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$$\sum \int_{-\infty}^{+\infty} f(x,y)dx$$

MATEMATIK ANALIZDAN MISOL VA MASALALAR

O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS
TA'LIM VAZIRLIGI

A. GAZIYEV, I. ISRAILOV, M.YAXSHIBOYEV

MATEMATIK ANALIZDAN MISOL VA MASALALAR

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan (5460100 — matematika, 5440200 — mexanika, 5480100 — amaliy matematika va informatika, 5440100 — fizika) bakalavriyat ta'lim yo'nalishi talabalari uchun matematik analiz fanidan o'quv qo'llanma sifatida tavsiya etilgan

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Ushbu qo'llanma matematik analizning sonli qatorlar, funksional qatorlar, xosmas integrallar, parametrga bog'liq bo'lgan integrallar bo'limlari bo'yicha talabalarda misol va masalalarni mustaqil yechish ko'nikmasini hosil qilishga mo'ljallangan.

Qo'llanmaning har bir paragrafida, avvalo mavzuning nazariy qismidan qisqacha ma'lumot berilgan, so'ngra mavzuga mos tipik misol va masalalar batafsil yechib ko'rsatilgan hamda mustaqil ishlash uchun yetarli miqdorda misol va masalalar javoblari bilan berilgan.

O'quv qo'llanma bakalavriatning «matematika», «mexanika», «amaliy matematika va informatika», «fizika» yo'nalishlari va texnika yo'nalishlarining «Oliy matematika kursi» chuqurlashtirilgan dastur asosida o'qitiladigan talabalari hamda o'qituvchilar uchun mo'ljallangan.

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SO‘ZBOSHI

Matematik analiz fanining funksional ketma-ketliklar, funksional qatorlar, xosmas integrallar, parametrga bog‘liq bo‘lgan integrallar bo‘limlarini o‘zlashtirishda talabalar ancha qiyinchiliklarga duch keladilar, bu ularning yuqoridagi bo‘limlar bo‘yicha misol va masalalarni yechishida yaqqol ko‘rinadi.

Mazkur qo‘llanmani yozishda talabalarda ko‘rsatilgan bo‘limlarga oid misol va masalalarni samarali yo‘l bilan yechish ko‘nikmalari hosil qilishni o‘z oldimizga maqsad qilib qo‘ydik va qo‘llanma talabalar uchun doimiy maslahatchi bo‘lib qolishiga umid qilamiz.

O‘quv qo‘llanma to‘rt bobdan iborat bo‘lib, ***birinchi bobda*** *sonli qatorlar,* ***ikkinchi bobda*** *funksional ketma-ketliklar va funksional qatorlar qaralgan. Qo‘llanmaning* ***uchinchi bobi*** *xosmas integrallarga bag‘ishlangan va, nihoyat,* ***to‘rtinchi bobda*** *parametrga bog‘liq xos va xosmas integrallar qaralgan. O‘z navbatida, har bir bob tegishli paragraflarga bo‘lingan bo‘lib, har bir paragraf mavzuga taalluqli asosiy ta’riflar, tasdiqlar va teoremalarni o‘z ichiga oladi, shuningdek, ularning har biri an’anaviy misollarni batafsil tahlil qilish bilan yechish orqali namoyish qilingan. Qo‘llanmada jami 220 ga yaqin misol va masala yechilgan, 800 ga yaqin misol va masala esa mustaqil yechish uchun tavsiya qilingan va ularning javoblari ham berilgan.*

Ushbu qo‘llanmani yozishga bizni undagan narsa ko‘p yillar mobaynida Alisher Navoiy nomidagi Samarqand Davlat universitetida matematik analiz kursidan olib borgan ma’ruza va amaliy mashg‘ulotlarimizda orttirgan tajribamiz natijasidir. O‘ylaymizki, qo‘llanma o‘z o‘quvchilarini topadi va boshqa mavjud o‘quv adabiyotlari qatorida matematik analiz kursining aytib o‘tilgan bo‘limlari bo‘yicha ularga bilimlarini oshirishga ko‘mak beradi.

O‘quv qo‘llanma haqidagi fikr-mulohazalar, undagi mavjud kamchiliklar bo‘yicha hamkasblarimizning takliflarini mamnuniyat bilan qabul qilamiz.

Mualliflar

I BOB

SONLI QATORLAR

1- §. Yaqinlashuvchi qatorlar va ularning yig'indisi

Ushbu sonlar ketma-ketligi berilgan bo'lsin:

$$a_1, a_2, ..., a_n, ...$$

1.1- ta'rif. Quyidagi $a_1 + a_2 + ... + a_n + ...$ ifodaga *sonli qator yoki cheksiz sonli qator* deyiladi va qisqacha $\sum_{n=1}^{\infty} a_n$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ..., \qquad (1.1)$$

bunda $a_1, a_2, ... a_n, ...$ *qatorning hadlari,* a_n esa *qatorning umumiy hadi* deyiladi.

(1.1) sonli qatorning hadlaridan ushbu

$$S_1 = a_1,\ S_2 = a_1 + a_2,\ S_3 = a_1 + a_2 + a_3, ..., S_n = a_1 + a_2 + ... + a_n, ...$$

yig'indilar ketma-ketligini tuzamiz. Bunday tuzilgan $\{S_n\}$ yig'indilar ketma-ketligi (1.1) sonli qatorning qismiy yig'indilar ketma-ketligi deyiladi. Bundan keyin sonli qator deyish o'rniga qator deymiz.

1.2- ta'rif. Agar $n \to \infty$ da (1.1) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligi chekli limitga ega, ya'ni

$$\lim_{n \to \infty} S_n = S$$

bo'lsa, u holda (1.1) *qator yaqinlashuvchi* deyiladi. Bu limitning qiymati S son esa (1.1) *qatorning yig'indisi* deyiladi va u quyidagicha yoziladi:

$$S = a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n .$$

1.3- ta'rif. Agar $n \to \infty$ da (1.1) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligining limiti cheksiz bo'lsa yoki mavjud bo'lmasa, u holda (1.1) qator *uzoqlashuvchi* deyiladi.

Qator qismiy yig‘indilari ketma-ketligining limitini topishda bu ketma-ketlikning umumiy hadini qulay shaklga keltirish muhim ahamiyatga ega. Buning uchun ko‘p hollarda quyidagi formulalardan foydalanish maqsadga muvofiq bo‘ladi:

$$A_n(x)=x+x^2+\ldots+x^n=\begin{cases}\dfrac{x}{1-x}-\dfrac{x^{n+1}}{1-x}, & x=1 \quad \text{bo‘lganda,}\\ n, & x=1 \quad \text{bo‘lganda;}\end{cases} \tag{A}$$

$$B_n(x)=1+2x+\ldots+nx^{n-1}=\begin{cases}\dfrac{1}{(1-x)^2}+\dfrac{nx^{n+1}-(1+n)x^n}{(1-x)^2}, & x\neq 1 \quad \text{bo‘lganda,}\\ \dfrac{n(n+1)}{2}, & x=1 \quad \text{bo‘lganda;}\end{cases} \tag{B}$$

$$C_n(x)=x+x^3+\ldots+x^{2n-1}=\begin{cases}\dfrac{x}{1-x^2}-\dfrac{x^{2n+1}}{1-x^2}, & x\neq \pm 1 \quad \text{bo‘lganda,}\\ \pm n, & x=\pm 1 \quad \text{bo‘lganda;}\end{cases} \tag{D}$$

$$D_n(x)=1+3x^2+\ldots+(2n-1)x^{2n-2}=$$

$$=\begin{cases}\dfrac{1+x^2}{(1-x^2)^2}+\dfrac{(2n-1)x^{2n+2}-(2n+1)x^{2n}}{(1-x^2)^2}, & x\neq \pm 1 \quad \text{bo‘lganda,}\\ n^2, & x=\pm 1 \quad \text{bo‘lganda.}\end{cases} \tag{E}$$

(A) formula maktab matematika kursidan ma’lum bo‘lganligi uchun uni isbotsiz, (B), (D) va (E) formulalarni esa isboti bilan keltiramiz.

Bizga ma’lumki, $x=1$ bo‘lsa,

$$B_n(1)=1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

bo‘ladi. Endi $x\neq 1$ bo‘lsin. Bu holda (B) tenglikning ikkala tomonini x ga ko‘paytirib, hosil bo‘lgan tenglikni $B_n(x)$ dan ayiramiz:

$$(1-x)B_n(x)=1+x+x^2+\ldots+x^{n-1}-nx^n=\frac{1-x^n}{1-x}-n\cdot x^n,$$

bundan

$$B_n(x)=\frac{1-x^n}{(1-x)^2}-\frac{n\cdot x^n}{1-x}=\frac{1-x^n-n\cdot x^n+n\cdot x^{n+1}}{(1-x)^2}=\frac{1}{(1-x)^2}+\frac{n\cdot x^{n+1}-(1+n)x^n}{(1-x)^2},$$

ya’ni talab qilingan formulani olamiz.

(D) formulani isbot qilamiz. $x=\pm1$ bo'lsin, u holda

$$C_n(1)=1+1+...+1=n,\ C_n(-1)=-1-1-...-1=-n$$

bo'ladi. $x\neq\pm1$ bo'lsin. Bu holda ham tenglikning ikkala tomonini x ga ko'paytiramiz:

$$xC_n(x)=x^2+x^4+...+x^{2n}.$$

Oxirgi tenglikning o'ng tomoniga (A) formulani qo'llab, quyidagiga ega bo'lamiz:

$$xC_n(x)=x^2\cdot\frac{1-x^{2n}}{1-x^2}.$$

Bundan (D) formula o'rinli ekanligi kelib chiqadi.

(E) formulaning o'rinli ekanligini isbotlaymiz. $x=\pm1$ bo'lsin, u holda

$$D_n(\pm1)=1+3+5...+(2n-1)=n^2.$$

Endi $x\neq\pm1$ bo'lsin. Bu holda $D_n(x)$ ning ikkala tomonini x^2 ga ko'paytirib, hosil bo'lgan tenglikni $D_n(x)$ dan ayiramiz:

$$(1-x^2)D_n(x)=1+2x^2+2x^4+...+2x^{2n-2}-(2n-1)x^{2n}=$$

$$=1+2x^2\cdot\frac{1-x^{2n-2}}{1-x^2}-(2n-1)x^{2n}.$$

Bu yerdan (E) formula o'rinli ekanligi kelib chiqadi.

1.1-teorema. Agar (1.1) *qator yaqinlashuvchi bo'lsa,* u holda

$$\lim_{n\to\infty}a_n=0 \qquad (1.2)$$

bo'ladi.

Eslatma. (1.2) shart qator yaqinlashuvchi bo'lishi uchun zaruriy shart bo'ladi, lekin yetarli shart bo'lmaydi. Agar qatorning umumiy hadi nolga intilmasa, ya'ni $\lim\limits_{n\to\infty}a_n\neq0$ bo'lsa, (1.1) qator uzoqlashuvchi bo'ladi.

1.1-misol. Ushbu $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning umumiy hadi $a_n=\frac{1}{\sqrt{n}}$ ning limitini topamiz: $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}=0$.

Qator yaqinlashuvchiligining zaruriy sharti (1.2) bajarilayapti, lekin berilgan qator uzoqlashuvchi, chunki

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq n\frac{1}{\sqrt{n}} = \sqrt{n}\ ,\ \lim_{n\to\infty}\sqrt{n} = \infty\ ,$$

bundan $\lim_{n\to\infty} S_n = \infty$. Demak, 1.3- ta'rifga asosan, berilgan qator uzoqlashuvchi.

1.2- misol. $\sum_{n=1}^{\infty}\left(\frac{2n^2-3}{2n^2+1}\right)^{n^2}$ qatorni yaqinlashishga tekshiring.

Yechillshi. Berilgan qatorning umumiy hadini quyidagi

ko'rinishda yozamiz: $a_n = \frac{\left(1-\frac{3}{2n^2}\right)^{n^2}}{\left(1+\frac{1}{2n^2}\right)^{n^2}}$.

Ikkinchi ajoyib limitdan foydalanib, $n\to\infty$ da a_n ning limitini

topamiz: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\left(1-\frac{3}{2n^2}\right)^{n^2}}{\left(1+\frac{1}{2n^2}\right)^{n^2}} = \frac{e^{-\frac{3}{2}}}{e^{\frac{1}{2}}} = e^{-2} \neq 0$.

Qator yaqinlashuvchi bo'lishligining zaruriy sharti (1.2) bajarilmaydi. Demak, berilgan qator uzoqlashuvchi.

1.3- misol. Hadlari geometrik progressiyadan iborat bo'lgan

$$\sum_{n=0}^{\infty} bq^n = b + bq + bq^2 + ... + bq^{n-1} + ...$$

qatorni yaqinlashuvchilikka tekshiring.

Yechilishi. $q \neq 1$ bo'lsin, (A) formulaga asosan, S_n qismiy yig'indini topamiz:

$$S_n = b + bq + bq^2 + ... + bq^{n-1} = \frac{b}{1-q} - \frac{bq^n}{1-q} .$$

Bu yerda berilgan qatorning yig'indisini topish uchun bir nechta holni qarash lozim:

1. $|q|<1$ bo‘lsa, $\lim\limits_{n\to\infty} q^n = 0$ bo‘ladi. $n\to\infty$ da S_n ning limitini topamiz:

$$S=\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{b}{1-q}-\frac{bq^n}{1-q}\right)=\frac{b}{1-q}.$$

Demak, $S=\dfrac{b}{1-q}$ chekli son bo‘lganligi uchun berilgan qator yaqinlashuvchi.

2. $|q|>1$ bo‘lsa, $\lim\limits_{n\to\infty} |q|^n = +\infty$. Bu holda

$$S=\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{b}{1-q}-\frac{bq^n}{1-q}\right)=\infty$$

bo‘lib, berilgan qator uzoqlashuvchi bo‘ladi.

3. $q=1$ bo‘lsa, $S_n = b\cdot n$ bo‘ladi. $n\to\infty$ da S_n ning limitini topamiz:

$$S=\lim_{n\to\infty} S_n = \lim_{n\to\infty} nb = \infty.$$

Demak, berilgan qator uzoqlashuvchi.

4. $q=-1$ bo‘lsa, $S_n = b-b+b-b+...+(-1)^{n-1}b$. $n\to\infty$ da S_n ning limitini topamiz:

$$\lim_{n\to\infty} S_n = \begin{cases} 0, & \text{agar } n-\text{juft bo‘sa,} \\ b, & \text{agar } n-\text{toq bo‘lsa.}\end{cases}$$

Bu holda S_n ning limiti mavjud emas. Demak, ta’rifga asosan, berilgan qator uzoqlashuvchi.

Shunday qilib, berilgan qator $|q|<1$ bo‘lganda yaqinlashuvchi, $|q|\geq 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

1.4- misol. Ushbu qatorni yaqinlashuvchilikka tekshiring:

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+...+(-1)^{n-1}\frac{1}{3^{n-1}}+... .$$

Yechilishi. Berilgan qatorning n- qismiy yig‘indisi

$$S_n = 1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+...+(-1)^{n-1}\frac{1}{3^{n-1}}$$

bo‘ladi. S_n ni (A) formuladan foydalanib, quyidagicha yozib olamiz:

$$S_n=1+\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^3+\ldots+\left(-\frac{1}{3}\right)^{n-1}=$$

$$=\frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)}=\frac{3}{4}-\frac{3}{4}\left(-\frac{1}{3}\right)^n.$$

Endi, $n\to\infty$ da limitga o‘tib, S ni topamiz:

$$S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{3}{4}-\frac{3}{4}\left(-\frac{1}{3}\right)^n\right]=\frac{3}{4}.$$

Demak, 1.2- ta’rifga asosan, berilgan qator yaqinlashuvchi.

1.5- misol. $1-6+27-\ldots+(-1)^{n-1}\cdot 3n+\ldots$ qatorni yaqinlashuvchilikka tekshiring

Yechilishi. Berilgan qatorning n- qismiy yig‘indisi

$$S_n=1-6+27-\ldots+(-1)^{n-1}\cdot 3n$$

bo‘ladi. S_n ni topish uchun (B) formulada x ni –3 bilan almashtiramiz:

$$S_n=\frac{1}{(1-(-3))^2}+\frac{n\cdot(-3)^{n+1}-(1+n)\cdot(-3)^n}{(1-(-3))^2}=\frac{1}{16}\left[1-(-3)^n(4n+1)\right].$$

Agar $n=2k, k\in N$ bo‘lsa, $S_{2k}=\frac{1}{16}[1-9^k(8k+1)]$ bo‘ladi.

Agar $n=2k-1, k\in N$ bo‘lsa, $S_{2k-1}=\frac{1}{16}[1+3^{2k-1}\cdot(8n-3)]$ bo‘ladi.

Bu yerdan

$$\lim_{k\to\infty}S_{2k}=-\infty,\quad \lim_{k\to\infty}S_{2k-1}=+\infty$$

bo‘lishi kelib chiqadi. Demak, 1.3- ta’rifga asosan, berilgan qator uzoqlashuvchi.

1.6- misol. Ushbu qatorni yaqinlashuvchilikka tekshiring:

$$\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2n}+\ldots.$$

Yechilishi. Berilgan qatorning n- qismiy yig‘indisini quyidagi ko‘rinishda yozamiz:

$$S_n=\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+...+\frac{2n-1}{2^n}=\frac{1}{2}\left[1+3\left(\frac{1}{\sqrt{2}}\right)^2+\right.$$

$$\left.+5\left(\frac{1}{\sqrt{2}}\right)^4+...+(2n-1)\left(\frac{1}{\sqrt{2}}\right)^{2n-2}\right].$$

(E) formulada $x=\frac{1}{\sqrt{2}}$ deb belgilasak, u holda qatorning n- qismiy yig'indisi

$$S_n=3+\frac{2n-1}{2^n}-\frac{2n-1}{2^{n-1}}$$

ko'rinishga ega bo'ladi. $n\to\infty$ da limitga o'tib, S ni topamiz:

$$S=\lim_{n\to\infty}S_n=3+\lim_{n\to\infty}\frac{2n+1}{2^{n+2}}-\lim_{n\to\infty}\frac{2n-1}{2^{n+1}}=3.$$

Demak, 1.2- ta'rifga ko'ra, berilgan qator yaqinlashuvchi.

1.7- misol. $\frac{1}{5}+\frac{1}{125}+\frac{1}{3125}+...$ qatorni yaqinlashuvchilikka tekshiring.

Yechilishi. Berilgan qatorning hadlari geometrik progressiyani tashkil etganligini e'tiborga olib, uning umimiy hadini topamiz:

$$a_n=\left(\frac{1}{5}\right)^{2n-1}.$$

Endi qatorning qismiy yig'indisini (D) formuladan foydalanib hisoblaymiz:

$$S_n=\frac{1}{5}+\left(\frac{1}{5}\right)^3+\left(\frac{1}{5}\right)^5+...+\left(\frac{1}{5}\right)^{2n-1}=$$

$$=\frac{\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}-\frac{\left(\frac{1}{5}\right)^{2n+1}}{1-\left(\frac{1}{5}\right)^2}=\frac{25}{24}\left[\frac{1}{5}-\left(\frac{1}{5}\right)^{2n+1}\right].$$

Bu yerdan

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{25}{24}\left[\frac{1}{5} - \left(\frac{1}{5}\right)^{2n+1}\right] = \frac{5}{24}.$$

Demak, 1.2- ta'rifga asosan, berilgan qator yaqinlashuvchi bo'ladi va uning yig'indisi $S = \frac{5}{24}$.

1.2- teorema. Agar istalgan $n \in N$ uchun (1.1) qatorning umumiy hadi $a_n = b_n - b_{n+1}$ ko'rinishda tasvirlansa va

$$\lim_{n\to\infty} b_n = b \qquad (1.3)$$

chekli limit mavjud bo'lsa, (1.1) qator yaqinlashuvchi va uning yig'indisi $S=b_1-b$ ga teng bo'ladi, ya'ni $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - b$.

Isboti. Teoremaning shartiga ko'ra (1.1) qatorning S_n qismiy yig'indisini quyidagi ko'rinishda yozamiz:

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_k - b_{k+1}) = (b_1 - b_2) + (b_2 - b_3) + \ldots + (b_n - b_{n+1}) = b_1 - b_{n+1}.$$

(1.3) shartga ko'ra $n \to \infty$ da limitga o'tib, S yig'indini topamiz:

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = b_1 - b.$$

Bundan esa teoremaning isboti kelib chiqadi.

1.8- misol. Ushbu qatorning yaqinlashuvchiligini ko'rsating va yig'indisini toping:

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)}, \quad (\alpha \neq -1, -2, -3, \ldots).$$

Yechilishi. Berilgan qatorning umumiy hadini

$$a_n = \frac{1}{(\alpha+n)(\alpha+n+1)} = b_n - b_{n+1}$$

ko'rinishda yozamiz, bunda $b_n = \frac{1}{\alpha+n}$. 1.2-teoremaning hamma shartlari bajariladi. Haqiqatan ham,

$$a_n = b_n - b_{n+1}, \ \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\alpha+n} = 0 = b, \ b_1 = \frac{1}{1+\alpha}.$$

Demak, berilgan qator yaqinlashuvchi bo‘lib, uning yig‘indisi

$$S=\sum_{n=1}^{\infty}\frac{1}{(\alpha+n)(\alpha+n+1)}=\frac{1}{1+\alpha}.$$

1.9- misol. $\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$ qatorning yaqinlashuvchiligini ko‘rsating va yig‘indisini toping.

Yechilishi. Berilgan qatorning umumiy hadini quyidagi ko‘rinishda shakl almashtiramiz:

$$a_n=\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}=\left(\sqrt{n}-\sqrt{n+1}\right)-$$

$$-\left(\sqrt{n+1}-\sqrt{n+2}\right)=b_n-b_{n+1},$$

bunda $b_n=\sqrt{n}-\sqrt{n+1}$. Endi $n\to\infty$ da b_n ning limitini topamiz:

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)=\lim_{n\to\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}=0=b.$$

1.2- teoremaning hamma shartlari bajariladi. Demak, berilgan qator yaqinlashuvchi va uning yig‘indisi $S=1-\sqrt{2}$ bo‘ladi.

Mustaqil yechish uchun misollar

Qatorlarning yaqinlashuvchiligini ko‘rsating va yig‘indisini toping:

1.1. $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots+\frac{1}{n\cdot(n+1)}+\ldots$.

1.2. $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\ldots+\frac{1}{(3n-2)\cdot(3n+1)}+\ldots$.

1.3. $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\ldots+\frac{1}{n(n+1)(n+2)(n+3)}+\ldots$.

1.4. $\frac{3}{1^2\cdot2^2}+\frac{5}{2^2\cdot3^2}+\ldots+\frac{2n+1}{n^2\cdot(n+1)^2}+\ldots$.

1.5. $\left(\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}\right)+\left(\frac{1}{10^2}+\frac{2}{10^3}+\frac{5}{10^4}\right)+...+$
$+\left(\frac{1}{10^n}+\frac{2}{10^{n+1}}+\frac{5}{10^{n+2}}\right)+...$.

1.6. $\frac{1}{1\cdot(1+m)}+\frac{1}{2\cdot(2+m)}+...+\frac{1}{n\cdot(n+m)}+...\quad (m\in N)$.

1.7. $1-\frac{1}{5}+\frac{1}{5^2}-...+\frac{(-1)^{n-1}}{5^{n-1}}+...$. **1.8.** $\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)^2}$.

1.9. $1+\frac{2}{\sqrt{3}}+1+\frac{4}{3\sqrt{3}}+...+n\left(\frac{1}{3}\right)^{\frac{n-1}{2}}+...$.

1.10. $\frac{1}{3}+\frac{1}{8}+...+\frac{1}{n^2-1}+...$.

1.11. $\sum_{n=1}^{\infty}\frac{1}{16n^2-8n-3}$. **1.12.** $\sum_{n=1}^{\infty}\frac{1}{36n^2+12n-35}$.

1.13. $\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$. **1.14.** $\sum_{n=2}^{\infty}\ln\left(1-\frac{2}{n(n+1)}\right)$.

1.15. $\sum_{n=1}^{\infty}\sin\frac{1}{2^n}\cos\frac{3}{2^n}$.

Qatorlar uchun qator yaqinlashuvchiligining zaruriy sharti bajarilmasligini ko'rsating:

1.16. $\sum_{n=1}^{\infty}\left(\frac{3n^3-2}{3n^3+4}\right)^3$. **1.17.** $\sum_{n=1}^{\infty}(n^2+2)\ln\frac{n^2+1}{n^2}$. **1.18.** $\sum_{n=2}^{\infty}\frac{1}{\sqrt[n]{\ln n}}$.

1.19. $\sum_{n=1}^{\infty}\sin n\alpha$, bunda $\alpha\neq\pi m,\ m\in Z$. **1.20.** $\sum_{n=1}^{\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$.

1.21. $\sum_{n=1}^{\infty}\sqrt[n]{0,002}$. **1.22.** $\sum_{n=1}^{\infty}(n+1)\operatorname{arctg}\frac{1}{n+2}$.

1.23. $\sum_{n=1}^{\infty}\frac{n^3+1}{n+3}\arcsin\frac{1}{n^2+2}$. 1.24. $\sum_{n=1}^{\infty}\sqrt{\frac{3n+5}{6n+7}}$.

1.25. $\sum_{n=1}^{\infty}(-1)^n\frac{n+4}{n+3}$.

Misollarning javoblari

1.1. $S=1$. **1.2.** $S=\frac{1}{3}$. **1.3.** $S=\frac{1}{18}$. **1.4.** $S=1$. **1.5.** $S=\frac{5}{36}$.

1.6. $S=\frac{1}{m}\left(1+\frac{1}{2}+...+\frac{1}{m}\right)$. **1.7.** $S=\frac{5}{6}$. **1.8.** $S=\frac{1}{8}$.

1.9. $S=\frac{3}{2}\left(\sqrt{3}+1\right)$. **1.10.** $S=\frac{3}{4}$. **1.11.** $S=\frac{1}{4}$. **1.12.** $S=\frac{2}{21}$.

1.13. $S=-\ln 2$. **1.14.** $S=-\ln 3$. **1.15.** $S=\frac{1}{2}\sin 2$.

