), \quad X'' - \lambda X = 0,\ Y'' + \lambda Y = 0,$$

$$Y(y) = c_1\sin\frac{\pi y}{b}, \quad X(x) = c_2\mathrm{ch}\frac{\pi x}{b} + c_3\mathrm{sh}\frac{\pi x}{b}.$$

Chegaraviy shartlarni ishlatish natijasida quyidagini olamiz:

$$v(x,y) = A\frac{\mathrm{sh}\frac{\pi(a-x)}{b}}{\mathrm{sh}\frac{\pi a}{b}}\sin\frac{\pi y}{b}.$$

w uchun masala ham xuddi shu yo'l bilan yechiladi. Umumiy javob:

$$u(x,y) == A\frac{\mathrm{sh}\frac{\pi(a-x)}{b}}{\mathrm{sh}\frac{\pi a}{b}}\sin\frac{\pi y}{b} + B\frac{\mathrm{sh}\frac{\pi(b-y)}{a}}{\mathrm{sh}\frac{\pi b}{a}}\sin\frac{\pi x}{a}.$$

8.1-mashq: Ikkala mashq bir xil yechimga ega. Sodda holdan boshlaymiz:

$$\int\limits_{-\infty}^{\infty}\delta[a(x-x_0)]f(x)dx = \int\limits_{-\infty}^{\infty}\delta(y)f(x_0 + y/a)\frac{dy}{|a|} = \frac{1}{|a|}f(x_0).$$

Demak, $\delta[a(x-x_0)] = \delta(x-x_0)/|a|$. Ko'p o'lchamli holda:

$$\delta^{(n)}(a(x-x_0)) = \frac{1}{|\det a|}\delta^{(n)}(x-x_0).$$

8.2-mashq: $f(x)$ ning teskarisi mavjud deb olamiz.

$$\int \delta(f(x))\varphi(x)dx = \int \delta(y)\varphi(x(y))dx = \int \delta(y)\tilde{\varphi}(y)\left|\frac{dx}{dy}\right|dy = \sum_i \frac{1}{|f'(x_i)|}\varphi(x_i),$$

bu yerda x_i nuqtalar $f(x) = 0$ tenglamaning yechimlari.

8.3-mashq: Paragrafning ichida ko'rsatilgan misollarga o'xshab bevosita hisoblanadi.

8.4-mashq: Sferik sistemada

$$d^3r = \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2dr\sin\theta d\theta d\varphi = r^2drd(\cos\theta)d\varphi.$$

$\int d^3rf(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) = f(\mathbf{r_0})$ bo'lishini ta'minlash uchun $\delta(r - r_0) = \frac{1}{r^2}\delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\varphi - \varphi_0)$ bo'lishi kerak.

8.5-mashq: $f(x)$ funksiya 2π davrli deb olamiz. Bu holda uning Fourier-qatori uchun

$$f(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} f_m e^{-imx}, \qquad f_m = \int_0^{2\pi} dx e^{imx} f(x)$$

ga egamiz. Mashqdagi munosabat quyidagicha tekshiriladi:

$$f(x_0) = \int_0^{2\pi} dx\delta(x - x_0)f(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-imx_0}\int_0^{2\pi} dx e^{imx} f(x) =$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} f_m e^{-imx_0} = f(x_0).$$

8.6-mashq: Jordan lemmasidan darhol kelib chiqadi.

8.7-mashq:

$$Q_1^m = \sqrt{\frac{4\pi}{3}}\int_0^{\infty} drr^3\int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\varphi\rho(r,\theta,\varphi)Y_1^m(\theta,\varphi)$$

I.4-misoldagi zaryadlar taqsimotiga (I.4-rasmning a) qismida ko'rsatilgan) quyidagi zaryadlar zichligi mos keladi:

$$\rho(r,\theta,\varphi) = \frac{q}{r^2}\delta(r - a)[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)].$$

$\cos\theta$ bo'yicha delta-funksiyalarni hisoblaganda ularning argumentlarini $\cos\theta \mp (1 - \varepsilon)$, $\varepsilon \to 0$ ma'nosida tushunish kerak.

VIII.8-mashq:

$$\Delta\frac{e^{\pm ikr}}{r} = \nabla\cdot\nabla\frac{e^{\pm ikr}}{r} = \nabla\left(\nabla\frac{1}{r}e^{\pm ikr} \pm ik\frac{\mathbf{r}}{r^2}e^{\pm ikr}\right) =$$

$$= \left(\Delta\frac{1}{r}\right)e^{\pm ikr} + \nabla\frac{1}{r}\cdot\nabla e^{\pm ikr} \pm ik\nabla\left(\frac{\mathbf{r}}{r^2}e^{\pm ikr}\right) = -4\pi\delta(\mathbf{r})e^{\pm ikr} - k^2\frac{e^{\pm ikr}}{r} =$$

$$= -4\pi\delta(\mathbf{r}) - k^2\frac{e^{\pm ikr}}{r}.$$

8.9-mashq:

$$\int \frac{\theta(R-r)}{\sqrt{R^2-r^2}}e^{i\mathbf{k}\cdot\mathbf{r}}d^2r = \int_0^R \frac{rdr}{\sqrt{R^2-r^2}}\int_0^{2\pi} d\varphi e^{ikr\cos\varphi} = 2\pi\int_0^R \frac{rdr}{\sqrt{R^2-r^2}}J_0(kr) =$$

$$= 2\pi R\int_0^1 \frac{du\, u}{\sqrt{1-u^2}}J_0(kRu) = 2\pi R\int_0^{\pi/2} d\theta\cos\theta\, J_0(kR\cos\theta).$$

Bu yerda, birinchidan, I.12-mashqning natijasi ishlatildi, ikkinchidan, $u = \cos\theta$ almashtirish bajarildi.

8.10-mashq:

$$\int_0^{\pi/2} J_0(x\cos\theta)\cos\theta d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\int_0^{\pi/2}\cos^{2k+1}\theta d\theta =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\frac{k!\sqrt{\pi}}{2(k+1/2)!} = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{2k}\frac{1}{k!(k+1/2)!} =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!} = \frac{\sin x}{x}.$$

Bu hisoblashda (10)-formula $\nu = 0$ hol uchun ishlatildi, undan tashqari, Legendrening ikkilash formulasi (22)-ham ishlatildi.

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Mundarija