2- §. *Yaqinlashuvchi qatorlarning xossalari*

2.1- xossa. Agar (1.1) qator yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty}ca_n$ qator ham yaqinlashuvchi va

$$\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n$$

tenglik o‘rinli bo‘ladi (c – ixtiyoriy o‘zgarmas son).

2.2-xossa. Agar $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty}(a_n\pm b_n)$ qator ham yaqinlashuvchi va

$$\sum_{n=1}^{\infty}(a_n\pm b_n)=\sum_{n=1}^{\infty}a_n\pm\sum_{n=1}^{\infty}b_n \qquad (2.1)$$

tenglik o‘rinli bo‘ladi.

2.1- eslatma. (2.1) tenglikning chap tomoni yaqinlashuvchiligidan uning o'ng tomonining yaqinlashuvchiligi har doim ham kelib chiqavermaydi. Masalan, umumiy hadlari $a_n = n + \frac{1}{3^n}$, $b_n = -n$ bo'lgan qatorlar uzoqlashuvchi, lekin $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} \frac{1}{3^n}$ yaqinlashuvchi geometrik qator.

2.1- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=0}^{\infty} Mx^n \ (|x| < 1,\ M \text{ — o'zgarmas son}).$$

Yechilishi. Berilgan qatorning n- qismiy yig'indisi

$$S_n = \sum_{k=0}^{n} Mx^k = M + Mx + Mx^2 + ... + Mx^n = M(1 + x + ... + x^n) = M\sum_{k=0}^{n} x^k.$$

Bundan, (A) formulaga asosan

$$S_n = \frac{M}{1-x} - \frac{Mx^{n+1}}{1-x}, \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{M}{1-x} - \frac{Mx^{n+1}}{1-x}\right) = \frac{M}{1-x}.$$

Demak, 2.1- xossaga asosan berilgan qator yaqinlashuvchi va

$$\sum_{n=0}^{\infty} Mx^n = M\sum_{n=0}^{\infty} x^n = \frac{M}{1-x}$$

o'rinli bo'ladi.

2.2- misol. $\sum_{n=1}^{\infty}\left(\frac{1}{3^n} + \frac{1}{5^n}\right)$ qatorni yaqinlashishga tekshiring.

Yechilishi. 2.2- xossaga asosan qatorni yaqinlashishga tekshiramiz. $\sum_{n=1}^{\infty}\frac{1}{3^n}$ va $\sum_{n=1}^{\infty}\frac{1}{5^n}$ qatorlarning n- qismiy yig'indilarini topamiz va ularning limitini hisoblaymiz:

$$S'_n = \frac{1}{3} + \frac{1}{3^2} + ... + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right), \quad S''_n = \frac{1}{5} + \frac{1}{5^2} + ... + \frac{1}{5^n} = \frac{1}{4}\left(1 - \frac{1}{5^n}\right),$$

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty}\frac{1}{2}\left(1 - \frac{1}{3^n}\right) = \frac{1}{2}, \qquad \lim_{n\to\infty} S''_n = \lim_{n\to\infty}\frac{1}{4}\left(1 - \frac{1}{5^n}\right) = \frac{1}{4}.$$

$\sum_{n=1}^{\infty}\frac{1}{3^n}$ va $\sum_{n=1}^{\infty}\frac{1}{5^n}$ qatorlar yaqinlashuvchi, u holda 2.2- xossaga asosan $\sum_{n=1}^{\infty}\left(\frac{1}{3^n}+\frac{1}{5^n}\right)$ qator ham yaqinlashuvchi va uning yig'indisi

$$S=\lim_{n\to\infty}\left(S_n'+S_n''\right)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}.$$

2.1- ta'rif. (1.1) qatorning birinchi m ta hadini tashlasak, u holda hosil bo'lgan ushbu

$$a_{m+1}+a_{m+2}+\ldots+a_n+\ldots=\sum_{n=m+1}^{\infty}a_n \qquad (2.2)$$

qator (1.1) qatorning m ta *hadidan keyingi qoldig'i* deyiladi.

2.3- xossa. Agar (1.1) qator yaqinlashuvchi bo'lsa, uning istalgan qoldig'i (2.2) ham yaqinlashuvchi bo'ladi va, aksincha (2.2) istalgan qoldig'ining yaqinlashuvchiligidan berilgan (1.1) qatorning ham yaqinlashuvchiligi kelib chiqadi.

Natija. Agar (1.1) qator yaqinlashuvchi bo'lsa, uning

$r_m=a_{m+1}+a_{m+2}+\ldots=\sum_{n=m+1}^{\infty}a_n$ qoldig'i $m\to\infty$ da nolga intiladi.

2.2- eslatma. Qatorning chekli sondagi hadlarini olib tashlash yoki qatorga chekli sondagi hadlarni qo'shish bilan uning yaqinlashish xarakteri o'zgarmaydi.

2.3- misol. $\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots$ qatorning qoldig'ini yaqinlashishga tekshiring.

Yechilishi. Ma'lumki, bu geometrik qator yaqinlashuvchi. Bu qatorning m - hadidan keyingi qoldig'ini tuzamiz:

$$r_m=\frac{1}{2^{m+1}}+\frac{1}{2^{m+2}}+\ldots+\frac{1}{2^n}+\ldots.$$

Qoldiq qatorning n-qismiy yig'indisini hisoblaymiz:

$$S_n=\frac{1}{2^{m+1}}+\frac{1}{2^{m+2}}+\ldots+\frac{1}{2^{m+n}}=\frac{1}{2^{m+1}}\left(1+\frac{1}{2}+\ldots+\frac{1}{2^{n-1}}\right)=\frac{1}{2^m}\left(1-\frac{1}{2^n}\right),$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2^m}\left(1-\frac{1}{2^n}\right)=\frac{1}{2^m},$$

bunda m — tayinlangan son. Shuning uchun qoldiq qator yaqinlashuvchi bo‘ladi. Natija tasdig‘ining to‘g‘riligini ko‘rsatish uchun

$$r_m = \frac{\frac{1}{2^{m+1}}}{1-\frac{1}{2}} = \frac{1}{2^m}$$

ni topib, $m \to \infty$ da limitga o‘tamiz:

$$\lim_{m\to\infty} r_m = \lim_{m\to\infty} \frac{1}{2^m} = 0.$$

(1.1) qator hadlarini guruhlab, quyidagi qatorni tuzamiz:

$$\left(a_1 + a_2 + ... + a_{n_1}\right) + \left(a_{n_1+1} + a_{n_1+2} + ... + a_{n_2}\right) + ... +$$

bunda $n_1, n_2, ... (n_1 < n_2 < ...)$ lar natural sonlar ketma-ketligining biror $\{n_k\}$ qismiy ketmaketligi bo‘lib, $k \to \infty$ da $n_k \to \infty$.

2.4- xossa. Agar (1.1) qator yaqinlashuvchi bo‘lib, uning yig‘indisi C bo‘lsa, uning hadlarini (o‘rinlarini o‘zgartirmasdan) guruhlash natijasida tuzilgan

$$\sum_{k=1}^{\infty} C_k = \sum_{k=1}^{\infty} \left(a_{n_{k-1}+1} + a_{n_{k-1}+2} + ... + a_{n_k}\right) \tag{2.3}$$

qator ham yaqinlashuvchi va uning yig‘indisi ham C ga teng bo‘ladi.

2.3- eslatma. Bu xossaning teskarisi o‘rinli emas, ya’ni (2.3) qatorning yaqinlashishidan (1.1) qatorning yaqinlashishi har doim ham kelib chiqavermaydi. Masalan, $\sum_{n=1}^{\infty} (-1)^{n+1}$ qator uzoqlashuvchi, lekin uning hadlarini ikkitalab guruhlashdan tuzilgan $(1-1)+(1-1)+...+(1-1)+...$ qator — yaqinlashuvchi.

2.5- xossa. Agar (1.1) qatorning hadlari musbat va uning hadlarini guruhlash natijasida tuzilgan

$$\sum_{n=1}^{\infty} A_n, \text{ bunda } A_n = \sum_{i=P_n}^{P_{n+1}-1} a_i \left(P_1 = 1,\ P_1 < P_2 < ...\right)$$

qator yaqinlashuvchi bo‘lsa, (1.1) qator ham yaqinlashuvchi bo‘ladi.

2.4- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{3}}+\frac{1}{3\sqrt{4}}+...+\frac{1}{n\sqrt{n+1}}+....$$

Yechilishi. Berilgan qatorning hadlarini guruhlash natijasida hosil bo'lgan

$$\frac{1}{\sqrt{2}}+\left(\frac{1}{2\sqrt{3}}+\frac{1}{3\sqrt{4}}\right)+\left(\frac{1}{4\sqrt{5}}+\frac{1}{5\sqrt{6}}+\frac{1}{6\sqrt{7}}+\frac{1}{7\sqrt{8}}\right)+$$

$$\left(\frac{1}{8\sqrt{9}}+...+\frac{1}{15\sqrt{16}}\right)+...+\left(\frac{1}{2^n\sqrt{2^n+1}}+.....+\frac{1}{(2^{n+1}-1)\sqrt{2^{n+1}}}\right)+... \quad (*)$$

qatorni qaraymiz. Bu (*) qatorning har bir hadini baholaymiz:

$$\frac{1}{2\sqrt{3}}+\frac{1}{3\sqrt{4}}<\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}<\frac{2}{\sqrt{2^3}}=\frac{1}{\sqrt{2}},$$

$$\frac{1}{4\sqrt{5}}+\frac{1}{5\sqrt{6}}+\frac{1}{6\sqrt{7}}+\frac{1}{7\sqrt{8}}<\frac{1}{4\sqrt{4}}+...+\frac{1}{7\sqrt{7}}<\frac{4}{\left(\sqrt{2^3}\right)^2}=\frac{1}{\left(\sqrt{2}\right)^2},$$

....

$$\frac{1}{2^n\sqrt{2^n+1}}+...+\frac{1}{(2^{n+1}-1)\sqrt{2^{n+1}}}<\frac{1}{\sqrt{(2^n)^3}}+...+\frac{1}{\sqrt{(2^{n+1}-1)^3}}<\frac{1}{\left(\sqrt{2}\right)^n}.$$

(*) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligi uchun quyidagi baholarga ega bo'lamiz:

$$S_n=\frac{1}{\sqrt{2}}+...+\frac{1}{(2^{n+1}-1)\sqrt{2^{n+1}}}<\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1}{\left(\sqrt{2}\right)^2}+\frac{1}{\left(\sqrt{2}\right)^3}+...+$$

$$+\frac{1}{\left(\sqrt{2}\right)^n}\leq\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\right)^2+...+\left(\frac{1}{\sqrt{2}}\right)^n+...=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}-1}.$$

$\{S_n\}$ ketma-ketlik monoton o'suvchi va yuqoridan chegaralangan, u holda (*) qatorning $\{S_n\}$ qismiy yig'indilar ketma-ketligi chekli limitga ega bo'ladi. 1.2-ta'rifga ko'ra, (*) qator yaqinlashuvchi.

Demak, 2.5- xossaga asosan berilgan qator yaqinlashuvchi.

2.1- teorema(Koshi kriteriysi). (1.1) qator yaqinlashuvchi bo'lishi uchun istalgan musbat $\varepsilon > 0$ son olinganda ham shunday $n_0(\varepsilon) \in N$ mavjud bo'lib, barcha $n > n_0(\varepsilon)$ va $p \in N$ lar uchun

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| < \varepsilon \qquad (2.4)$$

tengsizlikning bajarilishi zarur va yetarli.

2.1- teoremaga ixtiyoriy qator yaqinlashishining *Koshi kriteriysi* deyiladi.

2.4- eslatma. (2.4) shart bajarilmasa, ya'ni

$$\exists \varepsilon_0 > 0 : \forall k \in N \ \ \exists n \geq k \ \ \exists p \in N :$$

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+p}| \geq \varepsilon_0 \qquad (2.5)$$

tengsizlik o'rinli bo'lsa, (1.1) qator uzoqlashuvchi bo'ladi.

2.5- misol. Koshi kriteriysidan foydalanib, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ qatorning yaqinlashuvchiligini ko'rsating.

Yechilishi. Berilgan qator uchun (2.4) shartning bajarilishini tekshiramiz:

$$|S_{n+p} - S_n| = \left| \frac{\cos(n+1)}{2^{n+1}} + \frac{\cos(n+2)}{2^{n+2}} + ... + \frac{\cos(n+p)}{2^{n+p}} \right| \leq$$

$$\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + ... + \frac{1}{2^{n+p}} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + ... + \frac{1}{2^{n+p}} + ... = \frac{1}{2^n}.$$

Endi berilgan $\forall \varepsilon > 0$ songa ko'ra $n_0(\varepsilon) = [-\log_2 \varepsilon] + 1$ deb olinsa, barcha $n > n_0(\varepsilon)$ va $p \in N$ lar uchun $|S_{n+p} - S_n| < \varepsilon$ tengsizlik o'rinli bo'ladi. Demak, Koshi kriteriysiga asosan, berilgan qator yaqinlashuvchi bo'ladi.

2.6- misol. Koshi kriteriysidan foydalanib ushbu

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n+1)n}}$$

qatorning uzoqlashuvchi bo'lishini ko'rsating.

Yechilishi. $\varepsilon_0 = \frac{1}{4}$ deb olamiz. Faraz qilaylik, $p=n$ bo'lsin. U holda barcha $n \in N$ lar uchun

$$|S_{2n} - S_n| = \frac{1}{\sqrt{(n+1)(n+2)}} + \frac{1}{\sqrt{(n+2)(n+3)}} + \ldots +$$

$$+\frac{1}{\sqrt{2n(2n+1)}} > \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+1} > \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} > \frac{1}{4} = \varepsilon_0.$$

Demak, (2.5) shartga asosan, berilgan qator uzoqlashuvchi.

Mustaqil yechish uchun misollar

Qatorlarning qismiy yig'indilari ketma-ketligini va yig'indisini toping:

2.1. $\sum_{n=1}^{\infty}\left(\frac{3}{2^{n-1}} + \frac{(-1)^{n-1}}{2\cdot 3^{n-1}}\right)$. **2.2.** $\sum_{n=1}^{\infty}\frac{1}{5}\left(\frac{1}{5n-2} - \frac{1}{5n+3}\right)$.

2.3. $\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. **2.4.** $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

2.5. $\sum_{n=1}^{\infty}\frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}}$. **2.6.** $\sum_{n=1}^{\infty}\frac{1}{(4n+5)(4n+9)}$.

Qatorlarning yaqinlashuvchiligini Koshi kriteriysidan foydalanib, ko'rsating:

2.7. $a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} + \ldots (|a_n| < 10)$.

2.8. $\frac{\sin x}{3} + \frac{\sin 2x}{3^2} + \ldots + \frac{\sin nx}{3^n} + \ldots$.

2.9. $\frac{\cos x - \cos 2x}{1} + \frac{\cos 2x - \cos 3x}{2} + \ldots + \frac{\cos nx - \cos(n+1)x}{n} + \ldots$.

2.10. $\frac{\cos x}{1^2} + \frac{\cos x^2}{2^2} + \ldots + \frac{\cos x^n}{n^2} + \ldots$.

2.11. $1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+...$.

2.12. $\frac{\sin^3 1\cdot x}{2\cdot 4}+\frac{\sin^3 2x}{3\cdot 5}+...+\frac{\sin^3 nx}{(n+1)(n+3)}...$.

Koshi kriteriysidan foydalanib, quyidagi qatorlarning uzoqlashuvchiligini ko‘rsating:

2.13. $1+\frac{1}{2}+...+\frac{1}{n}+...$. **2.14.** $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$.

2.15. $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...+\frac{1}{2n+1}+...$.

2.16. $\frac{2}{5}+\frac{3}{8}+\frac{4}{13}+...+\frac{n+1}{n^2+4}+...$.

2.17. $\ln 2+\ln\frac{2}{3}+...+\ln\left(1+\frac{1}{n}\right)+...$.

2.18. $\frac{1}{\sqrt{1\cdot 3}}+\frac{1}{\sqrt{3\cdot 5}}+...+\frac{1}{\sqrt{(2n-1)(2n+1)}}+...$.

Misollarning javoblari

2.1. $S_n=\frac{51}{8}-\frac{3}{2^{n-1}}+\frac{(-1)^{n-1}}{8\cdot 3^{n-1}}$, $S=\frac{51}{8}$.

2.2. $S_n=\frac{1}{5}\left(\frac{1}{3}-\frac{1}{5n+3}\right)$, $S=\frac{1}{15}$.

2.3. $S_n=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$, $S=\frac{3}{4}$.

2.4. $S_n=1-\frac{1}{(n+1)^2}$, $S=1$. **2.5.** $S_n=\sqrt{\frac{n}{n+1}}$, $S=1$.

2.6. $S_n=\frac{1}{4}\left(\frac{1}{9}-\frac{1}{4n+9}\right)$, $S=\frac{1}{36}$

3- §. Musbat hadli qatorlar

3.1. Musbat qatorlarning yaqinlashuvchi bo'lishlik sharti. Biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (3.1)$$

qator berilgan bo'lsin.

Agar $a_n \geq 0, (n = 1, 2, \ldots)$ bo'lsa, (3.1) qator *musbat hadli qator* yoki qisqacha *musbat qator* deb ataladi.

3.1- teorema. (3.1) musbat qator yaqinlashuvchi bo'lishi uchun uning qismiy yig'indilar ketma-ketligining yuqoridan chegaralangan bo'lishi zarur va yetarlidir.

3.1- natija. Musbat hadli qatorning qismiy yig'indilari ketma-ketligi yuqoridan chegaralanmagan bo'lsa, qator uzoqlashuvchi bo'ladi.

3.1- misol. Ushbu qatorni yaqinlashuvchilikka tekshiring:

$$\frac{\cos^2 1}{1 \cdot 2} + \frac{\cos^2 2}{2 \cdot 3} + \ldots + \frac{\cos^2 n}{n(n+1)} + \ldots .$$

Yechilishi. Berilgan qatorning umumiy hadi uchun quyidagi tengsizlik o'rinli: $a_k = \dfrac{\cos^2 k}{k(k+1)} \leq \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$, chunki $\cos^2 n \leq 1$. U holda

$$S_n = \sum_{k=1}^{n} \frac{\cos^2 n}{k(k+1)} \leq \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} < 1.$$

$\{S_n\}$ qismiy yig'indilar ketma-ketligi yuqoridan 1 soni bilan chegaralangan. Demak, yuqoridagi 3.1- teoremaga asosan berilgan qator yaqinlashuvchi.

3.2- teorema. Agar (3.1) qatorning hadlari monoton kamayuvchi, ya'ni $a_n > a_{n+1} \geq 0$ $(n = 1, 2, 3, \ldots)$ bo'lsa, u holda (3.1) qator bilan

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + \ldots + 2^k a_{2^k} + \ldots \qquad (3.2)$$

qator bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo'ladi.

3.2- misol. Ushbu umumlashgan garmonik qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+...+\frac{1}{n^p}+....$$

Yechilishi. 1) $p\le 0$ bo'lganda berilgan qatorning uzoqlashuvchi ekanligi 1.1-teoremadan kelib chiqadi, chunki bu holda qatorning umumiy hadi $n\to\infty$ da nolga intilmaydi.

2) p>0 bo'lsin, u holda 3.2-teoremadan foydalanib, qator yaqinlashuvchiligini tekshiramiz, chunki uning hadlari monoton kamayuvchi, ya'ni $\frac{1}{n^p}>\frac{1}{(n+1)^p}$. Bu holda $\sum_{k=0}^{\infty}2^k\frac{1}{(2^k)^p}=\sum_{k=0}^{\infty}2^{k(1-p)}$

bo'ladi. Bu yerda bir necha hol bo'lishi mumkin:

a) $p>1$ bo'lsa, u holda $2^{(1-p)}<1$ va $\sum_{k=0}^{\infty}\frac{1}{2^{k(p-1)}}$ cheksiz kamayuvchi geometrik qator hosil bo'ladi. Bu qator yaqinlashuvchi va uning yigindisi $S=\sum_{k=0}^{\infty}\frac{1}{2^{k(p-1)}}=\frac{2^{p-1}}{2^{p-1}-1}$ bo'ladi.

b) $0<p<1$ bo'lsin, u holda $2^{1-p}>1$ bo'ladi. Shunga asosan, $\sum_{k=0}^{\infty}2^{k(1-p)}$ geometrik qator uzoqlashuvchi bo'ladi.

d) $p=1$ bo'lsa, u holda $2^{(1-p)}=1$. Natijada 1+1+...+1+... qator hosil bo'ladi. Bu qator esa uzoqlashuvchi.

Shunday qilib, berilgan umumlashgan garmonik qator, 3.2-teoremaga asosan, $p>1$ bo'lganda yaqinlashuvchi, $p\le 1$ bo'lganda esa uzoqlashuvchi bo'ladi.

3.2. Musbat hadli qatorlarni taqqoslash haqidagi teoremalar. Ikkita

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+..., \tag{3.3}$$

$$\sum_{n=1}^{\infty}b_n=b_1+b_2+...+b_n+... \tag{3.4}$$

musbat qatorlar berilgan bo'lsin.

3.3- teorema. Agar n ning biror $n_0 \left(n_0 \geq 1\right)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun $a_n \leq b_n$ tengsizlik o'rinli bo'lsa, (3.4) qatorning yaqinlashuvchi bo'lishidan (3.3) qatorning ham yaqinlashuvchi bo'lishi yoki (3.3) qatorning uzoqlashuvchi bo'lishidan (3.4) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

3.4- teorema. Agar $n \to \infty$ da $\frac{a_n}{b_n}$ $\left(a_n \geq 0,\ b_n > 0\right)$ nisbat

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k \ \left(0 \leq k \leq +\infty\right)$$

limitga ega bo'lsa, u holda:

a) $k < +\infty$ bo'lganda (3.4) qatorning yaqinlashuvchi bo'lishidan (3.3) qatorning yaqinlashuvchi bo'lishi;

b) $k > 0$ bo'lganda (3.4) qatorning uzoqlashuvchi bo'lishidan (3.3) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

3.2- natija. Agar

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k$$

limit o'rinli bo'lib, $0 < k < \infty$ bo'lsa, (3.3) va (3.4) qatorlar bir vaqtda yaqinlashuvchi yoki uzoqlashuvchi bo'ladi.

3.3- natija. Agar $n \to \infty$ da $a_n \sim b_n$ bo'lsa, (3.3) va (3.4) qatorlar bir vaqtda yaqinlashuvchi yoki bir vaqtda uzoqlashuvchi bo'ladi.

3.5- teorema. Agar n ning biror $n_0 \left(n_0 \geq 1\right)$ qiymatidan boshlab barcha $n \geq n_0$ lar uchun

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \quad \left(a_n > b_n, b_n > 0\right)$$

tengsizlik o'rinli bo'lsa, u holda (3.4) qatorning yaqinlashuvchi bo'lishidan (3.3) qatorning ham yaqinlashuvchi bo'lishi yoki (3.3) qatorning uzoqlashuvchi bo'lishidan (3.4) qatorning ham uzoqlashuvchi bo'lishi kelib chiqadi.

3.3- misol. $\sum\limits_{n=1}^{\infty} \frac{5+3\left(-1\right)^n}{2^{n+3}}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Ravshanki, barcha $n \in N$ lar uchun $2 \le 5+3(-1)^n \le 8$ bo'ladi. Bu holda

$$0 < a_n = \frac{5+3(-1)^n}{2^{n+3}} \le \frac{1}{2^n} = b_n$$

bo'lishi kelib chiqadi. Ma'lumki, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometrik qator $\left(q = \frac{1}{2} < 1\right)$ yaqinlashuvchi. Demak, 3.3- teoremaga ko'ra, berilgan qator ham yaqinlashuvchi bo'ladi.

3.4- misol. $\sum_{n=1}^{\infty} \frac{e^n + n^3}{3^n + \ln^2(n+1)}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Asimptotik formulalardan foydalanib, quyidagilarga ega bo'lamiz: $n \to \infty$ da $e^n + n^3 \sim e^n$, $3^n + \ln^2(n+1) \sim 3^n$.

Bu holda $n \to \infty$ da $a_n = \frac{e^n + n^3}{3^n + \ln^2(n+1)} \sim \left(\frac{e}{3}\right)^n = b_n$,

bunda $\frac{e}{3} < 1$. Ma'lumki, $\sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$ geometrik qator $\left(q = \frac{e}{3} < 1\right)$ yaqinlashuvchi. Yuqorida keltirilgan 3.3-natijaga asosan, berilgan qator yaqinlashuvchi bo'ladi.

3.5- misol. $\sum_{n=2}^{\infty} \frac{1}{(n+1)^{\frac{2}{\ln n}}}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorni $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator bilan solishtiramiz. Ravshanki, bu ikki qator umumiy hadlari nisbatining limiti

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{2}{\ln n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)^{\frac{2}{\ln n}}} = \lim_{n\to\infty} e^{\ln n} \cdot e^{-2\frac{\ln(n+1)}{\ln n}} = +\infty$$

bo'ladi. Yuqorida keltirilgan 3.4-teoremaga ko'ra berilgan qator uzoqlashuvchi bo'ladi.

3.3. Musbat hadli qatorlar uchun yaqinlashuvchilik alomatlari. Dalamber alomati. Biror

$$\sum_{n=1}^{\infty} a_n \ (\text{barcha } n \in N \text{ lar uchun } a_n > 0) \qquad (3.5)$$

qator berilgan bo'lsin. U holda:

a) agar shunday $q \in (0,1)$ son va $m(m \in N)$ nomer mavjud bo'lib va $\forall n \geq m$ dan boshlab

$$\frac{a_{n+1}}{a_n} \leq q$$

tengsizlik o'rinli bo'lsa, (3.5) qator yaqinlashuvchi bo'ladi.

b) agar shunday $m(m \in N)$ nomer mavjud bo'lib, barcha $n \geq m$ lar uchun

$$\frac{a_{n+1}}{a_n} \geq 1$$

tengsizlik o'rinli bo'lsa, (3.5) qator uzoqlashuvchi bo'ladi.

3.4- natija (Dalamber alomatining limit ko'rinishi). Agar (3.5) qator uchun

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda \qquad (3.6)$$

mavjud bo'lib, $\lambda < 1$ bo'lsa, (3.5) qator yaqinlashuvchi, $\lambda > 1$ bo'lganda esa, qator uzoqlashuvchi bo'ladi.

3.6- misol. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$, $a > 0$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qator uchun $a_n = \frac{a^n}{n!}, a_{n+1} = \frac{a^{n+1}}{(n+1)!}$ bo'ladi.

Dalamber alomatining limit ko'rinishidan foydalanib, qatorni yaqinlashishga tekshiramiz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{a^{n+1}}{(n+1)!}\cdot\frac{n!}{a^n}=\lim_{n\to\infty}\frac{a}{n+1}=0=\lambda .$$

Demak, $\lambda=0<1$ bo‘lganligi sababli, berilgan qator yaqinlashuvchi.

3.7-misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\frac{3\cdot 1!}{1}+\frac{3^2\cdot 2!}{2^2}+\ldots+\frac{3^n\cdot n!}{n^n}+\ldots .$$

Yechilishi. Berilgan qator uchun:

$$a_n=\frac{3^n n!}{n^n},\quad a_{n+1}=\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n\cdot n!}=\lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^n=\frac{3}{e}>1$$

bo‘lganligi sababli, yuqoridagi 3.4- natijaga ko‘ra, berilgan qator uzoqlashuvchidir.

3.8- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}nx\prod_{k=1}^{n}\frac{\sin^2 k\alpha}{1+x^2+\cos^2 k\alpha}\quad (x\geq 0)\cdot$$

Yechilishi. Agar x=0 bo‘lsa, berilgan qator yaqinlashuvchi bo‘ladi. $x\neq 0$ deb faraz qilamiz. Bu holda berilgan qatorning umumiy hadi

$$a_n=n\cdot x\prod_{k=1}^{n}\frac{\sin^2 k\alpha}{1+x^2+\cos^2 k\alpha}\leq\frac{nx}{(1+x^2)^n}=b_n$$

bo‘ladi. U holda $\sum_{n=1}^{\infty}\frac{nx}{(1+x^2)^n}$ qatorga ega bo‘lamiz. Dalamber alomatidan foydalanib hosil bo‘lgan qatorni yaqinlashishga tekshiramiz:

$$b_n=\frac{nx}{(1+x^2)^n},\ b_{n+1}=\frac{(n+1)x}{(1+x^2)^{n+1}},$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{1+x^2}=\frac{1}{1+x^2}<1,$$

hosil bo'lgan qator yaqinlashuvchi. Demak, taqqoslash alomatiga ko'ra, berilgan qator ham yaqinlashuvchi.

Koshi alomati. Ushbu qator berilgan bo'lsin.

$$\sum_{n=1}^{\infty} a_n \text{ (barcha } n\in N \text{ lar uchun } a_n \geq 0) \qquad (3.7)$$

U holda:

a) agar shunday $q\in(0;1)$ son va m nomer mavjud bo'lib, barcha $n\geq m$ dan boshlab

$$\sqrt[n]{a_n}\leq q$$

tengsizlik o'rinli bo'lsa, (3.7) qator yaqinlashuvchi bo'ladi;

b) agar shunday m nomer mavjud bo'lib, barcha $n\geq m$ lar uchun

$$\sqrt[n]{a_n}\geq 1$$

tengsizlik o'rinli bo'lsa, (3.7) qator uzoqlashuvchi bo'ladi.

3.5- natija (Koshi alomatining limit ko'rinishi). Agar (3.7) qator uchun

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lambda \qquad (3.8)$$

limit mavjud bo'lib, $\lambda<1$ bo'lsa, (3.7) qator yaqinlashuvchi, $\lambda>1$ bo'lganda esa, uzoqlashuvchi bo'ladi.

3.1- eslatma. Agar (3.6) va (3.8) shartlarda $\lambda=1$ bo'lsa, Dalamber va Koshi alomatlari qatorning yaqinlashuvchi va uzoqlashishi to'g'risida hech narsa ayta olmaydi: qator yaqinlashuvchi ham, uzoqlashuvchi ham bo'lishi mumkin.

Bizga ma'lumki, $\sum_{n=1}^{\infty}\frac{1}{n}$ va $\sum_{n=1}^{\infty}\frac{1}{n^2}$ qatorlarning ikkalasi ham (3.6) va (3.8) shartlarni qanoatlantirib, $\lambda=1$ bo'ladi, lekin ulardan birinchisi uzoqlashuvchi, ikkinchisi yaqinlashuvchi (3.2-misolga q.) bo'ladi.

3.2- eslatma. (3.6) limitning mavjudligidan (3.8) limitning mavjudligi kelib chiqadi, lekin buning teskarisi o'rinli emas, ya'ni (3.8) limitning mavjudligidan (3.6) limitning mavjudligi har doim ham kelib chiqavermaydi. Shuning uchun qatorlarni yaqinlashishga tekshirishda Koshi alomati Dalamber alomatiga qaraganda sezgirroq bo‘lib hisoblanadi.

3.9- misol. $\sum_{n=1}^{\infty} a_n$ qatorni yaqinlashishga tekshiring, bunda

$$a_n = \begin{cases} q^{n+\sqrt{n}}, & n\text{- juft bo‘lganda,} \\ q^{n-\sqrt{n}}, & n\text{- toq bo‘lganda,} \end{cases} \quad q \text{ — musbat.}$$

Yechilishi. Quyidagi nisbatni qaraymiz:

$$\frac{a_{n+1}}{a_n} = \begin{cases} q^{1+\sqrt{n+1}+\sqrt{n}}, & n - \text{toq bo‘lganda,} \\ q^{1-\sqrt{n+1}-\sqrt{n}}, & n - \text{juft bo‘lganda.} \end{cases}$$

$q \neq 1$ bo‘lganda $\frac{a_{n+1}}{a_n}$ chegaralanmagan bo‘ladi, shuning uchun u chekli limitga ega emas. Lekin $n \to \infty$ da $\sqrt[n]{a_n} = q^{1 \pm n^{-\frac{1}{2}}} \to q$ bo‘lganligi uchun, $q < 1$ bo‘lganda, qator yaqinlashuvchi, $q > 1$ bo‘lganda esa, qator uzoqlashuvchi bo‘ladi. Shunday qilib, Koshi alomati Dalamber alomatiga qaraganda sezgirroq ekanligiga ishonch hosil qilamiz.

3.10- misol. $\sum_{n=1}^{\infty} n^6 \left(\frac{4n+5}{5n+6}\right)^n$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qator uchun:

$$a_n = n^6 \cdot \left(\frac{4n+5}{5n+6}\right)^n, \quad \sqrt[n]{a_n} = n^{\frac{6}{n}} \cdot \frac{4n+5}{5n+6},$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} n^{\frac{6}{n}} \frac{4n+5}{5n+6} = \frac{4}{5} < 1$$

bo‘lganligi sababli, Koshi alomatiga ko‘ra, berilgan qator yaqinlashuvchi.

3.11- misol. $\sum_{n=1}^{\infty} \frac{n!}{n^{\sqrt{n}}}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Stirlingning asimptotik formulasidan foydalanamiz:

$$n \to \infty \text{ da } n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Bu holda $n \to \infty$ da $\sqrt[n]{a_n} \sim e^{-1}(2\pi)^{\frac{1}{2n}} \cdot n^{\frac{1}{2n}-\frac{1}{\sqrt{n}}} \cdot n \sim \frac{n}{e}$ bo'ladi. Yuqorida keltirilgan 3.3- natijaga ko'ra berilgan qator uzoqlashuvchi bo'ladi.

3.12- misol. $\sum_{n=2}^{\infty}\left(\frac{n-1}{n+1}\right)^{n(n-1)}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qator uchun

$$a_n = \left(\frac{n-1}{n+1}\right)^{n(n-1)}, \quad \sqrt[n]{a_n} = \left(\frac{n-1}{n+1}\right)^{n-1},$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{n-1} = \frac{1}{e^2} < 1$$

bo'lganligi sababli, Koshi alomatiga ko'ra, berilgan qator yaqinlashuvchi.

3.5- teorema (umumlashgan Koshi alomati). Agar $\overline{\lim_{n\to\infty}}\sqrt[n]{a_n} = q$, $a_n \geq 0$, $(n=1,2,...)$ bo'lsa, u holda: *a)* $q<1$ bo'lganda $\sum_{n=1}^{\infty} a_n$ qator yaqinlashadi; *b)* $q>1$ bo'lganda esa $\sum_{n=1}^{\infty} a_n$ uzoqlashadi.

3.13- misol. $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{2^n}$ qatorni yaqinlashuvchilikka tekshiring.

Yechilishi. Quyidagi ifodani qaraymiz:

$$K_n = \sqrt[n]{a_n} = \sqrt[n]{\frac{2+(-1)^n}{2^n}} = \frac{1}{2}\sqrt[n]{2+(-1)^n}.$$

Agar $n=2k,\ k\in N$ bo'lsa, $K_{2n} = \frac{1}{2}3^{\frac{1}{2k}}$ va $\lim_{k\to\infty} K_{2k} = \frac{1}{2}$.

Agar $n=2k-1,\ k\in N$ bo'lsa, $K_{2k-1} = \frac{1}{2}$ bo'ladi.

Umumlashgan Koshi alomatiga ko'ra $\overline{\lim_{n\to\infty}} K_n = \frac{1}{2} < 1$ va berilgan qator yaqinlashuvchidir.

Raabe alomati. Agar

$$\sum_{n=1}^{\infty} a_n \ (a_n > 0, \text{ barcha } n \in N \text{ lar uchun}) \qquad (3.9)$$

qatorda $n \in N$ ning biror n_0 $(n_0 \geq 1)$ qiymatidan boshlab barcha $n \geq n_0$ qiymatlar uchun

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq r > 1 \quad \left(n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1\right)$$

tengsizlik o'rinli bo'lsa, u holda (3.9) qator yaqinlashuvchi (uzoqlashuvchi) bo'ladi.

3.6- natija (Raabe alomatining limit ko'rinishi). Agar (3.9) qator uchun

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \rho \quad (\rho = \text{const})$$

limit mavjud bo'lib, $\rho > 1$ bo'lsa, (3.9) qator yaqinlashuvchi, $\rho < 1$ bo'lsa, qator uzoqlashuvchi bo'ladi.

3.3- eslatma. Agar (3.9) qator uchun $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \rho = 1$ bo'lsa, u holda qator yaqinlashuvchi bo'lishi ham, uzoqlashuvchi bo'lishi ham mumkin.

3.14-misol. $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2^n}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qator uchun

$$a_n = \frac{(2n-1)!!}{(2n)!!} \frac{1}{2^n}, \quad a_{n+1} = \frac{(2n+1)!!}{(2n+2)!!} \frac{1}{2^{n+1}},$$

$$\frac{a_n}{a_{n+1}} = \frac{2(2n+2)}{(2n+1)}, \quad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2(2n+2)}{2n+1} - 1\right) = \infty > 1$$

bo'lganligi sababli, Raabe alomatiga ko'ra, berilgan qator yaqinlashuvchi.

3.15- misol. $\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{n}{e}\right)^n$ qatorni yaqinlashishga tekshiring.

Yechilishi. Dalamber alomatiga ko‘ra, berilgan qatorning yaqinlashuvchi yoki uzoqlashuvchi bo‘lishi to‘g‘risida biror qarorga kelib bo‘lmaydi, chunki $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$. Ammo Raabe alomatiga ko‘ra, berilgan qator uzoqlashuvchi bo‘ladi.

Haqiqatan ham, berilgan qator uchun

$$a_n=\frac{1}{n!}\left(\frac{n}{e}\right)^n,\quad a_{n+1}=\frac{1}{(n+1)!}\left(\frac{n+1}{e}\right)^{n+1},$$

$$R_n=n\left(\frac{a_n}{a_{n+1}}-1\right)=n\left(\frac{e}{\left(1+\frac{1}{n}\right)^n}-1\right),\quad \lim_{n\to\infty}R_n=\lim_{n\to\infty}n\left(\frac{e}{\left(1+\frac{1}{n}\right)^n}-1\right).$$

Bu limitni hisoblash uchun $x\to 0$ da $\ln(1+x)$ va e^x larning Teylor formulasida yoyilmasidan foydalanib, R_n ni quyidagi ko‘rinishda ifodalaymiz:

$$R_n=n\left(\frac{e}{(1+\frac{1}{n})^n}-1\right)=n\left(e(1+\frac{1}{n})^{-n}-1\right)=n\left(e\cdot e^{-n\ln(1+\frac{1}{n})}-1\right)=$$

$$=n\left(e\cdot e^{-n(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))}-1\right)=n\left(e\cdot e^{-1+\frac{1}{2n}+o(\frac{1}{n})}-1\right)=n\left(e^{\frac{1}{2n}+o(\frac{1}{n})}-1\right)=$$

$$=n\left(1+\frac{1}{2n}+o(\frac{1}{n})-1\right)=\frac{1}{2}+o(\frac{1}{n}).$$

Demak, $\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left(\frac{1}{2}+o(\frac{1}{n})\right)=\frac{1}{2}$ bo‘ladi, ya’ni $\lambda=\frac{1}{2}<1$ bo‘lganligi sababli, Raabe alomatiga ko‘ra, qator uzoqlashuvchi.

Gauss alomati. Agar (3.9) qator uchun

$$\frac{a_n}{a_{n+1}}=\lambda+\frac{\mu}{n}+\frac{\Theta_n}{n^{1+\varepsilon}} \quad \left(|\Theta_n|<c,\ \varepsilon>0\right)$$

bo‘lsa, u holda:

a) $\lambda>1$ bo‘lganda, (3.9) qator yaqinlashuvchi;

b) $\lambda<1$ bo‘lganda, (3.9) qator uzoqlashuvchi;

d) $\lambda=1$ bo‘lib, $\mu>1$ bo‘lganda, (3.9) qator yaqinlashuvchi;

e) $\lambda=1$ bo‘lib, $\mu\le 1$ bo‘lganda, (3.9) qator uzoqlashuvchi bo‘ladi.

3.16- misol. $\sum\limits_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^p$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qator uchun

$$a_n=\left(\frac{(2n-1)!!}{(2n)!!}\right)^p, \quad a_{n+1}=\left[\frac{(2n+1)!!}{(2n+2)!!}\right]^p,$$

bo‘ladi. $\frac{a_n}{a_{n+1}}$ nisbatni qaraymiz:

$$\frac{a_n}{a_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^p=\left(1+\frac{1}{2n+1}\right)^p=1+\frac{p}{2n+1}+$$

$$+\frac{p(p-1)}{2(2n+1)^2}+o\left(\frac{1}{n^2}\right),\ n\to\infty.$$

Gauss alomatiga ko‘ra, berilgan qator $p>2$ bo‘lganda, yaqinlashuvchi, $p\le 2$ bo‘lganda esa uzoqlashuvchi bo‘ladi.

3.17- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty}\frac{p(p+1)...(p+n-1)}{n!}\cdot\frac{1}{n^q}.$$

Yechilishi. Ravshanki, p — butun manfiy yoki nol bo‘lganda berilgan qator yaqinlashuvchi bo‘ladi. Shuning uchun biz bu holni qaramaymiz. Bu qatorda $\frac{a_n}{a_{n+1}}$ nisbatni quyidagicha ifodalaymiz, bunda

$$a_n = \frac{p(p+1)...(p+n-1)}{n!} \cdot \frac{1}{n^q},$$

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{p+n}\left(1+\frac{1}{n}\right)^q = \left(1+\frac{p}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{q+1}.$$

Biz bu yerda $\frac{1}{1+x}$ va $(1+x)^\alpha$ funksiyalarning $x \to 0$ da Teylor formulasi yoyilmasidan foydalanib,

$$\frac{a_n}{a_{n+1}} = \left(1-\frac{p}{n}+o(\frac{1}{n})\right)\cdot\left(1+\frac{q+1}{n}+o(\frac{1}{n})\right) = 1+\frac{q-p+1}{n}+o(\frac{1}{n})$$

ni topamiz.

Demak, Gauss alomatiga asosan, $q-p+1>1$ yoki $q>p$ bo'lganda berilgan qator yaqinlashuvchi bo'ladi.

Koshining integral alomati. Agar $f(x)$ funksiya $[k;+\infty)$ ($k \in N$ — biror son) da aniqlangan, uzluksiz, o'smaydigan va manfiy bo'lmagan funksiya bo'lib, $F(x) = \int_k^x f(t)dt$ funksiya $f(x)$ funksiya uchun boshlang'ich funksiya va $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ bo'lsa,

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_k^x f(t)dt$$

mavjud va chekli bo'lganda (3.7) qator yaqinlashuvchi, bu limit mavjud bo'lmaganda yoki cheksiz bo'lganda (3.7) qator uzoqlashuvchi bo'ladi.

3.18- misol. $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning umumiy hadi $a_n = \frac{1}{n \ln n \cdot \ln(\ln n)}$.

U holda $f(x) = \frac{1}{x \ln x \cdot \ln(\ln x)}$ funksiya $[3\,;+\infty)$ da aniqlangan, uzluksiz, o'smaydigan va manfiy bo'lmagan funksiya. $f(x)$ funksiyaning boshlang'ich funksiyasi esa

$$F(x)=\int_3^x \frac{dt}{t\ln t\ln\ln t}=\ln\ln\ln x-\ln\ln\ln 3$$

bo'lib, $x\to\infty$ da $F(x)\to\infty$. Demak, Koshining integral alomatiga ko'ra, berilgan qator uzoqlashuvchi bo'ladi.

3.19- misol. $\sum_{n=1}^{\infty} n^3 e^{-n^4}$ qatorni yaqinlashishga tekshiring.

Yechilishi. $f(x)=x^3e^{-x^4}$ funksiya $x\geq 1$ da kamayuvchi va manfiy bo'lmagan funksiya. $f(x)=x^3e^{-x^4}$ funksiya uchun $F(x)$ boshlang'ich funksiyani topamiz:

$$F(x)=\int_1^x t^3e^{-t^4}dt=\frac{1}{4}\left(e^{-1}-e^{-x^4}\right).$$

$x\to\infty$ da $F(x)$ funksiyaning limiti mavjud va chekli, ya'ni $\lim_{x\to\infty} F(x)=\frac{1}{4e}$. Demak, Koshining integral alomatiga ko'ra, berilgan qator yaqinlashuvchi.

Mustaqil yechish uchun misollar

Qatorlarni 3.1- teoremadan foydalanib yaqinlashishga tekshiring

3.1. $\sum_{n=1}^{\infty}\frac{\sin^4 2n}{(n+1)(n+2)}$. **3.2.** $\sum_{n=1}^{\infty}\left(1+\frac{\ln n}{n}\right)q^n,\ 0<q<1.$

3.3. $\sum_{n=1}^{\infty}\frac{\pi-\mathrm{arctg}n}{(n+1)(n+2)(n+3)}$. **3.4.** $\sum_{n=1}^{\infty}\frac{n\cdot 2^{n+5}}{n\cdot 3^n+4}$.

Qatorlarni 3.2- teoremadan foydalanib yaqinlashishga tekshiring:

3.5. $\sum_{n=2}^{\infty}\frac{4}{n(\ln n)^p}$. **3.6.** $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p(\ln n)^q}$. **3.7.** $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}$.

3.8. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$. 3.9. $\sum_{n=2}^{\infty}\frac{3}{4+n^2}$. 3.10. $\sum_{n=3}^{\infty}\frac{1}{n\ln n\ln\ln n}$.

3.11. $\sum_{n=3}^{\infty}\frac{1}{n\ln n(\ln\ln n)^2}$.

Taqqoslash teoremalaridan foydalanib, qatorlarni yaqinlashishga tekshiring:

3.12. $\sum_{n=1}^{\infty}\frac{5+3(-1)^{n+1}}{3^n}$. 3.13. $\sum_{n=1}^{\infty}\frac{\sin^4 3n}{n\sqrt{n}}$. 3.14. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$.

3.15. $\sum_{n=1}^{\infty}\frac{\mathrm{arctg}n}{n^2+1}$. 3.16. $\sum_{n=1}^{\infty}\frac{\cos\frac{\pi}{4n}}{\sqrt[5]{2n^5-1}}$. 3.17. $\sum_{n=1}^{\infty}\frac{n^{n-1}}{(2n^2+n+1)^{\frac{n+1}{2}}}$.

3.18. $\sum_{n=1}^{\infty}\frac{n^5}{2^n+3^n}$. 3.19. $\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^{n+1}}$. 3.20. $\sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}$.

3.21. $\sum_{n=1}^{\infty}\frac{1}{n^2-4n+5}$. 3.22. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+2n}}$. 3.23. $\sum_{n=2}^{\infty}\frac{\ln n}{\sqrt[4]{n^5}}$.

3.24. $\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{n^2+n+1}-\sqrt{n^2-n+1}\right)$. 3.25. $\sum_{n=1}^{\infty}\mathrm{tg}\frac{\pi}{4n}$.

3.26. $\sum_{n=1}^{\infty}\left(\frac{1+n^2}{1+n^3}\right)^2$.

Dalamber alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring: $\sum_{n=1}^{\infty}a_n$.

3.27. $a_n=\frac{n^{12}}{(n+2)!}$. 3.28. $a_n=\frac{n^4}{4^n}$.

3.29. $a_n=\frac{n!a^n}{n^n}$, $a\neq e$, $a>0$. 3.30. $a_n=\frac{3\cdot 6\ldots(3n)}{(n+1)!}\arcsin\frac{1}{2^n}$.

3.31. $a_n = \dfrac{(2n)!}{(n!)^2}$. 3.32. $a_n = \dfrac{n!(2n+1)!}{(3n)!}$.

3.33. $a_n = \dfrac{2\cdot 5 ... (3n+2)}{2^n \cdot (n+1)!}$. 3.34. $a_n = \dfrac{1\cdot 3\cdot 5 ... (2n-1)}{2\cdot 5\cdot 8 ... (3n-1)}$.

3.35. $a_n = \dfrac{n!}{2^n+1}$. 3.36. $a_n = \dfrac{\ln^{101} n}{4!}$, $n \ge 2$.

Koshi alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty} a_n$.

3.37. $a_n = \dfrac{1}{(\ln n)^n}$, $n \ge 2$. 3.38. $a_n = \left(\dfrac{4}{n}\right)^n$. 3.39. $a_n = \left(\dfrac{n^2+5}{n^2+6}\right)^{n^3}$.

3.40. $a_n = 3^{n+1}\left(\dfrac{n}{n+1}\right)^{n^2}$. 3.41. $a_n = \left(\dfrac{6n+4}{5n-3}\right)^{\frac{n}{2}}\left(\dfrac{5}{6}\right)^{\frac{2n}{3}}$.

3.42. $a_n = \dfrac{n^\alpha}{(\ln(n+1))^{n/2}}$. 3.43. $a_n = \left(\dfrac{2n-1}{2n+1}\right)^{n(n-1)}$.

3.44. $a_n = \dfrac{3^n}{n\left(\sqrt{2}\right)^n}$. 3.45. $a_n = \left(\dfrac{n}{3n-1}\right)^{2n-1}$. 3.46. $a_n = \dfrac{2^{n-1}}{n^n}$.

Umumlashgan Koshi alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring.

3.47. $\sum\limits_{n=1}^{\infty} \dfrac{a\cos^2 \frac{\pi n}{3}}{2^n}$. 3.48. $\sum\limits_{n=1}^{\infty} \dfrac{2+(-1)^n}{2n}$. 3.49. $\sum\limits_{n=1}^{\infty} \dfrac{5+3(-1)^{n+1}}{4^n}$.

3.50. $\sum\limits_{n=1}^{\infty} \dfrac{n^3\left[\sqrt{2}+(-1)^n\right]^n}{5^n}$. 3.51. $\sum\limits_{n=1}^{\infty} \left(\dfrac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$.

3.52. Agar $a_n > 0$ $\forall n \in N$ larda $a_{n+1} < a_n$ bo'lib, $\sum\limits_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo'lsa, $\lim\limits_{n\to\infty} n a_n = 0$ ekanini isbotlang.

3.53. Agar $\lim\limits_{n\to\infty} na_n = a$ bo'lib, $a \neq 0$ bo'lsa, $\sum\limits_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo'lishini isbotlang.

3.54. $\sum\limits_{n=1}^{\infty} a_n$ qator ($a_n \geq 0$) berilgan bo'lib, $n \geq n_0$ larda $\left(1-\sqrt[n]{a_n}\right)\dfrac{n}{\ln n} \geq P > 1$ bo'lsa, berilgan qator yaqinlashuvchi, $n \geq n_0$ larda $\left(1-\sqrt[n]{a_n}\right)\dfrac{n}{\ln n} \leq 1$ bo'lsa, uzoqlashuvchi ekanini isbotlang.

3.55. Agar $\sum\limits_{n=1}^{\infty} a_n^2$, $\sum\limits_{n=1}^{\infty} b_n^2$ qatorlar yaqinlashuvchi bo'lsa, $\sum\limits_{n=1}^{\infty} |a_n b_n|$, $\sum\limits_{n=1}^{\infty} (a_n + b_n)^2$, $\sum\limits_{n=1}^{\infty} \dfrac{|a_n|}{n}$ qatorlar ham yaqinlashuvchi bo'lishini isbotlang

Raabe va Gauss alomatlaridan foydalanib, quyidagi qatorlarni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty} a_n$.

3.56. $a_n = \left(\dfrac{(2n+1)!!}{(2n+2)!!}\right)^{\alpha} \cdot \dfrac{1}{n^{\beta}}$.

3.57. $a_n = \dfrac{\sqrt{n!}}{\left(a+\sqrt{2}\right)\left(a+\sqrt{3}\right)\ldots\left(1+\sqrt{n+1}\right)}$, $a > 0$.

3.58. $a_n = \dfrac{(n+1)}{\beta(\beta+1)\ldots(\beta+n)^{n^{\alpha}}}$, $\beta > 0$. **3.59.** $a_n = \dfrac{n!e^n}{n^{n+\alpha}}$.

3.60. $a_n = \left(\dfrac{p(p+1)\ldots(p+n+1)}{q(q+1)\ldots(q+n-1)}\right)^{\alpha}$, $p > 0$, $q > 0$.

3.61. $a_n = \dfrac{1\cdot 4\ldots(3n-2)\cdot 2\cdot 5\ldots(3n+2)}{n!\cdot(n+1)!9^n}$.

3.62 $a_n=\dfrac{\ln 2\ln 3...\ln(n+1)}{\ln(2+a)\ln(3+a)...\ln(n+1+a)}$, $a>0$.

3.63. $a_n=\dfrac{1}{n\ln n}$, $n>2$. 3.64. $a_n=\dfrac{\ln\dfrac{n}{n-1}}{\sqrt{n}-\sqrt{n-1}}$, $n>2$.

3.65. $a_n=\dfrac{1}{n\ln^2 n}$, $n>2$.

Koshining integral alomatidan foydalanib, qatorlarni yaqinlashishga tekshiring:

3.66. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n(1+\ln n)}$. 3.67. $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{6n+5}}$. 3.68. $\sum\limits_{n=1}^{\infty}\dfrac{5}{4+n^2}$.

3.69. $\sum\limits_{n=1}^{\infty}\left(\dfrac{1+n}{1+n^2}\right)^2$. 3.70. $\sum\limits_{n=1}^{\infty}\dfrac{\ln(n+1)}{(n+1)^2}$. 3.71. $\sum\limits_{n=2}^{\infty}\dfrac{1}{\sqrt{n}}\ln\dfrac{n+1}{n-1}$.

3.72. $\sum\limits_{n=1}^{\infty}ne^{-n^2}$. 3.73. $\sum\limits_{n=1}^{\infty}\dfrac{e^{-\sqrt{n}}}{\sqrt{n}}$. 3.74. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n\ln^{\alpha+1}n}$, $(\alpha>0)$.

3.75. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n\ln n(\ln\ln n)^{1+\alpha}}$, $\alpha>0$. 3.76. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n^{\alpha}\ln^{\beta}n}$.

Misollarning javoblari

3.1. Yaqinlashuvchi. **3.2.** Yaqinlashuvchi. **3.3.** Yaqinlashuvchi. **3.4.** Yaqinlashuvchi. **3.5.** $p>1$ da yaqinlashuvchi, $p\le 1$ da uzoqlashuvchi. **3.6.** $\forall q$, $p>1$ da yaqinlashuvchi, $p=1$, $q>1$ da yaqinlashuvchi. **3.7.** Uzoqlashuvchi. **3.8.** Uzoqlashuvchi. **3.9.** Yaqinlashuvchi. **3.10.** Uzoqlashuvchi. **3.11.** Yaqinlashuvchi. **3.12.** Yaqinlashuvchi. **3.13.** Yaqinlashuvchi. **3.14.** Yaqinlashuvchi. **3.15.** Yaqinlashuvchi. **3.16.** Uzoqlashuvchi. **3.17.** Yaqinlashuvchi. **3.18.** Yaqinlashuvchi. **3.19.** Yaqinlashuvchi. **3.20.** Uzoqlashuvchi. **3.21.** Yaqinlashuvchi. **3.22.** Uzoqlashuvchi. **3.23.** Uzoqlashuvchi. **3.24.** Yaqinlashuvchi. **3.25.** Uzoqlashuvchi. **3.26.** Yaqinlashuvchi.

3.27. Yaqinlashuvchi. **3.28.** Yaqinlashuvchi. **3.29.** Uzoqlashuvchi. **3.30.** Uzoqlashuvchi. **3.31.** Uzoqlashuvchi. **3.32.** Yaqinlashuvchi. **3.33.** Uzoqlashuvchi. **3.34.** Yaqinlashuvchi. **3.35.** Uzoqlashuvchi. **3.36.** Yaqinlashuvchi. **3.37.** Yaqinlashuvchi. **3.38.** Yaqinlashuvchi. **3.39.** Yaqinlashuvchi. **3.40.** Uzoqlashuvchi **3.41.** Yaqinlashuvchi. **3.42.** Ixtiyoriy α lar uchun yaqinlashuvchi. **3.43.** Yaqinlashuvchi. **3.44.** Uzoqlashuvchi. **3.45.** Yaqinlashuvchi. **3.46.** Yaqinlashuvchi. **3.47.** Yaqinlashuvchi. **3.48.** Yaqinlashuvchi. **3.49.** Yaqinlashuvchi. **3.50.** Yaqinlashuvchi. **3.51.** Yaqinlashuvchi. **3.56.** $\frac{\alpha}{2}+\beta>1$ da yaqinlashuvchi, $\frac{\alpha}{2}+\beta\leq 1$ da uzoqlashuvchi. **3.57.** Yaqinlashuvchi. **3.58.** $\alpha+\beta>2$ da yaqinlashuvchi, $\alpha+\beta\leq 2$ da uzoqlashuvchi. **3.59.** $\alpha>1$ da yaqinlashuvchi, $\alpha\leq 1$ da uzoqlashuvchi. **3.60.** $\alpha(q-p)>1$ da yaqinlashuvchi. **3.61.** Uzoqlashuvchi. **3.62.** Uzoqlashuvchi. **3.63.** Uzoqlashuvchi. **3.64.**Uzoqlashuvchi. **3.65.** Yaqinlashuvchi. **3.66.** Uzoqlashuvchi. **3.67.** Uzoqlashuvchi. **3.68.** Yaqinlashuvchi. **3.69.** Yaqinlashuvchi. **3.70.** Yaqinlashuvchi. **3.71.** Yaqinlashuvchi. **3.72.** Yaqinlashuvchi. **3.73.** Yaqinlashuvchi. **3.74.** Yaqinlashuvchi. **3.75.** Yaqinlashuvchi. **3.76.** $\alpha>1, \beta\in R$ da yaqinlashuvchi, agar $\alpha=1, \beta>1$ bo‘lsa, yaqinlashuvchi, *a* va *b* ning boshqa qiymatlarida uzoqlashuvchi.

4 - §. Ixtiyoriy ishorali qatorlar va ularning yaqinlashuvchiligi

4.1. Qatorlarning absolut va shartli yaqinlashuvchiligi. Hadlari ixtiyoriy ishorali

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... a_n + ... \tag{4.1}$$

qator berilgan bo‘lsin. Bu qator hadlarining absolut qiymatlaridan ushbu qatorni tuzamiz.

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + ... |a_n| + ... \tag{4.2}$$

4.1- ta’rif. Agar (4.2) qator yaqinlashuvchi bo‘lsa, (4.1) qator *absolut yaqinlashuvchi qator* deyiladi.

4.2- ta’rif. Agar (4.1) qator yaqinlashuvchi bo‘lib, (4.2) qator uzoqlashuvchi bo‘lsa, (4.1) qator *shartli yaqinlashuvchi* deyiladi.

4.1- teorema. Agar (4.2) qator yaqinlashuvchi bo'lsa, (4.1) qator ham yaqinlashuvchi bo'ladi.

4.2- teorema. Agar (4.1) qator absolut yaqinlashuvchi bo'lib, $\{b_n\}$ ketma-ketlik esa chegaralangan bo'lsa, ya'ni $\exists M > 0 : \forall n \in \mathrm{N}$ uchun $|b_n| \le M$ bo'lsa, $\sum_{n=1}^{\infty} a_n b_n$ qator absolut yaqinlashuvchi bo'ladi.

4.3- teorema. Agar ixtiyoriy ishorali $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar absolut yaqinlashuvchi bo'lsa, barcha $\lambda, \mu \in R$ o'zgarmas sonlar uchun

$$\sum_{n=1}^{\infty} (\lambda a_n + \mu b_n)$$

qator ham absolut yaqinlashuvchi bo'ladi.

4.4- teorema. Agar (4.1) qator absolut yaqinlashuvchi bo'lsa, (4.1) qator hadlarining o'rinlarini almashtirish natijasida tuzilgan

$$\sum_{n=1}^{\infty} \tilde{a}_n$$

qator ham absolut yaqinlashuvchi bo'ladi va uning yig' indisi (4.1) qatorning yig'indisiga teng bo'ladi.

4.5- teorema. Agar (4.1) qator absolut yaqinlashuvchi bo'lsa, u holda $\sum_{n=1}^{\infty} Ca_n$ (C — o'zgarmas son) qator ham absolut yaqinlashuvchi bo'ladi.

4.6- teorema. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots a_n + \ldots, \tag{A}$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \ldots b_n + \ldots \tag{B}$$

qatorlar absolut yaqinlashuvchi bo'lib, ularning yig' indilari mos ravishda S', S'' ga teng bo'lsa, ular hadlarining istalgan tartibdagi $a_j \cdot b_j$ ko'paytmasidan tuzilgan qator ham absolut yaqinlashuvchi bo'ladi va uning yig'indisi $S' \cdot S''$ ga teng bo'ladi.

4.1- eslatma. (4.2) qatorning uzoqlashuvchi bo‘lishidan (4.1) qatorning uzoqlashuvchi bo‘lishi har doim ham kelib chiqavermaydi.

4.2- eslatma. Agar (A) va (B) qatorlarning biri yaqinlashuvchi, ikkinchisi absolut yaqinlashuvchi bo‘lsa, u holda qatorlarni ko‘paytirishda Koshi qoidasi o‘rinli bo‘ladi:

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n, \quad c_n = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1.$$

4.3- eslatma. (A) va (B) qatorlar shartli yaqinlashuvchi bo‘lganda, ularning ko‘paytmasi uzoqlashuvchi bo‘lishi ham mumkin. Masalan, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ qatorning Leybnis alomatiga ko‘ra shartli yaqinlashuvchi ekanligini ko‘rsatish qiyin emas.

Bu qatorni Koshi qoidasiga asosan o‘zini-o‘ziga ko‘paytiramiz:

$$c_n = 1 \cdot \frac{(-1)^{n-1}}{\sqrt{n}} - \frac{1}{\sqrt{2}} \frac{(-1)^{n-2}}{\sqrt{n-1}} + \frac{1}{\sqrt{3}} \cdot \frac{(-1)^{n-3}}{\sqrt{n-2}} + \ldots + \frac{(-1)^{n-1}}{\sqrt{n}} \cdot 1 =$$

$$= (-1)^{n-1} \left(\frac{1}{1 \cdot \sqrt{n}} + \frac{1}{\sqrt{2} \cdot \sqrt{n-1}} + \ldots + \frac{1}{\sqrt{k}\sqrt{n-k+1}} + \ldots + \frac{1}{\sqrt{n} \cdot 1} \right).$$

Qavs ichidagi har bir qo‘shiluvchi $\frac{1}{n}$ dan katta bo‘lganligi uchun $|c_n| > 1$ bo‘ladi. Demak, ko‘paytma qator uzoqlashuvchi bo‘ladi.

4.1- misol. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+4} - \sqrt{n+1}}{n}$ qatorning absolut yaqinlashuvchiligini ko‘rsating.

Yechilishi. Berilgan qator bilan birga $\sum_{n=1}^{\infty} \frac{\sqrt{n+4} - \sqrt{n+1}}{n}$ qatorni qaraymiz. Bu qatorning umumiy hadini quyidagicha shakl almashtiramiz:

$$\frac{\sqrt{n+4} - \sqrt{n+1}}{n} = \frac{1}{3} \frac{1}{n(\sqrt{n+4} + \sqrt{n+1})} \leq \frac{1}{3n^{\frac{3}{2}}}.$$

Demak, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ — qator yaqinlashuvchi, chunki $p=\frac{3}{2}>1$. U holda 4.1- ta'rifga ko'ra, berilgan qator absolut yaqinlashuvchi bo'ladi.

4.2- misol. $\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n^4}$ qatorni yaqinlashishga tekshiring.

Yechilishi. $b_n=\sin nx,\ \ a_n=\frac{(-1)^n}{n^4}$ deb belgilaymiz. $\forall n\in N$, $\forall x\in R$ uchun $|b_n|=|\sin nx|\le 1$, $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ — qator absolut yaqinlashuvchi, chunki $\sum_{n=1}^{\infty}\frac{1}{n^4}$ — qator yaqinlashuvchi. Demak, 4.2- teoremaga asosan, berilgan qator absolut yaqinlashuvchi bo'ladi.

4.3- misol. Ushbu qatorlar yig' indisining absolut yaqinlashuvchi ekanligini ko'rsating:

$$\sum_{n=1}^{\infty}\frac{\cos nx}{2n^2}\ \text{va}\ \sum_{n=1}^{\infty}\frac{\sqrt{3}\sin nx}{2n^2}\quad (x\neq 2n\pi,\ n\in Z).$$

Yechilishi. Ravshanki, berilgan qatorlar 4.2- teoremaga asosan, absolut yaqinlashuvchi bo'ladi. Endi quyidagi yig'indi qatorni qaraymiz:

$$\sum_{n=1}^{\infty}\frac{\cos nx}{2n^2}+\sum_{n=1}^{\infty}\frac{\sqrt{3}\sin nx}{2n^2}=\sum_{n=1}^{\infty}\frac{\sin(\frac{\pi}{6}+nx)}{n^2}.$$

Hosil bo'lgan qator 4.2- teoremaga asosan, absolut yaqinlashuvchi bo'ladi.

Demak, 4.3- teoremaga asosan, absolut yaqinlashuvchi qatorlarning yig' indisidan tuzilgan qator ham yaqinlashuvchi bo'lar ekan.

4.4- misol. Ushbu qatorlarning ko'paytmasini hisoblang:

$$\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n!}\ \text{va}\ \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{3})^n}{n!}.$$

Yechilishi. Ma'lumki, bu berilgan qatorlar absolut yaqinlashuvchi bo'lib, ularning yig' indilari mos ravishda e^{-2} va $e^{-\frac{1}{3}}$ ga teng bo'ladi.

Endi bu qatorlarni Koshi formulasi bo'yicha ko'paytiramiz. Ko'paytma qatorning umumiy hadi

$$c_n = 1\cdot\frac{(-\frac{1}{3})^n}{n!} + \frac{(-2)}{1!}\cdot\frac{(-\frac{1}{3})^{n-1}}{(n-1)!} + \ldots + \frac{(-2)^k}{k!}\frac{(-\frac{1}{3})^{n-k}}{(n-k)!} +$$

$$+\ldots+\frac{(-2)^n}{n!}\cdot 1 = \sum_{k=0}^{n}\frac{(-1)^n}{k!(n-k)!}\cdot 2^k(-\frac{1}{3})^{n-k} = \frac{(-2\frac{1}{3})^n}{n!}$$

Endi $\sum_{n=0}^{\infty}\frac{(-2\frac{1}{3})^n}{n!}$ qator hadlarining absolut qiymatlaridan tuzilgan $\sum_{n=0}^{\infty}\frac{(2\frac{1}{3})^n}{n!}$ qatorni absolut yaqinlashishga tekshiramiz: Dalamber alomatiga asosan, qator yaqinlashuvchi. Demak, 4.6- teoremaga asosan, ko'paytma qator absolut yaqinlashuvchi va uning yig'indisi $e^{-2}\cdot e^{-\frac{1}{3}} = e^{-2\frac{1}{3}}$ ga teng bo'ladi.

4.5- misol. $\sum_{n=1}^{\infty}\frac{(n+3)\cos 3n}{\sqrt[3]{n^7+3n+4}}$. qatorning absolut yaqinlashuvchiligini ko'rsating.

Yechilishi. Barcha $n\in N$ lar uchun $n+3\le 4n$, $|\cos 3n|\le 1$, $n^7+3n+4>n^7$ tengsizlik o'rinli bo'ladi. U holda, $|a_n|\le\frac{4}{n^{4/3}}=b_n$. $\frac{4}{3}>1$ bo'lib, $\sum_{n=1}^{\infty}b_n$ qator yaqinlashuvchi. Demak, taqqoslash alomatiga ko'ra berilgan qator absolut yaqinlashuvchi bo'ladi.

4.2. Ishorasi almashinuvchi qatorlar.

4.3- ta'rif. Ushbu (bunda $a_n\ge 0$ yoki $a_n\le 0$, $\forall n\in N$) qator *ishorasi almashinuvchi* yoki *Leybnis* qatori deyiladi.

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - ... + (-1)^{n+1}\cdot a_n + ... \quad (4.3)$$

4.7- teorema (Leybnis alomati). Agar ishorasi almashinuvchi (4.3) qatorning hadlari absolut qiymati bo'yicha monoton kamayuvchi, ya'ni

$$a_n \geq a_{n+1} > 0 \quad (\forall n \in N) \quad (4.4)$$

va

$$\lim_{n\to\infty} a_n = 0 \quad (4.5)$$

bo'lsa, (4.3) qator yaqinlashuvchi bo'ladi.

4.4- eslatma. Absolut yaqinlashuvchi qatorlar uchun Leybnis alomatining shartlari bajarilmasa ham ishorasi almashinuvchi qator yaqinlashuvchi bo'lishi mumkin.

4.5- eslatma. Absolut yaqinlashuvchi bo'lmagan ishorasi almashinuvchi, hadlari monoton kamayuvchi qatorlar yaqinlashuvchi bo'lishi uchun Leybnis alomatidagi shartlarning bajarilishi zarur va yetarli.

4.6- eslatma. Leybnis alomatidagi uchala shart ham, ya'ni qatorning hadlarini ishora almashinuvchiligi, absolut qiymati bo'yicha monotonligi va ularning nolga intilishi absolut yaqinlashuvchi bo'lmagan qatorlarning yaqinlashishi uchun muhim shart bo'lib hisoblanadi. Shulardan birortasi buzilsa, u holda qator uzoqlashuvchi bo'ladi.

Bundan keyin, Leybnis alomati shartlarini qanoatlantiruvchi qatorlarni *Leybnis tipidagi qatorlar* deb ataymiz.

Natija. Leybnis tipidagi qatorlarda $\forall n \in N$ uchun quyidagi tengsizliklar o'rinli bo'ladi:

$$S_{2n} < S \leq S_{2n-1}, \quad |S - S_n| \leq a_{n+1}, \quad 0 < S < a_1.$$

4.6- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$1 - \frac{1}{2^3} + \frac{1}{3^4} - \frac{1}{4^3} + \frac{1}{5^4} - .... + \frac{1}{(2n-1)^4} - \frac{1}{(2n)^3} + ...$$

Yechilishi. Berilgan qator hadlarining absolut qiymatlaridan tuzilgan

$$1 + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{4^3} + \frac{1}{5^4} + ... + \frac{1}{(2n-1)^4} + \frac{1}{(2n)^3} + ...$$

qator yaqinlashuvchi, chunki bu qator $\sum_{n=1}^{\infty}\frac{1}{n^3}$ qator bilan taqqoslanganda, berilgan qatorning har bir hadi bu qatorning mos hadlaridan katta emas. Lekin $\sum_{n=1}^{\infty}\frac{1}{n^3}$ —yaqinlashuvchi qator. Shuning uchun berilgan qator absolut yaqinlashuvchidir.

Endi Leybnis alomatining shartlarini tekshiramiz. $n\to\infty$ da $a_n\to 0$, lekin monotonlik sharti o‘rinli emas: $1>\frac{1}{2^3}>\frac{1}{3^4}$; $\frac{1}{3^4}<\frac{1}{4^3}$. Demak, 4.4- eslatmaga asosan, ishorasi almashinuvchi qatorlarda $\lim\limits_{n\to\infty}a_n=0$ shart bajarilib, monotonlik sharti bajarilmaganda ham qator yaqinlashuvchi bo‘lishi mumkin.

4.7- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{4}-\frac{1}{4}-\ldots .$$

Yechilishi. Qator hadlarining ishorasi almashmayapti, lekin hadlari absolut qiymati bo‘yicha monoton kamayib nolga intiladi. Qismiy yig‘indilarining ko‘rinishi ushbu

$$S_{\frac{n(n+1)}{2}}=\begin{cases}1, & n=2k-1,\ k\in N \text{ bo‘lganda}\\ 0, & n=2k,\ k\in N \text{ bo‘lganda,}\end{cases}$$

shaklda bo‘lgan ketma-ketlik limitga ega emas. Shuning uchun berilgan qator uzoqlashuvchi bo‘ladi.

4.8- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\ldots+\frac{1}{n}-\frac{1}{2n}+\ldots .$$

Yechilishi. Qator ishorasi almashinuvchi qator va umumiy had absolut qiymati bo‘yicha nolga intiladi, lekin monoton emas. Bu qator uzoqlashuvchi bo‘ladi, aks holda, bu qator yaqinlashuvchi bo‘lsa,

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{2n}\right)+\ldots=\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}+\ldots=$$
$$=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots\right)$$

qator ham yaqinlashuvchi bo‘lar edi, lekin qavs ichidagi qator uzoqlashuvchi – garmonik qator. Shunday qilib, Leybnis teoremasining ikkinchi sharti bajarilmasa ham qator uzoqlashuvchi bo‘lar ekan.

4.9- misol. Ushbu qatorni yaqinlashishga tekshiring:

$$1-\frac{3}{2}+\frac{5}{8}-...+(-1)^{n-1}\frac{2^{n-1}+1}{2^n}+... .$$

Yechilishi. Ravshanki, berilgan qator absolut yaqinlashuvchi emas, chunki

$$1+\frac{3}{2}+\frac{5}{8}+...+\frac{2^{n-1}+1}{2^n}+...$$

qatorda qator yaqinlashishing zaruriy sharti bajarilmaydi.

Leybnis alomati bo'yicha qatorni tekshiramiz. Berilgan qator ishorasi almashinuvchi va uning hadlari absolut qiymati bo'yicha monoton kamayuvchi, ya'ni $\forall n\in N$ lar uchun

$$a_n=\frac{2^{n-1}+1}{2^n}>\frac{2^n+1}{2^{n+1}}=a_{n+1}.$$

Lekin (4.5) shart bajarilmaydi, ya'ni

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{2^{n-1}+1}{2^n}=\frac{1}{2}\neq 0.$$

Demak, 4.6- eslatmaga ko'ra, berilgan qator uzoqlashuvchi bo'ladi.

4.10- misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ qatorni absolut va shartli yaqinlashishga tekshiring.

Yechilishi. 1) Berilgan qator hadlarining absolut qiymatlaridan tuzilgan

$$\frac{1}{1^p}+\frac{1}{2^p}+....+\frac{1}{n^p}+...$$

qatorni qaraymiz. Bu umumlashgan garmonik qator. 3.2- misolga asosan, $p>1$ bo'lganda yaqinlashuvchi, $p\leq 1$ bo'lganda esa uzoqlashuvchi bo'ladi.

2) Berilgan qatorni Leybnis alomatiga ko'ra, shartli yaqinlashishga tekshiramiz:

a) agar $p\leq 0$ bo'lsa, Leybnis alomatidagi $\lim_{n\to\infty}a_n=0$ shart bajarilmaydi. Demak, $p\leq 0$ bo'lganda qator uzoqlashuvchi;

b) agar $0 < p \le 1$ bo'lsa, u holda qator Leybnis alomati shartlarini qanoatlantiradi:

$$\frac{1}{n^p} > \frac{1}{(n+1)^p}, \ \forall n \in N, \ \lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Demak, berilgan qator $p > 1$ da absolut yaqinlashuvchi, $0 < p \le 1$ da shartli yaqinlashuvchi, $p \le 0$ da uzoqlashuvchi.

4.11- misol. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln^2 n}{n}$ qatorni yaqinlashishga tekshiring.

Yechilishi. $\varphi(x) = \frac{\ln^2 x}{x}$ deb belgilaymiz va Lopital qoidasiga asosan uning limitini topamiz:

$$\lim_{x\to\infty} \varphi(x) = \lim_{x\to\infty} \frac{\ln^2 x}{x} = 0.$$

Endi $\varphi(x)$ funksiyaning monotonligini tekshiramiz. Buning uchun $\varphi(x)$ funksiyaning hosilasini topamiz:

$$\varphi'(x) = \frac{\ln x}{x^2}(2 - \ln x).$$

Agar $x > e^2$ bo'lsa, $\varphi'(x) < 0$ bo'ladi. Demak, $\{a_n\} = \left\{\frac{\ln^2 n}{n}\right\}$ ketma-ketlik $n > e^2$ da (4.4) va (4.5) shartlarni qanoatlantiradi. Shunday qilib, Leybnis alomatiga asosan, berilgan qator yaqinlashuvchi.

Abel va Dirixle alomatlari. Biror

$$\sum_{n=1}^{\infty} a_n b_n = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n + \ldots \qquad (4.4)$$

ko'rinishdagi qator berilgan bo'lsin, bunda $\{a_n\}$ va $\{b_n\}$ — ixtiyoriy haqiqiy sonlar ketma-ketligi.

4.8- teorema (Dirixle alomati). Agar $\sum_{n=1}^{\infty} b_n$ qatorning qismiy yig'indisi chegaralangan, ya'ni

$$\exists M>0:\forall n\in N \text{ lar uchun } \Rightarrow \left|\sum_{k=1}^{n} b_k\right| \le M$$

va $\{a_n\}$ monoton ketma-ketlik bo‘lib, ya’ni $\exists n_0$ topilib, $\forall n \ge n_0$ lar uchun $a_{n+1} \ge a_n$ yoki $a_{n+1} \le a_n$ va $\lim\limits_{n\to\infty} a_n = 0$ bo‘lsa, (4.4) qator yaqinlashuvchi bo‘ladi.

4.9- teorema (Abel alomati). Agar $\{a_n\}$ ketma-ketlik monoton va chegaralangan bo‘lib, $\sum\limits_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo‘lsa, (4.4) qator yaqinlashuvchi bo‘ladi.

4.7- eslatma. Dirixle alomatidan xususiy holda Abel alomati kelib chiqadi.

Abel alomatiga ko‘ra, $\{a_n\}$ ketma-ketlik chekli limitga ega. (4.4) qatorni

$$\sum_{n=1}^{\infty}(a_n - a)\cdot b_n + a\sum_{n=1}^{\infty} b_n$$

ko‘rinishda yozib olsak, yig‘indidagi ikkinchi qo‘shiluvchi qator shart bo‘yicha yaqinlashuvchi, birinchi qatorga Dirixle alomatini qo‘llaymiz.

4.8- eslatma. Dirixle alomatidan xususiy holda Leybnis alomatini olish mumkin. Buning uchun $b_n = (-1)^{n+1}$ deb olish kifoya.

4.12- misol. $\sum\limits_{n=2}^{\infty} \frac{\ln^{100} n}{n} \sin\frac{n\pi}{4}$ qatorni yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorni Dirixle alomatiga asosan tekshiramiz. Ma’lumki, $B_n = \sum\limits_{k=1}^{n} \sin\frac{k\pi}{4}$ chegaralangan:

$$\left|\sum_{n=1}^{k} \sin\frac{n\pi}{4}\right| = \sin^{-1}\frac{\pi}{8}\left|\sin\frac{n\pi}{8}\sin\frac{n+1}{8}\pi\right| < \frac{1}{\sin\frac{\pi}{8}}.$$

$\{a_n\} = \left\{\frac{\ln^{100} n}{n}\right\}$ ketma-ketlik yetarlicha katta n larda nolga monoton intiladi. Haqiqatan ham,

$$\lim_{x\to+\infty} x^{-1}\ln^{100} x = 100\lim_{x\to+\infty} x^{-1}\ln^{99} x = 100\cdot 99\lim_{x\to+\infty} x^{-1}\cdot\ln^{98} x = \ldots =$$

$$= 100!\lim_{x\to+\infty}\frac{1}{x} = 0,\quad (x^{-1}\ln^{100} x)' < 0,\quad x > e^{100}.$$

Demak, Dirixle alomatiga asosan, berilgan qator yaqinlashuvchi.

4.13- misol. $\sum\limits_{n=2}^{\infty}\frac{\cos\frac{\pi n^2}{n+1}}{\ln^2 n}$ qatorni yaqinlashishga tekshiring.

Yechilishi. $\cos\frac{n^2\pi}{n+1}$ ifodani quyidagicha shakl almashtiramiz:

$$\cos\frac{\pi n^2}{n+1} = (-1)^n\cos\left(\pi\frac{n^2}{n+1} - \pi n\right) = (-1)^{n+1}\cos\frac{\pi}{n+1},$$

u holda qator quyidagi ko‘rinishga ega bo‘ladi: $\sum\limits_{n=2}^{\infty}\frac{(-1)^{n+1}\cos\frac{\pi}{n+1}}{\ln^2 n}$.

Abel alomatiga asosan qatorni yaqinlashishga tekshiramiz: $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln^2 n}$

qator Leybnis alomatiga ko‘ra yaqinlashuvchi, $\{a_n\} = \left\{\cos\frac{\pi}{n+1}\right\}$

ketma-ketlik esa, monoton o‘suvchi va chegaralangan. Demak, tekshirilayotgan qator Abel alomatiga ko‘ra yaqinlashuvchi.

4.10 - teorema (Riman teoremasi). Agar ixtiyoriy ishorali

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (4.5)$$

qator shartli yaqinlashuvchi bo‘lsa, har qanday A (chekli yoki cheksiz son) olinganda ham berilgan qator hadlarining o‘rinlarini shunday almashtirish mumkinki, hosil bo‘lgan qatorning yig‘ indisi xuddi shu A songa teng bo‘ladi.

4.14- misol. Quyidagi shartlar bajarilsin:

a) $\sum\limits_{n=1}^{\infty} a_n$ qatorning umumiy hadi $n\to\infty$ da $a_n \to 0$ bo‘lsin;

b) $\sum_{n=1}^{\infty} a_n$ qator hadlarining o'rinlarini o'zgartirmasdan guruhlab tuzilgan $\sum_{n=1}^{\infty} A_n$ qator yaqinlashuvchi bo'lsin;

d) $A_n = \sum_{i=p_n}^{p_{n+1}-1} a_i$ $(1 = p_1 < p_2 < ...)$ ga kiruvchi a_i qo'shiluvchi hadlarning soni chegaralangan bo'lsin. U holda $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi ekanligini isbotlang.

Isboti. $\sum_{n=1}^{\infty} A_n$ qatorning qismiy yig'indilar ketma-ketligini $\{S_{nk}^A\}$ deb belgilaymiz. Unda

$$S_{nk}^A = a_1 + a_2 + ... + a_{p_2-1} + a_{p_2} + a_{p_2+1} + ... + a_{p_3-1} + ... + a_{p_n} +$$

$$+a_{p_n+1} + ... + a_k + a_{k+1} + ... + a_{p_{n+1}-1} = S_k + a_{k+1} + ... + a_{p_{n+1}-1},$$

$$(p_n \le k < p_{n+1} - 1),$$

bu yerda $\{S_k\} - \sum_{n=1}^{\infty} a_n$ qatorning qismiy yig'indilari ketma-ketligi.

a) shartga ko'ra $n \to \infty$ da $a_n \to 0$ hamda $\{a_{k+1} + ... a_{p_{n+1}-1}\} = \{c_k\}$ ketma-ketlik;

d) shartga asosan chegaralangan bo'lganligi uchun, $k \to \infty$ da $c_k \to 0$. Shuning uchun $\lim_{n\to\infty} S_{nk}^A = \lim_{k\to\infty} S_k$ bo'ladi. Demak, berilgan $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo'ladi.

4.15- misol. Ushbu

$$a_1 + a_2 + ... + a_{p_2-1} - a_{p_2} - ... - a_{p_3-1} + ... + a_{p_3} + ... \quad (4.6)$$

qator bilan $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\sum_{i=p_n}^{i=p_{n+1}-1} a_i \right)$ $(a_i > 0,\ 1 = p_1 < p_2 < ...)$ (4.7) qator bir vaqtda yaqinlashuvchi yoki uzoqlashuvchi bo‘lishini isbotlang.

Isboti. a) (4.6) qator yaqinlashuvchi bo‘lsin. U holda uning istalgan qismiy yig‘indilar ketma-ketligi yaqinlashuvchi bo‘ladi. Jumladan, (4.7) qatorning

$$\left\{\sum_{k=1}^{n}(-1)^{k-1}\left(\sum_{i=p_k}^{i=p_{k+1}-1} a_i\right)\right\}=\{S_{nk}^{A}\}$$

qismiy yig‘indilar ketma-ketligi ham yaqinlashuvchi bo‘ladi. Bundan (4.7) qatorning yaqinlashuvchiligi kelib chiqadi.

b) (4.7) qator yaqinlashuvchi bo‘lsin. U holda qator yaqinlashishining zaruriy shartiga asosan, $A_n = \sum_{i=p_n}^{i=p_{n+1}-1} a_i \to 0$. Bu yerdan a_i musbat bo‘lganligini e’tiborga olganda, 4.11- misolga asosan, $a_{k+1}+...+a_{p_{n+1}-1}$ yig‘indi ham nolga intiladi va

$$\lim_{n\to\infty} S_{nk}^{A} = \lim_{k\to\infty} S_k$$

o‘rinli bo‘ladi. Bundan (4.6) qatorning yaqinlashuvchi bo‘lishi kelib chiqadi.

4.16- misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ qator yig‘indisining *ln*2 ga tengligini bilgan holda $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...$ qatorning avvalo ketma-ket *p* ta musbat va *q* ta manfiy hadlarini, so‘ngra yana *p* ta musbat va *q* ta manfiy hadlarini joylashtiramiz va hokazo. Natijada hosil bo‘lgan qatorning yig‘indisi $\ln\left(2\sqrt{\frac{p}{q}}\right)$ ga tengligini isbotlang.

Isboti. Berilgan qatorning hadlarini shartda aytilgandek guruhlab quyidagi qatorni tuzamiz:

$$1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2p-1}-\frac{1}{2}-\frac{1}{4}-...-\frac{1}{2q}+\frac{1}{2p+1}+\frac{1}{2p+3}+...+$$

$$+\frac{1}{4p-1}-\frac{1}{-2q+2}-\frac{1}{2q+4}-...-\frac{1}{4q}+... \quad (4.8)$$

(4.8) qatorning hadlaridan quyidagi qatorni tuzamiz:

$$\begin{aligned}&\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2q}\right)+\\&+\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\ldots+\frac{1}{4p-1}\right)-\ldots-\\&-\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\ldots+\frac{1}{4q}\right)+\ldots\,.\end{aligned} \tag{4.9}$$

Agar (4.9) qator yaqinlashuvchi bo‘lsa, u holda 4.12- misolga asosan (4.8) qatorning yigindisi (4.9) qator yig‘indisiga teng bo‘ladi. Ushbu

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\frac{1}{2(n-1)p+1}+\frac{1}{2(n-1)p+3}+\ldots+\\&+\frac{1}{2np-1}-\frac{1}{2(n-1)q+2}-\frac{1}{2(n-1)q+4}-\ldots-\frac{1}{2nq}\Big)\end{aligned} \tag{4.10}$$

qatorni qaraymiz. (4.10) qator (4.9) qator hadlarini ikkitadan guruhlash natijasida tuzilgan. Agar (4.10) qatorning yaqinlashuvchiligini ko‘rsatsak va yig‘indisini topsak, u holda 4.11- misolga asosan (4.9) qator ham yaqinlashuvchi bo‘lib, uning yig‘indisi (4.10) qatorning yig‘indisiga teng bo‘ladi.

a) $p>q$ bo‘lsin. U holda (4.10) ning qismiy yig‘indisi

$$S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2nq}+\frac{1}{2nq+1}+\frac{1}{2nq+3}+\ldots+\frac{1}{2np-1} \tag{4.11}$$

ko‘rinishda bo‘ladi. (4.11) ifodaga

$$\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}=\frac{1}{2}\left(\frac{1}{nq+1}+\frac{1}{nq+2}+\ldots+\frac{1}{np}\right)$$

yig‘indini ham qo‘shib, ham ayiramiz va

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots.+\frac{1}{n}=\ln\frac{m}{n}+E_{mn},\ \left(\begin{matrix}E_{mn}\to 0\\ m\to\infty\end{matrix}\right)$$

asimptotik formuladan foydalanib, (4.11) dan

$$S_n=C_{2np}+\ln\frac{2np}{2nq}-\frac{1}{2}\ln\frac{np}{nq}+E'_{n.}\left(\begin{matrix}E'_{n\to 0}\\ n\to\infty\end{matrix}\right) \tag{4.12}$$

tenglikni hosil qilamiz, bunda $\{C_{2np}\}$ ketma-ketlik $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ yaqinlashuvchi qatorning juft qismiy yig'indilar ketma-ketligidir. Shunday qilib, (4.12) dan

$$S=\lim_{n\to\infty}S_n=\ln 2+\frac{1}{2}\ln\frac{p}{q}=\ln\left(2\sqrt{\frac{p}{q}}\right)$$ ni hosil qilamiz.

b) $p\le q$ bo'lgan holni ham xuddi yuqoridagidek isbot qilish mumkin.

Xususiy holda, agar $p=2,\ q=1$ bo'lsa, u holda

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+.....=\frac{3}{2}\ln 2;$$

agar $p=1, q=2$ bo'lsa, u holda: $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+.....=\frac{1}{2}\ln 2$ bo'ladi.

4.17-misol. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ shartli yaqinlashuvchi qator hadlarining o'rinlarini shunday almashtiringki, natijada hosil bo'lgan qator uzoqlashuvchi bo'lsin.

Yechilishi. Berilgan qatorning hadlarini guruhlab quyidagi qatorni tuzamiz:

$$\left(1+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{7}}+\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{4}}\right)+...+$$

$$+\left(\frac{1}{\sqrt{6n-5}}+\frac{1}{\sqrt{6n-3}}+\frac{1}{\sqrt{6n-1}}-\frac{1}{\sqrt{2n}}\right)+$$

$$+...=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{6n-5}}+\frac{1}{\sqrt{6n-3}}+\frac{1}{\sqrt{6n-1}}-\frac{1}{\sqrt{2n}}\right)\equiv\sum_{n=1}^{\infty}a_n.$$

Bu qatorning uzoqlashuvchi ekanligini ko'rsatamiz:

$$a_n=\frac{1}{\sqrt{6n-5}}+\frac{1}{\sqrt{6n-3}}+\frac{1}{\sqrt{6n-1}}-\frac{1}{\sqrt{2n}}>\frac{2}{\sqrt{6n-1}}>\frac{1}{\sqrt{2n}}>0$$

tengsizlik o'rinli. Bu tengsizlikka asosan,

$$a_n = \frac{1}{\sqrt{6n-5}} + \frac{1}{\sqrt{6n-3}} + \frac{1}{\sqrt{6n-1}} - \frac{1}{\sqrt{2n}} > \frac{1}{\sqrt{6n-5}}$$

deb yozish mumkin. Ravshanki, $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{6n-5}}$ qator uzoqlashuvchi. U holda, 3.3-taqqoslash teoremasiga asosan, $\sum\limits_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo'ladi.

Mustaqil yechish uchun misollar

Qatorlarning absolut yaqinlashuvchiligini isbotlang.

4.1. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{ne^{\sqrt{n}}}$. **4.2.** $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^5}{2^n + 3^n}$.

4.3. $\sum\limits_{n=1}^{\infty} (-1)^n \ln\left(1 + \sin^2 \frac{\pi}{n}\right)$. **4.4.** $\sum\limits_{n=1}^{\infty} (-1)^n \sqrt[3]{n} \operatorname{arctg} \frac{2n+1}{n^3+2}$.

4.5. $\sum\limits_{n=1}^{\infty} \frac{(-n)^n}{(2n)!}$. **4.6.** $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^3}{\left(3 + \frac{1}{n}\right)^n}$. **4.7.** $\sum\limits_{n=1}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{n^{100}}{2^n}$.

4.8. $\sum\limits_{n=1}^{\infty} \frac{\sin \frac{n\pi}{4}}{n^3 + \sin \frac{\pi n}{4}}$. **4.9.** $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n \sin \frac{1}{n}} - \cos \frac{1}{n} \right) \cos \pi n$.

4.10. $\sum\limits_{n=1}^{\infty} \sin \frac{1}{2^n}$. **4.11.** $\sum\limits_{n=1}^{\infty} n^3 \sin n e^{-\sqrt{n}}$. **4.12** $\sum\limits_{n=1}^{\infty} 2^n \sin \frac{1}{3^n}$

4.13. $\sum\limits_{n=1}^{\infty} (-1)^n \left(\operatorname{arctg} \frac{1}{\sqrt{n}} - \arcsin \frac{1}{\sqrt{n}} \right)$.

4.14. $\sum\limits_{n=1}^{\infty} \left(\frac{\sin n}{\sqrt[3]{n^2}} - \sin\left(\frac{\sin n}{\sqrt[3]{n^2}} \right) \right)$. **4.15.** $\sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1} \ln^2 n}{2^n}$.

Ishorasi almashinuvchi qatorlarni absolut, shartli yaqinlashishga yoki uzoqlashishga tekshiring.

4.16. $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n^2+1}$. **4.17.** $\sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}$.

4.18. $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{3n+2}$. **4.19.** $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n+2}}{2^{n^2}}$.

4.20. $\sum_{n=3}^{\infty}(-1)^n \frac{\sqrt{n+1}-\sqrt{n-2}}{n}$. **4.21.** $\sum_{n=1}^{\infty}(-1)^n \left[\frac{(2n)!!}{(2n-1)!!}\right]^2$.

4.22. $\sum_{n=1}^{\infty}(-1)^n e^{-\sqrt[3]{n}}$. **4.23.** $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n\ln^2 n}$.

4.24. $\sum_{n=1}^{\infty}(-1)^n \frac{\left(2+\frac{1}{n}\right)^n}{n^2}$. **4.25.** $\sum_{n=2}^{\infty}(-1)^n \sqrt{n}\sin\frac{\pi}{n}$.

4.26. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n+(-1)^n}$. **4.27.** $\sum_{n=1}^{\infty}(-1)^n \operatorname{tg}\frac{1}{4\sqrt{n}}$.

4.28. $\sum_{n=1}^{\infty}(-1)^n \frac{(n!)^2}{(2n)!}4^n$. **4.29.** $\sum_{n=1}^{\infty}(-1)^n \frac{\ln(2n+1)}{(2n-1)^2}$.

Qatorlarni Dirixle va Abel alomatlari bo'yicha yaqinlashishga tekshiring.

4.30. $\sum_{n=1}^{\infty}(-1)^n \frac{\sin^2 n}{n}$. **4.31.** $\sum_{b=1}^{\infty}\frac{\sin nx}{n^{\alpha}}, \alpha>0$. **4.32.** $\sum_{n=2}^{\infty}\frac{\cos\frac{\pi n^2}{n+1}}{\ln^2 n}$.

4.33. $\sum_{n=1}^{\infty}\frac{\sin n\sin n^2}{n}$. **4.34.** $\sum_{n=1}^{\infty}\frac{\sin n\alpha}{\ln(\ln(n+2))}\cos\frac{1}{n}$.

4.35. $\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)\frac{\sin nx}{n}$.

Qatorlarni yaqinlashuvchi ekanligini isbotlang va ularning yig'indisini toping:

4.36. $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots$. **4.37.** $1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\ldots$.

4.38. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$.

4.39. $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\ldots$.

4.40. $1+\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\ldots$.

4.41. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[4]{n}}$ yaqinlashuvchi qatorning hadlarini shunday almashtiringki, hosil bo'lgan qator uzoqlashuvchi bo'lsin.

Qatorlarni yaqinlashishga tekshiring.

4.42. $\frac{1}{1^p}-\frac{1}{2^q}+\frac{1}{3^p}-\frac{1}{4^q}+\frac{1}{5^p}-\frac{1}{6^q}+\ldots$.

4.43. $1+\frac{1}{3^p}-\frac{1}{2^p}+\frac{1}{5^p}+\frac{1}{7^p}-\frac{1}{4^p}+\ldots$.

4.44. $1+\frac{1}{3^p}-\frac{1}{1^p}+\frac{1}{5^p}+\frac{1}{7^p}-\frac{1}{3^p}+\frac{1}{9^p}+\frac{1}{11^p}-\frac{1}{5^p}+\ldots$.

4.45. $1-\frac{2}{2^q}+\frac{1}{3^p}+\frac{1}{4^p}-\frac{2}{5^q}+\frac{1}{6^p}+\frac{1}{7^p}-\frac{2}{8^q}+\frac{1}{9^p}+\ldots$.

4.46. $\sum_{n=1}^{\infty}(-1)^n\frac{n+2}{\sqrt{n^2+4}}\operatorname{arctg}\frac{\pi}{\sqrt{n}}$.

4.47. $\sum_{n=1}^{\infty}(-1)^n\frac{\cos^2 2n}{\sqrt{n}}$. **4.48.** $\sum_{n=2}^{\infty}\frac{\sin\frac{n\pi}{12}}{\ln n}$. **4.49.** $\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n^p}$.

4.50. $\sum_{n=1}^{\infty}\frac{(-1)^{[\ln n]}}{n}$. **4.51.** $\sum_{n=1}^{\infty}\sin n^2$.

4.52. $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{(2n-1)!!}{(2n)!!}\right]^p$. **4.53.** $\sum_{n=1}^{\infty}\frac{\sin n}{\sqrt{n}+\sin n}$.

Misollarning javoblari

4.16. Absolut yaqinlashuvchi. **4.17.** Shartli yaqinlashuvchi. **4.18.** Shartli yaqinlashuvchi. **4.19.** Absolut yaqinlashuvchi. **4.20.** Shartli yaqinlashuvchi. **4.21.** Uzoqlashuvchi. **4.22.** Absolut yaqinlashuvchi. **4.23.** Absolut yaqinlashuvchi. **4.24.** Uzoqlashuvchi. **4.25.** Shartli yaqinlashuvchi. **4.26.** Absolut yaqinlashuvchi. **4.27.** Shartli yaqinlashuvchi. **4.28.** Uzoqlashuvchi. **4.29.** Absolut yaqinlashuvchi. **4.30.** Yaqinlashuvchi. **4.31.** $\forall x \in R$ lar uchun yaqinlashuvchi. **4.32.** Yaqinlashuvchi. **4.33.** Yaqinlashuvchi. **4.34.** $\forall \alpha$ lar uchun yaqinlashuvchi. **4.35.** Yaqinlashuvchi. **4.36.** $\frac{2}{9}$. **4.37.** $\frac{10}{3}$. **4.38.** ln 2 **4.39.** 0. **4.40.** ln 2. **4.42.** $p>1, q>1$ da absolut yaqinlashuvchi, $0<p=q=1$ da shartli yaqinlashuvchi. **4.43.** $p>1$ da absolut yaqinlashuvchi, $p=1$ da shartli yaqinlashuvchi. **4.44.** $p>1$ da absolut yaqinlashuvchi, $p=1$ da shartli yaqinlashuvchi. **4.45.** $p>1$, $q>1$ da absolut yaqinlashuvchi, $0<p=q\leq 1$ da shartli yaqinlashuvchi. **4.46.** Yaqinlashuvchi. **4.47.** Yaqinlashuvchi. **4.48.** Shartli yaqinlashuvchi. **4.49.** $p>1$ da absolut yaqinlashuvchi, $\frac{1}{2}<p\leq 1$ da esa shartli yaqinlashuvchi. **4.50.** Uzoqlashuvchi. **4.51.** Uzoqlashuvchi. **4.52.** $p>2$ da absolut yaqinlashuvchi, $0<p\leq 2$ da shartli yaqinlashuvchi. **4.53.** Uzoqlashuvchi.

II BOB

FUNKSIONAL KETMA-KETLIKLAR VA QATORLAR

5- §. *Funksional ketma-ketliklar, funksional qatorlar va ularning yaqinlashuvchiligi*

5.1. Funksional ketma-ketliklar va ularning yaqinlashuvchiligi. Elementlari biror $X \subset R$ to'plamda aniqlangan

$$f_1(x), f_2(x), ..., f_n(x), ... \qquad (5.1)$$

funksiyalar ketma-ketligi berilgan bo'lsin. Bu ketma-ketlik *funksional ketma-ketlik* deb ataladi va qisqacha $\{f_n(x)\}$ kabi belgilanadi. Umumiy holda $\{f_n(x)\}$ ketma-ketlik turli hadlarining aniqlanish sohasi, umuman aytganda, turlicha bo'lishi ham mumkin. Biz bu yerda X sifatida shu sohalarning umumiy qismini olamiz. (5.1) ketma-ketlikdagi $f_n(x)$ funksiya shu ketma-ketlikning *umumiy hadi* deyiladi. X to'plamdan x_0 $(x_0 \in X)$ nuqtani olib va (5.1) ketma-ketlik har bir hadining shu nuqtadagi qiymatini hisoblab, natijada

$$f_1(x_0), f_2(x_0), ..., f_n(x_0), ...$$

sonli ketma-ketlikni hosil qilamiz.

5.1- ta'rif. Agar $\{f_n(x_0)\}$ sonli ketma-ketlik yaqinlashuvchi (uzoqlashuvchi) bo'lsa, $\{f_n(x)\}$ funksional ketma-ketlik x_0 nuqtada *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

5.2- ta'rif. Agar $\{f_n(x)\}$ funksional ketma-ketlik X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, u X to'plamda *yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

5.1- eslatma $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish sohasi $\{f_n(x)\}$ funksional ketma-ketlikning aniqlanish sohasiga teng, yoki uning bir qismi, yoki bo'sh to'plam ham bo'lishi mumkin.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik $X \subset R$ to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan

$$f_1(x_0), f_2(x_0), ..., f_n(x_0), ...$$

ketma-ketlik chekli limitga ega bo'ladi, ya'ni

$$\lim_{n\to\infty} f_n(x_0) = f(x_0).$$

Agar X to'plamdan olingan har bir x ga, unga mos kelgan $f_1(x), f_2(x), ..., f_n(x), ...$ ketma-ketlikning limitini mos qo'ysak, ya'ni

$$f : x \to \lim_{n\to\infty} f_n(x)$$

deb olsak, unda X to'plamda aniqlangan biror $f(x)$ funksiya hosil bo'ladi. $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning *limit funksiyasi* deb ataladi va uni

$$\lim_{n\to\infty} f_n(x) = f(x) \ (x \in X) \qquad (5.2)$$

kabi yozamiz yoki, qisqacha, $f_n(x) \xrightarrow{X} f(x_0)$ deb belgilaymiz. (5.2) ni "ε" tilida quyidagicha ham yozish mumkin:

$$\forall \varepsilon > 0 \quad \exists n_0 = n_0(\varepsilon, x) \quad \forall n \geq n_0, \forall x \in X \ \Rightarrow \ |f_n(x) - f(x)| < \varepsilon.$$

5.1-misol. $\{f_n(x)\} = \left\{\dfrac{n^2+1}{n^2+x^2}\right\}$ funksional ketma-ketlikning limit funksiyasini toping.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $X = R$ da aniqlangan. Shuning uchun bu ketma-ketlikning aniqlanish sohasi $X = R$ dan iborat. Umumiy hadi $f_n(x) = \dfrac{n^2+1}{n^2+x^2}$. Endi limit funksiyani topamiz:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1+\dfrac{1}{n^2}}{1+\dfrac{x^2}{n^2}} = 1.$$

Demak, berilgan funksional $\left\{\dfrac{n^2+1}{n^2+x^2}\right\}$ ketma-ketlikning limit funksiyasi $f(x) = 1$ bo'ladi, ya'ni $\dfrac{n^2+1}{n^2+x^2} \xrightarrow{R} 1.$

5.2- misol. $\{f_n(x)\} = \left\{ n^2 \sin \frac{1}{n^2 x} \right\}$ funksional ketma-ketlikning yaqinlashish sohasi va limit funksiyasini toping.

Yechilishi. Berilgan ketma-ketlikning hamma hadlari $(-\infty;0) \cup (0;+\infty)$ to'plamda aniqlangan. Bu ketma-ketlikning umumiy hadi $f_n(x) = n^2 \sin \frac{1}{n^2 x}$. Limit funksiyani topamiz: $\forall x \in (-\infty;0) \cup (0;+\infty)$ lar uchun

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin \frac{1}{n^2 x}}{\frac{1}{xn^2}} \cdot \frac{1}{x} = \frac{1}{x}.$$

Demak, berilgan funksional ketma-ketlik $(-\infty;0)\cup(0;+\infty)$ da yaqinlashuvchi, uning limit funksiyasi $\frac{1}{x}$ ga teng:

$$n^2 \sin \frac{1}{n^2 x} \xrightarrow{(-\infty;0)\cup(0;+\infty)} \frac{1}{x}.$$

5.3- misol. $\{f_n(x)\} = \left\{x^n + 3x^{n+1}\right\}$ $(n = 1,2,...)$ funksional ketma-ketlikning yaqinlashish sohasi va limit funksiyasini toping.

Yechilishi. Berilgan $\left\{x^n + 3x^{n+1}\right\}$ ketma-ketlikning hamma hadlari $(-\infty;+\infty)$ da aniqlangan. Bu ketma-ketlikning umumiy hadi $f_n(x) = x^n + 3x^{n+1}$. Ketma-ketlikning yaqinlashish sohasini topish uchun quyidagi hollarni qaraymiz:

a) $\forall x \in (1;+\infty)$ da $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}(x^n + 3x^{n+1}) = +\infty$;

b) $\forall x \in (-1;1)$ da $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}(x^n + 3x^{n+1}) = 0$;

d) x=1 da $\lim\limits_{n\to\infty} f_n(1) = \lim\limits_{n\to\infty}(1^n + 3\cdot 1^{n+1}) = 4$;

e) $\forall x \in (-\infty;-1]$ bo'lganda esa berilgan funksional ketma-ketlikning limiti mavjud emas.

Shunday qilib, $\{f_n(x)\} = \{x^n + 3x^{n+1}\}$ funksional ketma-ketlikning yaqinlashish sohasi $X = (-1;1]$ bo'lib, limit funksiyasi esa

$$f(x) = \begin{cases} 0, & -1 < x < 1 \text{ bo'lganda} \\ 4, & x = 1 \text{ bo'lganda} \end{cases}$$

bo'ladi. Bu yerda funksional ketma-ketlikning yaqinlashish sohasi aniqlanish sohasining qismi bo'ladi, ya'ni $(-1;1] \subset (-\infty, +\infty)$.

5.4- misol. $\{f_n(x)\} = \{\cos nx\}$ funksional ketma-ketlikning yaqinlashish sohasi va limit funksiyasini toping.

Yechilishi. Berilgan $\{\cos nx\}$ ketma-ketlikning hamma hadlari $(-\infty, +\infty)$ da aniqlangan bo'lib, bu ketma-ketlik limit funksiyaga ega emas.

Demak, $\{f_n(x)\} = \{\cos nx\}$ ketma-ketlikning yaqinlashish sohasi bo'sh to'plamdan iborat.

5.5- misol. $\{f_n(x)\} = \left\{\ln\left(5 + \frac{n^4 \cdot e^x}{n^6 + e^{3x}}\right)\right\}$ funksional ketma-ketlikning $X = [0;+\infty)$ oraliqda limit funksiyasini toping.

Yechilishi. $\{f_n(x)\}$ funksional ketma-ketlikni quyidagi ko'rinishda yozamiz:

$$f_n(x) = \ln 5 + \ln\left(1 + \frac{n^4 e^x}{5(n^6 + e^{3x})}\right).$$

$t \to 0$ da $\ln(1+t) \sim t$, u holda $n \to \infty$ da

$$f_n(x) \sim \ln 5 + \frac{n^4 e^x}{5(n^6 + e^{3x})} \sim \ln 5 + \frac{e^x}{5n^2} = g_n(x).$$

Bu yerdan $[0;+\infty)$ da limit funksiyani topamiz:

$$\lim_{n\to\infty} g_n(x) = \lim\left(\frac{e^x}{5n^2} + \ln 5\right) = \ln 5 = f(x).$$

5.6- misol. $\{f_n(x)\} = \left\{\frac{n^2 x^2}{1 + n^4 x^4}\right\}$ funksional ketma-ketlikning $X = (-\infty;+\infty)$ oraliqda limit funksiyasini toping.

Yechilishi. Agar $x \neq 0$ bo'lsa, u holda

$$|f_n(x)| = \left| \frac{n^2x^2}{1+x^4n^4} \right| < \frac{n^2x^2}{x^4n^4} = \frac{1}{x^2n^2} \xrightarrow{n\to\infty} 0$$

bo'ladi, $x = 0$ bo'lgan holda esa barcha $n \in N$ lar uchun $f_n(x) = 0$ bo'ladi. Demak, $(-\infty;+\infty)$ da limit funksiya $f(x) = 0$ bo'ladi.

5.2. Funksional qatorlar va ularning yaqinlashuvchiligi. Biror X ($X \subset R$) to'plamda $u_1(x), u_2(x), ..., u_n(x), ...$ funksiyalar ketma-ketligi berilgan bo'lsin.

5.3- ta'rif. Ushbu

$$u_1(x) + u_2(x) + ... + u_n(x) + ...$$

ifoda *funksional qator* deyiladi va u $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$u_1(x) + u_2(x) + ... + u_n(x) + ... = \sum_{n=1}^{\infty} u_n(x). \tag{5.3}$$

Bunda $u_1(x), u_2(x), ..., u_n(x), ...$ lar qatorning *hadlari,* $u_n(x)$ esa funksional qatorning *umumiy hadi* deb ataladi. (5.3) funksional qatorning hadlaridan tuzilgan ushbu

$$\begin{aligned} &S_1(x) = u_1(x) \\ &S_2(x) = u_1(x) + u_2(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &S_n(x) = u_1(x) + u_2(x) + ... + u_n(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{5.4}$$

yig'indilar ketma-ketligi (5.3) funksional qatorning qismiy yig'indilari ketma-ketligi deyiladi va u $\{S_n(x)\}$ kabi belgilanadi.

5.2- eslatma. $\sum_{n=1}^{\infty} u_n(x)$ funksional qator turli hadlarining aniqlanish sohalari (to'plamlari), umuman aytganda, turlicha bo'ladi. Biz bu yerda X to'plam sifatida shu sohalarning umumiy qismini tushunamiz.

X to'plamdan x_0 ($x_0 \in X$) nuqtani olib va (5.3) funksional qator har bir $u_n(x)$ ($n = 1, 2, ...$) hadlarining shu nuqtadagi qiymatini hisoblab, ushbu sonli qatorni hosil qilamiz:

$$\sum_{n=1}^{\infty} u_n(x_0) = u_1(x_0) + u_2(x_0) + ... + u_n(x_0) + ... \qquad (5.5)$$

5.4- ta'rif. Agar (5.5) sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (5.3) funksional qator x_0 *nuqtada yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

5.5- ta'rif. Agar (5.3) funksional qator X to'plamning har bir nuqtasida yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (5.3) funksional qator X *to'plamda yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

Faraz qilaylik, (5.3) funksional qator X to'plamda yaqinlashuvchi bo'lsin. U holda $\forall x_0 \in X$ uchun unga mos kelgan (5.5) qator yaqinlashuvchi bo'ladi va uning yig'indisi biror S_0 songa teng bo'ladi.

Agar X to'plamdan olingan har bir x ga

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$

qatorning unga mos yig'indisini mos qo'ysak, u holda X to'plamda aniqlangan biror $S(x)$ funksiya hosil bo'ladi. Bu $S(x)$ funksiya $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning *yig'indisi* deyiladi va u $S(x) = \sum_{n=1}^{\infty} u_n(x)$ kabi yoziladi.

Sonli qatorlarning yaqinlashish (uzoqlashish) ta'rifiga asosan, funksional qatorning x_0 nuqtadagi yaqinlashish (uzoqlashish) ta'rifini quyidagicha ham berish mumkin.

5.6- ta'rif. Agar $n \to \infty$ da (5.4) funksional ketma-ketlik x_0 nuqtada yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (5.3) funksional qator x_0 *nuqtada yaqinlashuvchi (uzoqlashuvchi)* deyiladi.

Agar $n \to \infty$ da $\{S_n(x)\}$ funksional ketma-ketlik X to'plamda $S(x)$ limit funksiyaga ega, ya'ni $\lim_{n\to\infty} S_n(x) = S(x)$ bo'lsa, $S(x)$ funksiya (5.3) qatorning *yig'indisi* deyiladi.

5.7- ta'rif Agar

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + ... + |u_n(x)| + ... \qquad (5.6)$$

funksional qator $x=x_0$ nuqtada yaqinlashuvchi bo'lsa, (5.3) funksional qator x_0 nuqtada *absolut yaqinlashuvchi* deyiladi.

5.8- ta'rif. Agar X to'plamning har bir nuqtasida (5.6) qator yaqinlashuvchi bo'lsa, (5.3) funksional qator X to'plamda *absolut yaqinlashuvchi* deb ataladi.

Agar $x=x_0$ nuqtada (5.6) qator uzoqlashuvchi bo'lib, (5.3) qator yaqinlashuvchi bo'lsa, (5.3) qator $x=x_0$ *nuqtada shartli yaqinlashuvchi* deyiladi.

(5.3) va (5.6) qatorlar yaqinlashadigan nuqtalar to'plami mos ravishda (5.3) qatorning *yaqinlashish* va *absolut yaqinlashish sohasi* deyiladi.

5.3- eslatma. Berilgan (5.3) funksional qatorning yaqinlashish va absolut yaqinlashish sohasini topishda sonli qatorlar mavzusida ko'rib o'tilgan Dalamber va Koshi alomatlaridan foydalanish mumkin.

5.6- misol. Ushbu funksional qatorning $X=(0;+\infty)$ da yig'indisini toping:

$$\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}.$$

Yechilishi. 1.2- teoremaga ko'ra funksional qatorning umumiy hadini quyidagi ko'rinishda ifodalaymiz:

$$u_n(x)=\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}=$$

$$=\frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)}-\frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)(1+nx)}.$$

U holda funksional qatorning n- qismiy yig'indisi

$$S_n(x)=1-\frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$$

bo'ladi. Endi $n\to\infty$ da limitga o'tamiz:

$$\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\left(1-\frac{1}{(1+x)(1+2x)\cdots(1+nx)}\right)=1.$$

Demak, berilgan funksional qatorning yig'indisi $S(x)=1$ bo'ladi.

5.7- misol. $\sum_{n=0}^{\infty}(1-x)x^n$ funksional qatorning yaqinlashish sohasi va yig'indisini toping.

Yechilishi. Berilgan funksional qatorning n- qismiy yig'indisini topamiz:

$$S_n(x)=1-x+(1-x)x+.\cdots+(1-x)x^n=$$
$$=1-x+x-x^2+\cdots+x^n-x^{n+1}=1-x^{n+1}.$$

Qatorning yaqinlashish sohasi va yig'indisini topish uchun quyidagi hollarni qaraymiz:

a) agar $x \in (-1;1)$ bo'lsa, $S(x) = \lim\limits_{n\to\infty} S_n(x) = 1$ bo'ladi;

b) agar $x=1$ bo'lsa, $S(x) = \lim\limits_{n\to\infty} S_n(x) = 0$ bo'ladi;

d) agar $x \in (1;+\infty)$ bo'lsa, $S(x) = \lim\limits_{n\to\infty} S_n(x) = -\infty$ bo'ladi;

e) agar $x \in (-\infty;-1]$ bo'lsa, u holda $S_n(x)$ ketma-ketlik limitga ega emas.

Shunday qilib, berilgan funksional qatorning yaqinlashish sohasi $X = (-1;\ 1]$ to'plamdan iborat, yig'indisi esa

$$S(x) = \begin{cases} 0, \ x = 1 \ \text{bo'lganda}, \\ 1, \ x \in (-1;1) \ \text{bo'lganda} \end{cases}$$

5.8- misol. Funksional qatorning (shartli va absolut) yaqinlashish sohasini toping:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{2x-1}{x-1} \right)^n. \tag{5.7}$$

Yechilishi. Berilgan qatorning absolut va shartli yaqinlashish sohalarini topish uchun (5.7) qator hadlarining absolut qiymatlaridan tuzilgan ushbu

$$\sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} \left(\frac{2x-1}{x-1} \right)^n \right| \tag{5.8}$$

qatorga Dalamber alomatini qo'llaymiz, bunda x ni parametr deb hisoblaymiz. Ravshanki,

$$u_n(x) = \frac{1}{\sqrt{n}} \left(\frac{2x-1}{x-1} \right)^n, \quad u_{n+1}(x) = \frac{1}{\sqrt{n+1}} \left(\frac{2x-1}{x-1} \right)^{n+1}$$

bo'lib,

$$D_n(x) = \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \frac{\sqrt{n}}{\sqrt{n+1}} \left| \frac{2x-1}{x-1} \right|$$

bo'ladi. $n \to \infty$ da $D_n(x)$ ifodaning limitini topamiz:

$$\lim_{n\to\infty} D_n(x) = \left| \frac{2x-1}{x-1} \right|.$$

Ma'lumki, $\left|\frac{2x-1}{x-1}\right|<1$ bo'lsa, (5.8) qator yaqinlashuvchi bo'ladi, u holda 5.8- ta'rifga asosan, $0<x<\frac{2}{3}$ da (5.7) qator absolut yaqinlashuvchi bo'ladi. $\left|\frac{2x-1}{x-1}\right|>1$ bo'lsa, ya'ni $(-\infty;0)\cup\left(\frac{2}{3};+\infty\right)$ da esa, (5.8) qator uzoqlashuvchi bo'ladi. Endi x=0 va $x=\frac{2}{3}$ chegaraviy nuqtalarda (5.7) funksional qatorni yaqinlashishga tekshiramiz. x=0 bo'lganda bizga ma'lum bo'lgan $\sum_{n=1}^{\infty}u_n(0)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ uzoqlashuvchi sonli qator hosil bo'ladi. $x=\frac{2}{3}$ bo'lganda esa

$$\sum_{n=1}^{\infty}u_n\left(\frac{2}{3}\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

sonli qator hosil bo'ladi. Ma'lumki, Leybnis alomatiga ko'ra, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ sonli qator shartli yaqinlashuvchi, chunki uning hadlarining absolut qiymatidan tuzilgan qator uzoqlashuvchi.

Shunday qilib, berilgan (5.7) funksional qator $(0;\frac{2}{3})$ da absolut yaqinlashuvchi, $(-\infty;0]\cup(\frac{2}{3};+\infty)$ da uzoqlashuvchi, $x=\frac{2}{3}$ da shartli yaqinlashuvchi, $\left[0;\frac{2}{3}\right]$ esa uning yaqinlashish sohasi bo'ladi.

5.9- misol. $\sum_{n=1}^{\infty}\left[\frac{x(x+n)}{n}\right]^n$ funksional qatorning (absolut va shartli) yaqinlashish sohasini toping.

Yechilishi. Berilgan qatorning absolut va shartli yaqinlashish sohalarini topish uchun

$$\sum_{n=1}^{\infty}\left|\left(\frac{x(x+n)}{n}\right)^n\right|$$

qatorga Koshi alomatini qo'llaymiz, bunda x ni parametr deb olamiz.

Ravshanki, $u_n(x)=\left(\frac{x(x+n)}{n}\right)^n$ bo'lib,

$$K_n(x)=\sqrt[n]{|u_n(x)|}=\sqrt[n]{\left|\left(\frac{x(x+n)}{n}\right)^n\right|}=|x|\left|1+\frac{x}{n}\right|$$

bo'ladi. $n\to\infty$ da $K_n(x)$ ifodaning limitini topamiz:

$$\lim_{n\to\infty}K_n(x)=|x|.$$

Ma'lumki, $|x|<1$ bo'lganda, ya'ni $(-1;1)$ da berilgan funksional qator absolut yaqinlashuvchi, $|x|>1$ da esa funksional qator uzoqlashuvchi.

Endi $x=1$ va $x=-1$ bo'lgan hollarni qaraymiz.

Agar $x=1$ bo'lsa, u holda

$$\sum_{n=1}^{\infty}u_n(1)=\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$$

sonli qator uzoqlashuvchi bo'ladi, chunki qator yaqinlashishining zaruriy sharti bajarilmaydi, ya'ni

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0.$$

Agar $x=-1$ bo'lsa, u holda

$$\sum_{n=1}^{\infty}u_n(-1)=\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n}\right)^n$$

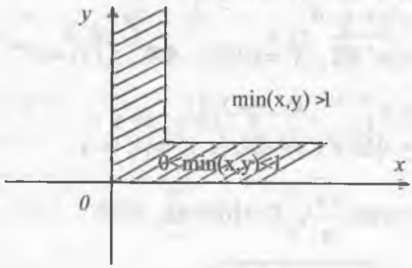
ishorasi almashinuvchi qator hosil bo'ladi. Bu qator uzoqlashuvchi, chunki Leybnis alomatining $\lim\limits_{n\to\infty}a_n=0$ sharti bajarilmaydi, ya'ni

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n=e^{-1}\neq 0.$$

Demak, berilgan funksional qator (–1;1) da absolut yaqinlashuvchi.

5.10- misol. $\sum_{n=1}^{\infty} \frac{x^n y^n}{x^n + y^n}$ $(x > 0,\ y > 0)$ funksional qatorning (absolut va shartli) yaqinlashish sohasini toping.

Yechilishi. Berilgan funksional qatorning umumiy hadi $u_n(x,y) = \frac{x^n y^n}{x^n + y^n}$. Bu qatorga Dalamber alomatini qo'llaymiz (bunda x va y ni parametrlar deb hisoblaymiz):



$$D_n(x,y) = \left|\frac{u_{n+1}(x,y)}{u_n(x,y)}\right| = |xy| \left|\frac{x^n + y^n}{x^{n+1} + y^{n+1}}\right|.$$

$$\lim_{n\to\infty} D_n(x,y) = \lim_{n\to\infty} xy \frac{x^n + y^n}{x^{n+1} + y^{n+1}} = \begin{cases} y, & x > y > 0 \text{ bo'lganda}, \\ x, & 0 < x < y \text{ bo'lganda}. \end{cases}$$

Demak, $0 < \min(x,y) < 1$ bo'lsa, u holda funksional qator chizmadagi shtrixlangan sohada absolut yaqinlashuvchi bo'ladi, $\min(x,y) \geq 1$ sohada esa uzoqlashuvchi bo'ladi.

Mustaqil yechish uchun misollar

X to'plamda $\{f_n(x)\}$ funksional ketma-ketliklarning limit funksiyasi $f(x)$ ni toping.

5.1. $f_n(x) = \frac{1}{x^n + 2n}$, $X = (-\infty; +\infty)$. **5.2.** $f_n(x) = \frac{n^3 + 1}{x^2 + n^3}$, $X = (-\infty; +\infty)$.

5.3. $f_n(x)=x^n-4x^{n+3}+3x^{n+4}, \quad X=[0;1]$.

5.4. $f_n(x)=x^4\cos\frac{1}{xn}, X=(0;+\infty)$.

5.5. $f_n(x)=\sqrt{x^2+\frac{1}{\sqrt{n}}}, X=(-\infty;+\infty)$.

5.6. $f_n(x)=n(x^{\frac{1}{n}}-1), X=[1;3]$.

5.7. $f_n(x)=n^2x^2\sin\frac{x}{n^2}, \ X=(-\infty;+\infty)$.

5.8. $f_n(x)=\sqrt[n]{x^n+1}, \ X=[0;2]$. **5.9.** $f_n(x)=e^{-nx^4}, \ X=[1;+\infty)$.

5.10. $f_n(x)=e^{-(x-n)^2}, X=[-2;2]$.

5.11. $f_n(x)=n[\ln(x+n)-\ln n], \ X=[1;+\infty)$.

5.12. $f_n(x)=n\sin\frac{\sqrt{x}}{n}, \ X=[0;+\infty)$.

5.13. $f_n(x)=\sqrt[n]{1+x^n+\left(\frac{x^2}{2}\right)^n}, \ X=[0;+\infty)$.

5.14. $f_n(x)=\frac{\ln nx}{nx^2}, X=[1;+\infty)$.

5.15. $f_n(x)=\frac{2}{\pi}\operatorname{arctg}nx, \ X=(-\infty;+\infty)$.

5.16. $f_n(x)=x\operatorname{arctg}nx, X=(0;+\infty)$.

5.17. $f_n(x)=n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right), X=(0;+\infty)$.

5.18. $f_n(x)=\frac{\operatorname{arctg}n^2x}{\sqrt{n^3+x^2}}, X=(-\infty;+\infty)$.

5.19. $f_n(x)=\ln\left(x^2+\frac{1}{n}\right), \quad X=[1;+\infty)$.

5.20. $f_n(x)=\sqrt[n]{\sin x}, X=[0;\pi]$.

Funksional qatorlarning (absolut va shartli) yaqinlashish sohalarini toping.

5.21. $\sum_{n=1}^{\infty} \ln^n x$. **5.22.** $\sum_{n=1}^{\infty} \frac{1}{n^x}$. **5.23.** $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\left(\frac{2x}{1+x^2}\right)^n$.

5.24. $\sum_{n=1}^{\infty} \frac{n^p \sin nx}{1+n^q}$ $(q>0; 0<x<\pi)$. **5.25.** $\sum_{n=1}^{\infty} x^n \operatorname{tg}\frac{x}{2^n}$.

5.26. $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$. **5.27.** $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^{\ln x}}$. **5.28.** $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^x}$.

5.29. $\sum_{n=1}^{\infty} 2^n \sin\frac{x}{3^n}$. **5.30.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(x-2)^n}$. **5.31.** $\sum_{n=1}^{\infty} \frac{\operatorname{tg}^n x}{n}$.

5.32. $\sum_{n=1}^{\infty} \frac{\ln(1+x^n)}{n^y}$, $(x\ge 0)$. **5.33.** $\sum_{n=1}^{\infty} \frac{x^n}{n+y^n}$ $(y\ge 0)$.

5.34. $\sum_{n=1}^{\infty} \sqrt[n]{|x|^{n^2}+|y|^{n^2}}$. **5.35.** $\sum_{n=1}^{\infty} \operatorname{tg}^n\left(x+\frac{y}{n}\right)$.

5.36. $\sum_{n=1}^{\infty} \frac{n!}{(x+1)\cdots(x+n)}$. **5.37.** $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$. **5.38.** $\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$.

5.39. $\sum_{n=1}^{\infty} \left(\cos\frac{\pi x}{n}\right)^{n^3}$. **5.40.** $\sum_{n=1}^{\infty} e^{-nx}\sin nx$.

Misollarning javoblari

5.1. $f(x)=0$. **5.2.** $f(x)=1$. **5.3.** $f(x)=0$. **5.4.** $f(x)=x^4$. **5.5.** $f(x)=|x|$.
5.6. $f(x)=\ln x$. **5.7.** $f(x)=x^3$.

5.8. $f(x)=\begin{cases} 1, & 0\le x<1 \text{ bo‘lganda}, \\ x, & 1\le x\le 2 \text{ bo‘lganda}. \end{cases}$

5.9. $f(x)=0$. **5.10.** $f(x)=0$. **5.11.** $f(x)=x$.

5.12. $f(x)=\sqrt{x}$. **5.13.** $f(x)=\begin{cases} 1, & 0\le x<1 \text{ bo‘lganda}, \\ x, & 1\le x<2 \text{ bo‘lganda}, \\ \frac{x^2}{2}, & x\ge 2 \text{ bo‘lganda}. \end{cases}$

5.14. $f(x)=0$. **5.15.** $f(x)=\operatorname{sgn} x$. **5.16.** $f(x)=\dfrac{\pi x}{2}$.

5.17. $f(x)=\dfrac{1}{2\sqrt{x}}$. **5.18.** $f(x)=0$. **5.19.** $f(x)=\ln x^2$.

5.20. $f(x)=\begin{cases} 1, & x\in(0;\pi) \text{ bo‘lganda}, \\ 0, & x=0, x=\pi \text{ bo‘lganda}. \end{cases}$ **5.21.** $(\frac{1}{e},e)$ — absolut yaqinlashuvchi. **5.22.** $(1,+\infty)$ — absolut yaqinlashuvchi. **5.23.** $x\neq 1$ — absolut yaqinlashuvchi; $x=-1$ — shartli yaqinlashuvchi. **5.24.** $q>p+1$ — absolut yaqinlashuvchi; $p<q\leq p+1$ — shartli yaqinlashuvchi. **5.25.** $(-2;2)$ — absolut yaqinlashuvchi. **5.26.** $(-\infty,+\infty)$ — absolut yaqinlashuvchi. **5.27.** $(e;+\infty)$ — absolut yaqinlashuvchi; $(1;e]$ — shartli yaqinlashuvchi. **5.28.** $(1;+\infty)$ — absolut yaqinlashuvchi, $(0;1]$ — shartli yaqinlashuvchi. **5.29.** $(-\infty;+\infty)$ — absolut yaqinlashuvchi. **5.30.** $(-\infty;1)\cup(3;+\infty)$ — absolut yaqinlashuvchi. **5.31.** $|x-\pi k|<\frac{\pi}{4}$ — absolut yaqinlashuvchi; $x=-\frac{\pi}{4}+\pi k$ — shartli yaqinlashuvchi, $k\in Z$. **5.32.**

$$\left.\begin{array}{l} 0\leq x<1,\ -\infty<y<\infty \\ x=1,\ y>1 \\ x>1,\ y>2 \end{array}\right\}$$

— absolut yaqinlashuvchi. **5.33.** $\left.\begin{array}{r} x=-1 \\ 0\leq y\leq 1 \end{array}\right\}$ — shartli yaqinlashuvchi, $\left.\begin{array}{l} |x|<1,\ 0\leq y<+\infty \\ |x|>1,\ y>|x| \end{array}\right\}$ — absolut yaqinlashuvchi. **5.34.** $\max(|x|,|y|)<1$ — absolut yaqinlashuvchi.

5.35. $|x-k\pi|<\frac{\pi}{4}$, $k\in Z$ — absolut yaqinlashuvchi. **5.36.** $(1;+\infty)$ — absolut yaqinlashuvchi. **5.37.** $(-\infty;-1)\cup(1;+\infty)$ — absolut yaqinlashuvchi. **5.38.** $(-1;1)$ — absolut yaqinlashuvchi. **5.39.** $x \neq 0$ — absolut yaqinlashuvchi. **5.40.** $x \geq 0$ va $x = -\pi k,\ k \in N$ — absolut yaqinlashuvchi.

6-§. Funksional ketma-ketlik va qatorlarning tekis yaqinlashuvchiligi

6.1. Funksional ketma-ketliklarning tekis yaqinlashuvchiligi. Ushbu

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots \qquad (6.1)$$

funksional ketma-ketlik X $(X \subseteq R)$ to‘plamda yaqinlashuvchi va uning limit funksiyasi $f(x)$ bo‘lsin.

6.1- ta’rif. Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\exists m_\varepsilon \in N$ nomer topilib, $\forall n > m$ va $\forall x \in X$ lar uchun bir vaqtda

$$|f_n(x) - f(x)| < \varepsilon$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to‘plamda $f(x)$ ga *tekis yaqinlashadi* deyiladi va u qisqacha

$$f_n \overset{X}{\rightrightarrows} f$$

kabi belgilanadi.

6.1- eslatma. 6.1-ta’rifdagi m natural son faqat ε ga bog‘liq bo‘lib, x larga bog‘liq bo‘lmaydi.

6.2- ta’rif. $\forall m \in N$ olinganda ham, $\exists \varepsilon_0 > 0$, $\exists n \geq m$ va $x_0 \in X$ mavjud bo‘lib,

$$|f_n(x_0) - f(x_0)| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\{f_n(x)\}$ funksional ketma-ketlik X to‘plamda $f(x)$ ga *tekis yaqinlashmaydi* deyiladi va u qisqacha

$$f_n \overset{X}{\not\rightrightarrows} f$$

kabi belgilanadi.

6.3- ta’rif. Agar $f_n(x) \overset{X}{\rightarrow} f(x)$ bo‘lib, lekin $f_n(x) \overset{X}{\not\rightrightarrows} f(x)$ bo‘lsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga *tekis yaqinlashmaydi (notekis yaqinlashadi*) deyiladi.

Xususiy holda, agar $f_n(x) \overset{X}{\longrightarrow} f(x)$ va $\exists \varepsilon_0 > 0,\ \forall m \in N\ \exists n \geq m$

va $$\exists x_n \in X\ \left|f_n(x_n) - f(x_n)\right| \geq \varepsilon_0 \qquad (6.2)$$

shart bajarilsa, $\{f_n(x)\}$ ketma-ketlik X da $f(x)$ ga *tekis yaqinlashmaydi* deyiladi.

Tekis yaqinlashuvchi funksional ketma-ketliklar quyidagi xossalarga ega.

1- xossa. Agar $\{f_n(x)\}$ va $\{g_n(x)\}$ funksional ketma-ketliklar $f_n \overset{X}{\rightrightarrows} f$, $g_n \overset{X}{\rightrightarrows} g$ bo'lsa, u holda $\{\lambda f_n(x) + \mu g_n(x)\}$ (bunda λ, μ — ixtiyoriy haqiqiy sonlar) funksional ketma-ketlik ham

$$\lambda \cdot f_n(x) + \mu \cdot g_n(x) \overset{X}{\rightrightarrows} \lambda \cdot f(x) + \mu \cdot g(x)$$

bo'ladi.

2- xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlik $f_n(x) \overset{X}{\rightrightarrows} f(x)$ bo'lib, $g(x)$ funksiya esa X to'plamda chegaralangan bo'lsa, ya'ni

$$\exists M > 0:\ \forall x \in X\ \ |g(x)| \leq M$$

bo'lsa, u holda $\{g(x) f_n(x)\}$ ketma-ketlik ham $g(x) f_n(x) \overset{X}{\rightrightarrows} g(x) f(x)$ bo'ladi.

6.1- teorema. (6.1) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0 \qquad (6.3)$$

shartning bajarilishi zarur va yetarli.

6.2- teorema. (6.1) funksional ketma-ketlik X to'plamda $f(x)$ ga tekis yaqinlashishi uchun shunday $\{a_n\}$ sonli ketma-ketlik (bunda $\lim\limits_{n\to\infty} a_n = 0$) va shunday m nomer mavjud bo'lib, barcha $n>m$ va barcha $x \in X$ lar uchun

$$|f_n(x) - f(x)| < a_n$$

tengsizlikning bajarilishi zarur va yetarli.

6.1- misol. $\{f_n(x)\}=\left\{\frac{\cos n\sqrt{x}}{n}\right\}$ funksional ketma-ketlikning $X=[0;+\infty)$ to'plamda tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}\frac{\cos n\sqrt{x}}{n}=0$$

bo'lib, $X=[0;+\infty)$ da yaqinlashuvchi bo'ladi.

Endi yaqinlashish xarakterini aniqlaymiz. $\forall\varepsilon>0$ son olinganda ham $m=\left[\frac{1}{\varepsilon}\right]$ deyilsa, unda barcha $n>m$ va $\forall x\in X$ lar uchun

$$|f_n(x)-f(x)|=\left|\frac{\cos n\sqrt{x}}{n}-0\right|=\left|\frac{\cos n\sqrt{x}}{n}\right|\le\frac{1}{n}<\frac{1}{m+1}<\varepsilon$$

bo'ladi. Yuqoridagi 6.1- ta'rif va 6.1- teoremaga ko'ra, $\{f_n(x)\}=\left\{\frac{\cos n\sqrt{x}}{n}\right\}$ ketma-ketlik $f(x)=0$ limit funksiyaga tekis yaqinlashadi: $\frac{\cos n\sqrt{x}}{n}\overset{X}{\rightrightarrows}0$.

6.2- misol. Ushbu funksional ketma-ketlikni $X=[0;+\infty)$ da tekis yaqinlashuvchilikka tekshiring:

$$\{f_n(x)\}=\left\{\frac{\operatorname{arctg} n\sqrt{x}}{n+\sqrt{x}}\right\}.$$

Yechilishi. $\{f_n(x)\}$ ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{\operatorname{arctg} n\sqrt{x}}{n+\sqrt{x}}=0$$

bo'lib, u $X=[0;+\infty)$ da yaqinlashuvchi bo'ladi.

Endi yaqinlashish xarakterini aniqlaymiz. Bizga ma'lumki, barcha $n\in N$ va $\forall x\in X$ lar uchun

$$0 \le \operatorname{arctg} n\sqrt{x} < \frac{\pi}{2}, \quad n+\sqrt{x} > n$$

bo‘ladi. U holda, $\forall \varepsilon > 0$ son olinganda ham $m = \left[\frac{2}{\varepsilon}\right]$ deyilsa, barcha $n > m$ va $\forall x \in X$ lar uchun

$$\left|f_n(x) - f(x)\right| = \left|\frac{\operatorname{arctg} n\sqrt{x}}{n+\sqrt{x}}\right| < \frac{\pi}{2n} < \frac{2}{n} < \varepsilon$$

bo‘ladi. Yuqoridagi 6.1-ta’rifga asosan, $\{f_n(x)\} = \left\{\frac{\operatorname{arctg} n\sqrt{x}}{n+\sqrt{x}}\right\}$ ketma-ketlik $f(x)=0$ limit funksiyaga tekis yaqinlashadi:

$$\frac{\operatorname{arctg} n\sqrt{x}}{n+\sqrt{x}} \overset{X}{\rightrightarrows} 0.$$

6.3- misol. $\{f_n(x)\} = \left\{\sqrt[4]{x+\frac{1}{n}} - \sqrt[4]{x}\right\}$ funksional ketma-ketlikni $X = [0;+\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Barcha $x \in X$ va $\forall n \in N$ lar uchun $x + \frac{1}{n} \le \left(\sqrt{x} + \frac{1}{\sqrt{n}}\right)^2$ tengsizlik o‘rinli bo‘ladi. U holda

$$0 \le \sqrt[4]{x+\frac{1}{n}} - \sqrt[4]{x} = \frac{\sqrt{x+\frac{1}{n}} - \sqrt{x}}{\sqrt[4]{x+\frac{1}{n}} + \sqrt[4]{x}} \le$$

$$\le n^{\frac{1}{4}}\left(\sqrt{(\sqrt{x}+\frac{1}{\sqrt{n}})^2} - \sqrt{x}\right) = \frac{n^{\frac{1}{4}}}{\sqrt{n}} = \frac{1}{\sqrt[4]{n}} = a_n.$$

Demak, 6.2- teoremaga asosan, berilgan funksional ketma-ketlik $[0;+\infty)$ da $f(x)=0$ limit funksiyaga tekis yaqinlashadi.

6.4- misol. $\{f_n(x)\}=\left\{\frac{x^2}{1+n^4x^4}\right\}$ funksional ketma-ketlikni $X=[-1;\,1]$ da tekis yaqinlashishga tekshiring.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}\frac{x^2}{1+n^4x^4}=0$$

bo‘lib, $f_n\xrightarrow{X}f$ bo‘ladi. Bu yerda

$$0\le\frac{1}{2n^2}\cdot\frac{2n^2x^2}{1+n^4x^4}\le\frac{1}{2n^2}$$

bo‘lganligi sababli, x ning har qanday qiymatida $f_n(x)<\varepsilon$ tengsizlikning o‘rinli bo‘lishi uchun $n>\sqrt{\frac{1}{2\varepsilon}}$ qilib olish kifoya. Shunday qilib, bu holda $m=\left[\sqrt{\frac{1}{2\varepsilon}}\right]$ deyilsa, bir vaqtning o‘zida x ning barcha qiymatlari uchun u yaroqlidir. Demak, $f_n \overset{X}{\text{I}}$ 0.

6.5- misol. $\{f_n(x)\}=\left\{\frac{\sqrt{nx}}{2+nx}\right\}$ funksional ketma-ketlikning $X=[0;\,2]$ da notekis yaqinlashuvchiligini ko‘rsating.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}\frac{\sqrt{nx}}{2+nx}=0$$

bo‘ladi.

Endi berilgan ketma-ketlikning $f(x)=0$ limit funksiyaga notekis yaqinlashishini ko‘rsatamiz. $\forall\varepsilon>0$ son olinganda ham m natural son sifatida $m=\left[\frac{1}{\varepsilon^2x}\right]$, $x\neq0$ olinsa, u holda $\forall m<n$ uchun

$$|f_n(x)-f(x)|=\left|\frac{\sqrt{nx}}{2+nx}-0\right|=\frac{\sqrt{nx}}{2+nx}<\frac{1}{\sqrt{nx}}\le\frac{1}{\sqrt{(m+1)x}}<\varepsilon$$

bo'ladi. Ravshanki, x=0 da $\forall n\in N$ lar uchun $f_n(0)=f(0)=0$ bo'ladi. Bu holda m ning x ga bog'liqligi evaziga, ixtiyoriy natural n son uchun $\varepsilon_0=\frac{1}{4}$ va $x_n=\frac{1}{n}\in(0;2]$ deb olsak,

$$|f_n(x_n)-f(x_n)|=\frac{\sqrt{\frac{1}{n}n}}{2+\frac{1}{n}n}=\frac{1}{3}>\varepsilon_0$$

bo'ladi. Bu esa berilgan $\{f_n(x)\}$ funksional ketma-ketlikning $f(x)$=0 limit funksiyaga notekis yaqinlashishini bildiradi, ya'ni $x\in[0;2]$ ning barcha qiymatlari uchun bir vaqtda yaraydigan nomer mavjud emas.

6.2- ta'rifga ko'ra berilgan ketma-ketlik $f_n(x)=\frac{\sqrt{nx}}{2+nx}\overset{[0;2]}{\not\rightrightarrows}0$.

6.6- misol. $\{f_n(x)\}=\{x^n\}$ funksional ketma-ketlikning $x\in[0;1)$ da notekis yaqinlashuvchiligini ko'rsating.

Yechilishi. Bu ketma-ketlik $x\in[0;1)$ da yaqinlashuvchi $f_n\overset{X}{\to}f$, ya'ni $f(x)=\lim\limits_{n\to\infty}x^n=0$.

Endi (6.2) shartning bajarilishini ko'rsatamiz. $x_n=\frac{1}{2^{\frac{1}{n}}}$ ketma-ketlikni olamiz, u holda barcha $n\in N$ lar uchun $x_n\in[0;1)$ bo'lib va $|f_n(x_n)-f(x_n)|=|x_n^n-0|=\frac{1}{2}=\varepsilon_0$ bo'ladi.

Demak, $\{x^n\}$ ketma-ketlik [0;1) da $f(x)$=0 funksiyaga notekis yaqinlashadi.

6.7- misol. $\{f_n(x)\}=\left\{\ln\left(5+\dfrac{n^4e^x}{n^6+e^{3x}}\right)\right\}$ funksional ketma-ketlikning:

a) $X=[0;+\infty)$ da notekis; *b)* $X_1=[0;a]\ (a>0)$ da tekis yaqinlashuvchiligini ko'rsating.

Yechilishi. *a)* 5.4- misolda bu ketma-ketlikning limit funksiyasi $f(x)=\ln 5$ ekanligi ko'rsatilgan. Endi berilgan ketma-ketlik uchun (6.2) shartning bajarilishini ko'rsatamiz. Agar $x_n=2\ln n$ deb olsak, u holda

$$|f_n(x_n)-f(x_n)|=\ln\left(5+\frac{n^4e^{2\ln n}}{n^6+e^{6\ln n}}\right)-\ln 5=$$

$$=\ln(5+\frac{n^6}{2\cdot n^6})-\ln 5=\ln\frac{11}{10}=\varepsilon_0\,.$$

Shunday qilib, $\varepsilon_0=\ln\dfrac{11}{10}$ va $\forall n\in N$ lar uchun (6.2) shart bajariladi. Demak, $\{f_n(x)\}=\left\{\ln\left(5+\dfrac{n^4e^x}{n^6+e^{3x}}\right)\right\}$ ketma-ketlik $X=[0;+\infty)$ da $f(x)=\ln 5$ ga notekis yaqinlashadi.

b) Berilgan ketma-ketlik $[0,a]$ da tekis yaqinlashuvchi ekanligini ko'rsatamiz. Ravshanki, barcha $t\geq 0$ lar uchun $\ln(1+t)\leq t$ tengsizlik o'rinli bo'ladi, u holda

$$0<f_n(x)-f(x)=\ln\left(1+\frac{n^4e^x}{5(n^6+e^{3x})}\right)\leq\frac{n^4\cdot e^x}{5(n^6+e^{3x})}<$$

$$<\frac{n^4\cdot e^x}{5\cdot n^6}\leq\frac{e^a}{5n^2}=a_n.$$

Demak, 6.2- teoremaga asosan, berilgan ketma-ketlik $X_1=[0;a]$ da $f(x)=\ln 5$ ga tekis yaqinlashadi, ya'ni $f_n \overset{X}{\rightrightarrows} \ln 5$.

6.8- misol. $\{f_n(x)\}=\{x^n-x^{n+1}\}$ funksional ketma-ketlikni $X=[0;1]$ to'plamda tekis yaqinlashishga tekshiring.

Yechilishi. Ravshanki, $x=1$, $x=0$ da $f_n(1)=f(0)=0$, $0<x<1$ da esa $\lim\limits_{n\to\infty}(x^n-x^{n+1})=0$ bo'ladi. Demak, berilgan funksional ketma-ketlik X da yaqinlashuvchi, uning limit funksiyasi $f(x)=0$ bo'ladi. Endi (6.3) shartning bajarilishini tekshiramiz. Buning uchun [0;1] da $f(x)$ funksiyaning ekstremum nuqtalarini topamiz. $f(x)$ funksiyaning hosilasi

$$f_n'(x)=x^{n-1}(n-x(n+1))$$

bo'ladi. [0;1] kesma ichida $x^{n-1}[n-x(n+1)]=0$ tenglama yagona $x=x_n=\dfrac{n}{n+1}$ ildizga ega. Agar $x\in(0;x_n)$ bo'lsa, $f_n'(x)>0$; $x\in(x_n;1)$ bo'lganda esa, $f_n'(x)<0$ bo'ladi. Shuning uchun, $\forall x\in X$ da $\sup f_n(x)=\max f_n(x)=f_n(x_n)$ bo'ladi. Demak,

$$\lim_{n\to\infty}\sup_{x\in X}|f_n(x)-f(x)|=\lim_{n\to\infty}\sup_{x\in X}f_n(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}\cdot\frac{1}{n+1}=0.$$

Bunda (6.3) shart bajarilayapti, shuning uchun $f_n \overset{X}{\rightrightarrows} 0$.

6.9- misol. $\{f_n(x)\}=\{x^n-x^{2n}\}$ funksional ketma-ketlikni $X=[0;1]$ to'plamda tekis yaqinlashishga tekshiring.

Yechilishi. Ravshanki, $x=1$ va $x=0$ nuqtalarda $f_n(0)=f(1)=0$, $0<x<1$ da esa $\lim\limits_{n\to\infty}(x^n-x^{2n})=0$ bo'ladi. Demak, berilgan funksional ketma-ketlik X da yaqinlashuvchi, uning limit funksiyasi $f(x)=0$. Bu yaqinlashishning xarakterini aniqlaymiz. Buning uchun [0;1] da $f_n(x)$ funksiyaning ekstremum nuqtalarini topamiz. $f_n(x)$ funksiyaning hosilasi

$$f_n'(x)=nx^{n-1}(1-2x^n).$$

[0;1] kesma ichida $nx^{n-1}(1-2x^n)=0$ tenglama $x=x_n=\dfrac{1}{\sqrt[n]{2}}$

yagona ildizga ega. Agar $x \in (0; x_n)$ bo‘lsa, $f_n'(x) > 0$; $x \in (x_n; 1)$ bo‘lganda esa $f_n'(x) < 0$ bo‘ladi. Bu yerdan $f_n(x) = x^n - x^{2n}$ funksiya $x_n = \frac{1}{\sqrt[n]{2}}$ nuqtada o‘zining maksimum qiymatiga erishishi kelib chiqadi. Natijada

$$\sup_{x \in X} |f_n(x) - f(x)| = \sup_{x \in X} f_n(x) = \max_{x \in X} f_n(x) = f_n(x_n) = \frac{1}{4}$$

bo‘lib, $\lim_{n \to \infty} \sup_{x \in X} |f_n(x) - f(x)| = \frac{1}{4} \neq 0$ bo‘ladi.

Demak, (6.3) shart bajarilmayapti, berilgan ketma-ketlik $X = [0; 1]$ da notekis yaqinlashadi.

6.10- misol. $\{f_n(x)\} = \{\mathrm{arctg}\, nx\}$ funksional ketma-ketlikni $X = (0; +\infty)$ to‘plamda tekis yaqinlashishga tekshiring.

Yechilishi. Berilgan ketma-ketlikning limit funksiyasini topamiz:

$$f(x) = \lim_{n \to \infty} \mathrm{arctg}\, nx = \frac{\pi}{2}.$$

Endi $\{f_n(x)\}$ ketma-ketlikning $f(x) = \frac{\pi}{2}$ limit funksiyaga yaqinlashish xarakterini aniqlash uchun (6.3) shartning bajarilishini tekshiramiz:

$$\sup_{x \in X} \left| \mathrm{arctg}\, nx - \frac{\pi}{2} \right| = \lim_{x \to +0} \left| \frac{\pi}{2} - \mathrm{arctg}\, nx \right| = \frac{\pi}{2} \neq 0 .$$

Bu yerda (6.3) shart bajarilmayapti. Demak, $\{f_n(x)\} = \{\mathrm{arctg}\, nx\}$ ketma-ketlik $(0; +\infty)$ da $f(x) = \frac{\pi}{2}$ ga tekis yaqinlashmaydi.

6.11- misol. $\{f_n(x)\} = \left\{ \frac{x}{n} \ln \frac{x}{n} \right\}$ funksional ketma-ketlikni $X = (0; 1)$ to‘plamda tekis yaqinlashishga tekshiring.

Yechilishi. Bu ketma-ketlikning limit funksiyasi

$$f(x) = \lim_{n \to \infty} \frac{x}{n} \ln \frac{x}{n} = x \lim_{n \to \infty} \frac{\ln x - \ln n}{n} = 0$$

bo'ladi. Endi berilgan ketma-ketlikning X da $f(x)=0$ limit funksiyaga yaqinlashish xarakterini aniqlash uchun (6.3) shartning bajarilishini tekshiramiz:

$$\sup_{x\in X}|f_n(x)-f(x)|=\sup_{x\in X}\left|\frac{x}{n}\cdot\ln\frac{x}{n}\right|=\frac{\ln n}{n},$$

$$\lim_{n\to\infty}\sup_{x\in X}|f_n(x)-f(x)|=\lim_{n\to\infty}\frac{\ln n}{n}=0.$$

(6.3) shart bajarilayapti, shuning uchun berilgan $\{\frac{x}{n}\ln\frac{x}{n}\}$ ketma-ketlik $X=(1;0)$ da $f(x)=0$ funksiyaga tekis yaqinlashadi.

Faraz qilaylik, X to'plamda $\{f_n(x)\}$ funksional ketma-ketlik berilgan bo'lsin.

6.3- teorema (funksional ketma-ketlikning tekis yaqinlashishi uchun Koshi kriteriysi). $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi uchun, ixtiyoriy $\varepsilon>0$ da x ga bog'liq bo'lmagan shunday nomer $m(\varepsilon)$ mavjud bo'lib, $n>m$ bo'lganda va istalgan $p\in N$ da x ning X dagi hamma qiymatlari uchun bir vaqtning o'zida

$$|f_{n+p}(x)-f_n(x)|<\varepsilon$$

tengsizlikning o'rinli bo'lishi zarur va yetarli.

6.3-teoremadagi Koshi shartini, qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall\varepsilon>0\ \exists m(\varepsilon):\ \forall n\geq m,\forall p\in N,\forall x\in X:|f_{n+p}(x)-f_n(x)|<\varepsilon, \quad (6.4)$$

yoki boshqacha ko'rinishda:

$$\forall\varepsilon>0\ \exists m(\varepsilon):\ \forall n\geq m,\forall k\geq m,\forall x\in X:|f_n(x)-f_k(x)|<\varepsilon. \quad (6.5)$$

Agar Koshi sharti bajarilmasa, ya'ni

$$\exists\varepsilon_0>0:\ \forall k\in N,\ \exists n\geq k\ \exists p\in N\ \exists x\in X:|f_{n+p}(x)-f_n(x)|\geq\varepsilon_0 \quad (6.6)$$

bo'lsa, u holda $\{f_n(x)\}$ funksional ketma-ketlik X to'plamda *notekis yaqinlashuvchi* deyiladi.

Xususiy holda, agar

$$\exists\varepsilon_0>0:\exists m\in N\ \forall n\geq m\ \exists p\in N\ \exists x_n\in X:|f_{n+p}(x_n)-f(x_n)|\geq\varepsilon_0$$

bo'lsa, $\{f_n(x)\}$ ketma-ketlik X to'plamda tekis yaqinlashuvchi bo'lmaydi.

6.12- misol. $\{f_n(x)\} = \{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\}$ funksional ketma-ketlikning $X = [0;+\infty)$ to‘plamda tekis yaqinlashuvchiligini Koshi teoremasidan foydalanib ko‘rsating.

Yechilishi. $\{f_n(x)\} = \{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\}$ ketma-ketlik uchun $[0;+\infty)$ da (6.4) shartni bajarilishini ko‘rsatamiz:

$$\left|f_{n+p}(x) - f_n(x)\right| = \left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}} - \frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right| \le \left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}}\right| + \left|\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right|$$

$$\le \sup_{x\in X}\left|\frac{\cos\sqrt{(n+p)x}}{\sqrt{n+p+2x}}\right| + \sup_{x\in X}\left|\frac{\cos\sqrt{(nx}}{\sqrt{n+2x}}\right| = \frac{1}{\sqrt{n+p}} + \frac{1}{\sqrt{n}}.$$

Agar $\forall\varepsilon > 0$ songa ko‘ra natural m son $m = \left[\frac{4}{\varepsilon^2}\right] + 1$ deb olinsa, u holda barcha $n>m$ va barcha $p \in N$ lar uchun

$$\frac{1}{\sqrt{n+p}} + \frac{1}{\sqrt{n}} < \frac{2}{\sqrt{m}} < \varepsilon$$

bo‘ladi. Demak, $\forall\varepsilon > 0$ son olinganda ham shunday natural m son mavjudki, $\forall n > m, \forall p \in N$ va $\forall x \in [0;+\infty)$ lar uchun

$$\left|f_{n+p}(x) - f_n(x)\right| < \varepsilon$$

tengsizlik o‘rinli bo‘ladi. Bu esa $\left\{\frac{\cos\sqrt{nx}}{\sqrt{n+2x}}\right\}$ ketma-ketlikning, Koshi teoremasiga ko‘ra, X da tekis yaqinlashuvchi ekanligini ko‘rsatadi.

6.13- misol. $\{f_n(x)\} = \left\{\frac{\ln nx}{\sqrt[4]{nx}}\right\}$ funksional ketma-ketlikning X=(1;0) to‘plamda notekis yaqinlashuvchiligini ko‘rsating.

Yechilishi. Shunday $\varepsilon_0 > 0$ son topiladiki, barcha $k \in N$ nomer uchun $n = k, p = k = n, \bar{x} = \frac{1}{k} = \frac{1}{n}$ deb olamiz, u holda (6.6) shart bajariladi, ya’ni

$$\left|f_{n+p}(x)-f_n(x)\right|=\left|f_{2n}\left(\frac{1}{n}\right)-f_n\left(\frac{1}{n}\right)\right|=\left|\frac{\ln 2}{\sqrt[4]{2}}-\ln 1\right|=\frac{\ln 2}{\sqrt[4]{2}}=\varepsilon_0 .$$

Demak, berilgan $\left\{\frac{\ln nx}{\sqrt[4]{nx}}\right\}$ ketma-ketlik X to'plamda tekis yaqinlashuvchi bo'lmaydi.

6.2. Funksional qatorlarning tekis yaqinlashuvchiligi. Ushbu

$$\sum_{n=1}^{\infty} u_n(x)=u_1(x)+u_2(x)+\cdots+u_n(x)+\cdots \qquad (6.8)$$

funksional qator $X\left(X\in R\right)$ to'plamda yaqinlashuvchi va uning yig'indisi $S(x)$ bo'lsin, ya'ni

$$\lim_{n\to\infty} S_n(x)=S(x)=\sum_{n=1}^{\infty} u_n(x).$$

6.4- ta'rif. Agar (6.8) funksional qatorning $\{S_n(x)\}$ qismiy yig'indilari ketma-ketligi X to'plamda $S(x)$ ga *tekis yaqinlashsa*, (6.8) funksional qator X to'plamda $S(x)$ ga tekis yaqinlashadi deyiladi va u qisqacha

$$S_n(x)\overset{X}{\rightrightarrows} S(x) \qquad (6.9)$$

kabi belgilanadi.

6.1- eslatma. Funksional qatorlarning tekis yaqinlashuvchiligi (yaqinlashmovchiligi) tushunchasi ham ularning oddiy yaqinlashuvchiligi singari, funksional ketma-ketliklarning tekis yaqinlashuvchilik (yaqinlashmovchiligi) tushunchasi orqali kiritiladi.

6.4- ta'rifni qisqacha, kvantor belgisidan foydalanib, quyidagicha yozish mumkin:

$$\forall\varepsilon>0\exists m(\varepsilon):\forall n>m\forall x\in X\to\left|S_n(x)-S(x)\right|<\varepsilon. \qquad (6.10)$$

6.5- ta'rif. (6.8) qatorning dastlabki n ta hadini tashlab yuborgandan so'ng, hosil bo'lgan

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+\cdots=\sum_{k=n+1}^{\infty} u_k(x)$$

qator (6.8) funksional qatorning n ta hadidan keyingi qoldig'i deyiladi. Bunda

$$r_n(x)=S(x)-S_n(x)$$

bo‘ladi. U holda (6.9) shartni quyidagi ko‘rinishda ifodalash mumkin:

$$r_n(x)\overset{X}{\text{I}}\ 0. \qquad (6.11)$$

(6.9) va (6.11) shartlar teng kuchli.

6.6- ta’rif. Agar X to‘plamda $S_n(x)$ ketma-ketlikning limit funksiyasi mavjud bo‘lsa va (6.10) shart bajarilsa, ya’ni

$$\forall\varepsilon_0>0:\forall k\in N\ \exists n\geq k\ \forall x\in X\to|S_n(\bar{x})-S(\bar{x})|\geq\varepsilon_0$$

bo‘lsa, $S(x)$ ketma-ketlik to‘plamda $S(x)$ ga *notekis yaqinlashadi* deyiladi.

6.14- misol. $\sum_{n=0}^{\infty}(e^{-nx}-e^{-(n+1)x})$ funksional qatorni, ta’rifga ko‘ra,

$a)X=(0;+\infty)$; $b)X_1=[\delta;+\infty),\delta>0$ sohalarda tekis yaqinlashishga tekshiring.

Yechilishi. Berilgan qatorning n- qismiy yig‘indisini topamiz:

$$S_n(x)=(1-e^{-x})\cdot(1+e^{-x}+e^{-2x}+\cdots+e^{-(n-1)x})=$$

$$=(1-e^{-x})\frac{(1-e^{-nx})}{1-e^{-x}}=1-e^{-nx}.$$

Bundan $\lim\limits_{n\to\infty}S_n(x)=1$, $S(x)=1$.

a) Funksional qatorni ta’rifga ko‘ra tekis yaqinlashishga tekshirish uchun $r_n(x)=S(x)-S_n(x)$ ayirmani qaraymiz:

$$|r_n(x)|=|1+e^{-nx}-1|=e^{-nx},$$

$\forall\varepsilon>0$ son olganda $m=\left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right]+1$ $(x\neq0)$ deyilsa, u holda $n>m$ lar uchun

$$|r_n(x)|=|S_n(x)-S(x)|=e^{-nx}<\varepsilon \qquad (6.12)$$

tengsizlik bajariladi.

Agar x=0 bo‘lsa, ravshanki, $\forall n\in N$ lar uchun S_n(0)=0; S(0)=1 bo‘lib, $|r_n(0)|=|S_n(0)-S(0)|=1$ bo‘ladi. m natural son $\varepsilon>0$ va x $(0<x<+\infty)$ larga bog‘liq bo‘lib, u barcha x lar uchun umumiy bo‘la olmaydi, chunki $m=\left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right]+1$ ning $(0;+\infty)$ da x bo'yicha maksimumi chekli son bo'lmaydi, ya'ni son $\forall n\in N$ olsak ham $\exists\varepsilon_0>0\left(\varepsilon_0=\frac{1}{e^2}\right)$ va $\exists\ x_n=\frac{1}{n}\in(0,+\infty)$ nuqta topiladiki,

$$\left|S\left(\frac{1}{n}\right)-S_n\left(\frac{1}{n}\right)\right|=e^{-1}>\varepsilon_0$$

bo'ladi. Demak, berilgan funksional qator, 6.6- ta'rifga ko'ra, $X=(0;+\infty)$ sohada notekis yaqinlashuvchi bo'ladi.

b) $\forall\varepsilon>0$ son olinganda ham $m_1=\max\limits_{x\in[\delta;+\infty)}\left\{\left[\frac{1}{x}\ln\frac{1}{\varepsilon}\right]+1\right\}=\left[\frac{1}{\delta}\ln\frac{1}{\varepsilon}\right]+1$ deb olinsa, $\forall n>m_1$ va $\forall x\in[\delta;+\infty)$ lar uchun birdaniga (6.12) tengsizlik bajariladi. Shunday qilib, (6.10) shartga asosan, berilgan funksional qator $[\delta;+\infty)$ sohada x ga nisbatan tekis yaqinlashuvchi bo'ladi, ya'ni $S_n(x)\overset{[\delta;+\infty)}{\rightrightarrows}1$.

6.15- misol. Ushbu funksional qatorni $X=(0;+\infty)$ da tekis yaqinlashuvchilikka tekshiring:

$$\sum_{n=1}^{\infty}\frac{x}{\left[(n+1)x+1\right](nx+1)}.$$

Yechilishi. 1.2-teoremaga asosan, berilgan funksional qatorning n- qismiy yig'indisi

$$S_n(x)=\frac{1}{x+1}-\frac{1}{1+(n+1)x}$$

bo'ladi, uning yig'indisi esa

$$S(x)=\lim_{n\to\infty}\left(\frac{1}{1+x}-\frac{1}{1+(n+1)x}\right)=\frac{1}{1+x}.$$

Ta'rifga ko'ra, $\forall\varepsilon>0$ son olinganda $m=\left[\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)-1\right]\ (x\neq 0)$ deyilsa, barcha $n>m$ uchun

$$|S_n(x)-S(x)|=\left|\frac{1}{1+x}-\frac{1}{1+(n+1)x}-\frac{1}{1+x}\right|=\frac{1}{1+(n+1)x}\leq$$

$$\leq\frac{1}{(m+2)x+1}<\varepsilon$$

bo'ladi. Agar x=0 bo'lsa, ravshanki, $\forall n$ uchun $S_n(0)=0,\ S(0)=1$ bo'lib, $|S_n(0)-S(0)|=1$ kelib chiqadi. Bundagi m natural son $\varepsilon>0$ va x $(0<x<+\infty)$ nuqtalarga bog'liq bo'lib, u barcha x $(0<x<+\infty)$ lar uchun umumiy bo'la olmaydi (bu holda $m=\left[\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)-1\right]$ ning $(0;+\infty)$ da x bo'yicha maksimumi chekli son emas). Boshqacha qilib aytganda, istalgan n natural son olinganda ham $\exists\varepsilon_0>0$ (masalan $\varepsilon_0=\frac{1}{4}$) va $x_n=\frac{1}{n+1}\in(0;+\infty)$ nuqta topiladiki,

$$\left|S_n\left(\frac{1}{n+1}\right)-S\left(\frac{1}{n+1}\right)\right|=\frac{1}{2}>\varepsilon_0.$$

Demak, berilgan funksional qator, 6.6- ta'rifga asosan, notekis yaqinlashadi.

6.3- teorema. (6.8) funkional qatorning X da tekis yaqinlashishi uchun

$$\lim_{n\to\infty}\sup_{x\in X}|r_n(x)|=0 \qquad (6.13)$$

shartning bajarilishi zarur va yetarlidir.

6.4- teorema (zaruriy shart). Agar (6.8) funksional qator X da tekis yaqinlashuvchi bo'lsa, u holda uning umumiy hadi $u_n(x)$ $(n=1,2,...)$ $u_n(x)\overset{X}{\text{I}}\ 0$ bo'ladi.

6.16- misol. Ushbu funksional qatorni: *a)* $X=(-q;\,q)$, bunda $0<q<1$; *b)* $X=(-1;1)$ to'plamlarda tekis yaqinlashuvchilikka tekshiring:

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^n + \cdots.$$

Yechilishi. (A) formulaga asosan funksional qatorning n- qismiy yig'indisi

$$S_n(x) = 1 + x^2 + \cdots + x^n = \frac{1-x^n}{1-x}$$

ga teng bo'ladi. Endi $(-1;1)$ da berilgan funksional qatorning yig'indisini topamiz:

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x},$$

va $r_n(x) = S(x) - S_n(x) = \dfrac{x^n}{1-x} = |r_n(x)| \le \dfrac{|x|^n}{1-|x|}$.

a) $X = (-q;q)$ da berilgan funksional qatorni tekis yaqinlashuvchilikka tekshiramiz: $\sup\limits_{x\in X} |r_n(x)| = \sup\limits_{x\in X} \left|\dfrac{x^n}{1-x}\right| \le \dfrac{q^n}{1-q}$,

$$\lim_{n\to\infty} \sup_{x\in X} |r_n(x)| \le \lim_{n\to\infty} \frac{q^n}{1-q} = 0.$$

Demak, 6.3- teoremaga asosan, berilgan funksional qator $X = (-q;\, q)$ $(0 < q < 1)$ da tekis yaqinlashuvchi bo'ladi.

b) $X = (-1;1)$ da funksional qatorni tekis yaqinlashuvchilikka tekshiramiz:

$$\sup_{x\in X} |r_n(x)| = \sup_{x\in X} \left|\frac{x^n}{1-x}\right| = \lim_{x\to 1-0} \left|\frac{x^n}{1-x}\right| = +\infty.$$

Demak, $\lim\limits_{n\to\infty} \sup\limits_{x\in X} |r_n(x)| = 0$ shart bajarilmayapti, shuning uchun berilgan funksional qator $(-1;1)$ da notekis yaqinlashuvchi.

6.17- misol. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt[5]{n+x}}$ funksional qatorni $X = [0; +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. $X=[0;+\infty)$ da $\left\{\frac{(-1)^n}{\sqrt[5]{n+x}}\right\}$ ketma-ketlik barcha $x\in X$ lar uchun monoton kamayuvchi va nolga intiladi. Leybnis alomatiga asosan, berilgan funksional qator X da yaqinlashuvchi va $S(x)$ yig'indiga ega. Bizga ma'lumki, barcha $n\in N$ va berilgan $x\in X$ lar uchun ishorasi almashinuvchi qatorlarda

$$|r_n(x)|\le|u_{n+1}(x)|=\frac{1}{\sqrt[5]{n+x+1}}\le\frac{1}{\sqrt[5]{n+1}}$$

tengsizlik o'rinli bo'ladi. Bu yerdan (6.13) shartning bajarilishi kelib chiqadi.

Demak, berilgan funksional qator $[0;+\infty)$ da tekis yaqinlashuvchi.

6.5- teorema (funksional qatorning tekis yaqinlashishi uchun Koshi kriteriysi). (6.8) funksional qator X da tekis yaqinlashishi uchun $\forall\varepsilon>0$ son olinganda ham $\exists m(\varepsilon)(m\in N)$ nomer topilib, $\forall n\ge m(\varepsilon)$, barcha butun $p\ge 0$ sonlar va $\forall x\in X$ lar uchun

$$|S_{n+p}-S_n|=\left|\sum_{k=n+1}^{n+p}u_k(x)\right|<\varepsilon$$

shartning bajarilishi zarur va yetarli.

6.6- eslatma. Koshi kriteriysidan, ya'ni 6.5- teoremadan, xususiy holda, p=0 bo'lganda, 6.4- teorema kelib chiqadi.

Agar 6.5-teoremaning shartlari bajarilmasa, ya'ni

$$\exists\varepsilon_0>0\ \ m\in N\ \exists\ n\ge m\ \ \exists p\in N\ \exists x\in X\to\left|\sum_{k=n+1}^{n+p}u_k(x)\right|\ge\varepsilon_0 \qquad (6.14)$$

bo'lsa, (6.8) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

Xususiy holda, agar

$$\exists\varepsilon_0>0\ \exists n_0\in N\ \forall n\ge N_0\ \exists x_n\in X\to|u_n(x_n)|\ge\varepsilon_0 \qquad (6.15)$$

bajarilsa, u holda (6.8) funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

6.18- misol. $\sum_{n}^{\infty}\frac{\sin nx}{3^n}$ funksional qatorning Koshi kriteriysiga ko'ra, $X=[0,+\infty)$ da tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. $S_{n+p}-S_n$ ayirmani tuzamiz va uni baholaymiz:

$$\left|S_{n+p}-S_n\right| = \left|\frac{\sin(n+1)x}{3^{n+1}}+\frac{\sin(n+2)x}{3^{n+2}}+\ldots+\frac{\sin(n+p)x}{3^{n+p}}\right| \le$$

$$\le \frac{1}{3^{n+1}}+\frac{1}{3^{n+2}}+\ldots\frac{1}{3^{n+p}} = \frac{1}{3^{n+1}}\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{p-1}}\right) =$$

$$\frac{1}{3^{n+1}}\cdot\frac{1-\left(\frac{1}{3}\right)^p}{1-\frac{1}{3}} = \frac{1}{3^n\cdot 2}\left(1-\left(\frac{1}{3}\right)^p\right) < \frac{1}{3^n}.$$

$\forall\varepsilon>0$ son olinganda ham $m=[\log_3\frac{1}{\varepsilon}]+1$ deb olinsa, u holda $\forall x\in[0,+\infty)$, $\forall n>m$ va $\forall p$ lar uchun $\left|S_{n+p}-S_n\right|<\varepsilon$ tengsizlik bajariladi.

Demak, Koshi kriteriysiga asosan, berilgan funksional qator $X=[0,+\infty)$ da tekis yaqinlashuvchi bo‘ladi.

6.19- misol. $\sum_{n=1}^{\infty}2^n\sin\frac{1}{3^n x}$ funksional qatorni $X=[0,+\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Har bir $x\in X$ uchun $n\to\infty$ da quyidagiga ega bo‘lamiz:

$$u_n(x)=2^n\sin\frac{1}{3^n x}\sim\left(\frac{2}{3}\right)^n\frac{1}{x}.$$

Bu yerdan, taqqoslash alomatiga ko‘ra, berilgan funksional qatorning X da yaqinlashuvchi ekanligi kelib chiqadi. Berilgan funksional qatorni X da tekis yaqinlashishga tekshirish uchun Koshi teoremasini qo‘llaymiz. $\varepsilon_0=1, p=n$ va $\bar{x}=\frac{1}{3^n}$ bo‘lsin. U holda barcha $n>1$ lar uchun

$$\left|S_{n+p}(\bar{x})-S_n(\bar{x})\right| = \left|2^{n+1}\sin\frac{1}{3}+\ldots+2^{n+p}\sin\frac{1}{3^n}\right| > 2^{n+1}\sin\frac{1}{3} > \varepsilon_0$$

bo‘ladi. Demak, (6.14) shart bajarilayapti, shuning uchun berilgan funksional qator X da tekis yaqinlashuvchi bo‘lmaydi.

6.20- misol. $\sum_{n+1}^{\infty} \frac{\sqrt{n}}{x} e^{-\frac{\sqrt{n}}{x}}$ funksional qatorni $X = [0, +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Shunday $\varepsilon_0 = \frac{1}{3}$ son va $\exists m \in N$ topilib, barcha $n \geq m$ hamda $\exists\, x_n = \sqrt{n} \in X$ uchun

$$|u_n(x_n)| = \frac{\sqrt{n}}{x} e^{-\frac{\sqrt{n}}{x}} = e^{-1} > \varepsilon_0$$

bo'ladi. Demak, (6.15) shart bajarilayapti, shuning uchun berilgan funksional qator X da tekis yaqinlashuvchi bo'lmaydi.

6.21- misol. $\sum_{n=1}^{\infty} n^3 x^2 e^{-n^2 x}$ funksional qatorni $X = [0, +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Bizga ma'lumki, barcha $t > 0$ lar uchun $e^t > \frac{t^3}{3!}$ tengsizlik o'rinli. Agar $x>0$ bo'lsa, u holda

$$0 < u_n(x) < n^3 x^2 \frac{3!}{(n^2 x)^3} = \frac{6}{n^2 x}.$$

Taqqoslash alomatiga asosan, berilgan funksional qator $[0, +\infty)$ da yaqinlashuvchi bo'ladi. Endi X da (6.14) shartning bajarilishini ko'rsatamiz. Har qanday $m \in N$ uchun $n = m$, $p = n$, $\bar{x} = \frac{1}{n^2} \in X$ va $\varepsilon_0 = e^{-4}$ deb olamiz. U holda

$$\sum_{k=n+1}^{2n} u_k(\bar{x}) = \sum_{k=n+1}^{2n} k^3 \bar{x}^2 e^{-k^2 \bar{x}} > n^3 \frac{1}{n^4} e^{-4n^2 \cdot \frac{1}{n^2}} \cdot n = e^{-4} = \varepsilon_0$$

bo'ladi. Demak, berilgan funksional qator X da notekis yaqinlashuvchi.

6.6- teorema (Veyershtrass alomati). Agar (6.8) funksional qatorning har bir hadi X da aniqlangan bo'lib, $\forall x \in X$ va $\forall n > n_0$ uchun $|u_n(x)| \leq c_n$ tengsizlikni qanoatlantirsa va

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + ... + c_n + ...$$

sonli qator yaqinlashuvchi bo'lsa, u holda (6.8) funksional qator X da absolut va tekis yaqinlashuvchi bo'ladi.

Natija. Agar $\sum_{n=1}^{\infty} a_n$ bunda $a_n = \sup_{x \in R} |u_n(x)|$, sonli qator yaqinlashuvchi bo'lsa, (6,8) funksional qator tekis yaqinlashuvchi bo'ladi.

6.22- misol. $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ funksional qatorni $X = [0; +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Berilgan funksional qator har bir belgilangan $x \in X$ uchun Leybnis alomatiga ko'ra, yaqinlashuvchi bo'ladi va $S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ yig'indiga ega.

Endi, Veyershtrass alomatiga ko'ra, berilgan qatorni tekis yaqinlashuvchilikka tekshiramiz. Bu funksional qator uchun X da majorantlovchi yaqinlashuvchi sonli qator mavjud emas. Haqiqatan ham,

$$\max_{x \in X} |u_n(x)| = \max_{x \in X} \left| \frac{(-1)^n}{x+n} \right| \le \frac{1}{n} = c_n,$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ sonli qator uzoqlashuvchi. Berilgan funksional qatorni, tekis yaqinlashish ta'rifiga ko'ra, tekis yaqinlashuvchilikka tekshiramiz. U holda $\forall \varepsilon > 0$ uchun $\exists m(\varepsilon)$ nomer topilib, barcha $n > m(\varepsilon)$ va $\forall x \in [0; +\infty)$ lar uchun

$$|S(x) - S_n(x)| = \left| \sum_{k=n}^{\infty} \frac{(-1)^k}{x+k} \right| < \frac{1}{x+n} \le \frac{1}{n} < \varepsilon$$

tengsizlik bajariladi. Bunda $m(\varepsilon) = \left[\frac{1}{\varepsilon} \right] + 1$ deb olinsa, berilgan funksional qator $[0; +\infty)$ da tekis yaqinlashadi.

6.7- eslatma. 6.21- misoldan ko'rinib turibdiki, Veyershtrass alomati funksional qatorning tekis yaqinlashishi uchun faqat yetarli shart bo'lib, lekin zaruriy shart bo'la olmaydi.

6.23- misol. $\sum_{n=1}^{\infty} \frac{1}{x^4 + n^4}$ funksional qatorni $X = (-\infty; +\infty)$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Bu funksional qatorning umumiy hadi $u_n(x) = \frac{1}{x^4 + n^4}$ funksiyadan iborat. X dagi barcha x va $n \in N$ lar uchun

$$|u_n(x)| = \left| \frac{1}{x^4 + n^4} \right| \le \frac{1}{n^4}$$

tengsizlik o'rinli bo'ladi. Bunda $\sum_{n=1}^{\infty} \frac{1}{n^4}$ sonli qator yaqinlashuvchi. Demak, Veyershtrass alomatiga ko'ra, berilgan funksional qator X da absolut va tekis yaqinlashuvchi bo'ladi.

6.24- misol. $\sum_{n=1}^{\infty} \frac{n^2 x}{n^4 + x^2} \operatorname{arctg} \frac{x}{n^2}$ funksional qatorni X=[-1;1] da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Ma'lumki, $\forall t \in R$ lar uchun $|\operatorname{arctg} t| \le |t|$ bo'ladi. $\forall x \in X$ uchun $n^4 + x^2 > n^4$ o'rinli ekanligini e'tiborga olib,

$$|u_n(x)| = \left| \frac{n^2 x}{n^4 + x^2} \operatorname{arctg} \frac{x}{n^2} \right| \le \frac{|n^2 x|}{n^4 + x^2} \frac{|x|}{n^2} \le \frac{1}{n^4}$$

tengsizlikni hosil qilamiz. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ sonli qator yaqinlashuvchi bo'lganligidan, Veyershtrass alomatiga asosan, berilgan funksional qator X da absolut va tekis yaqinlashuvchi bo'ladi.

6.25- misol. $\sum_{n=1}^{\infty} \frac{x}{1 + x^2 n^4}$ funksional qatorni X=[0; +∞) da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Berilgan funksional qatorning umumiy hadi $u_n(x) = \frac{x}{1 + x^2 n^4}$ funksiyadan iborat. Bu funksiyani X da ekstremumga tekshiramiz. $u_n(x)$ funksiyaning hosilasi

$$u_n'(x)=\frac{1-n^4x^2}{(1+x^2n^4)^2}\cdot u_n(x)$$

$x=\pm\frac{1}{n^2}$ nuqtada nolga aylanadi va $x=-\frac{1}{n^2}\notin X$, $x=\frac{1}{n^2}\in X$ bo'ladi. Bu $x=\frac{1}{n^2}$ nuqtada

$$u_n''(x_n)=u_2''\left(\frac{1}{n^2}\right)<0$$

bo'ladi. Demak, $u_n(x)$ funksiya $x=\frac{1}{n^2}\in X$ nuqtada maksimumga erishadi. Uning maksimum qiymati esa $\frac{1}{2n^2}$ ga teng, ya'ni

$$\sup_{x\in X}|u_n(x)|=\max_{x\in X}|u_n(x)|=u_n(x)=\frac{1}{2n^2}=a_n$$

bo'ladi va $\sum_{n=1}^{\infty}\frac{1}{2n^2}$ qator yaqinlashuvchi. Demak, Veyershtrass alomatiga asosan, berilgan funksional qator X da absolut va tekis yaqinlashuvchi.

Quyidagi funksional qatorni qaraymiz:

$$\sum_{n=1}^{\infty}a_n(x)b_n(x)=a_1(x)b_1(x)+\ldots+a_n(x)b_n(x)+\cdots, \qquad (6.\ 15)$$

bunda $a_n(x)$, $b_n(x),(n=1,2,\ldots)$ funksiyalar $X\,(X\subset R)$ da aniqlangan.

Dirixle alomati. Agar: 1) $\{a_n(x)\}$ funksional ketma-ketlik $\forall x\in X$ lar va $\forall n\in N$ lar uchun monoton bo'lib, $a_{n+1}(x)\le a_n(x)$ yoki $a_{n+1}(x)\ge a_n(x)$ va $a_n(x)\overset{X}{\rightrightarrows} 0$ bo'lsa;

2) $\sum_{n=1}^{\infty}b_n(x)$ qatorning qismiy yig'indilari ketma-ketligi

$B_n(x)=\sum_{k=1}^{n}b_k(x)$ $\forall n\in N,\forall x\in X$ larda chegaralangan bo'lsa, ya'ni $\exists M>0$: $\forall n\in N$, $\forall x\in X$ uchun

$$|B_n(x)| \leq M$$

bo‘lsa, u holda (6. 15) qator to‘plamda tekis yaqinlashuvchi bo‘ladi.

Abel alomati. Agar: 1) $\sum_{n=1}^{\infty} b_n(x)$ qator X to‘plamda tekis yaqinlashuvchi bo‘lsa, ya’ni $B_n(x) \overset{X}{\rightrightarrows} B(x)$;

2) $\{a_n(x)\}$ ketma-ketlik X to‘plamda monoton bo‘lsa, ya’ni $\forall n \in N$, $\forall x \in X$ uchun $a_{n+1}(x) \leq a_n(x)$ yoki $a_{n+1}(x) \geq a_n(x)$ va $\exists M > 0: \forall n \in N$, $\forall x \in X$ lar uchun $|a_n(x)| \leq M$ bo‘lsa, u holda (6. 15) funksional qator X da yaqinlashuvchi bo‘ladi.

6.8- eslatma. Dirixle alomatidan xususiy holda Abel alomati kelib chiqadi.

6.26- misol. $\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\alpha}}$ funksional qatorni $X = [\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$ da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Agar $\alpha > 1$ bo‘lsa, Veyershtrass alomatiga ko‘ra, berilgan funksional qator X da tekis yaqinlashuvchi bo‘ladi.

Haqiqatan ham, $\forall x \in X$ va $\forall n \in N$ lar uchun

$$|\cos nx| \leq 1, \quad |u_n(x)| = \left|\frac{\cos nx}{n^{\alpha}}\right| \leq \frac{1}{n^{\alpha}}$$

bo‘lib, $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ sonli qator yaqinlashuvchi bo‘ladi.

Agar $0 < \alpha < 1$ bo‘lsa, $\{a_n\} = \left\{\frac{1}{n^{\alpha}}\right\}$ ketma-ketlik monoton kamayuvchi va $a_n \overset{X}{\rightrightarrows} 0$. Ushbu $\sum_{n=1}^{\infty} \cos nx$ funksional qatorning qismiy yig‘indilari ketma-ketligi uchun

$$\{B_n(x)\} = \left\{\sum_{k=1}^{n} \cos kx\right\} = \left\{\frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right\}, \quad x \neq 2\pi m, \ m \in Z$$

formula o‘rinli. Bundan $\forall x \in X$ va $\forall n \in N$ lar uchun

$$\left|\sum_{k=1}^{n}\cos kx\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\sin\frac{\varepsilon}{2}}, \ x \ne 2\pi m, \ m \in Z$$

tengsizlik o‘rinli bo‘ladi. Demak, Dirixle alomatiga ko‘ra, berilgan qator X da tekis yaqinlashuvchi bo‘ladi.

Agar $\alpha \le 0$ bo‘lsa, u holda berilgan funksional qator X da uzoqlashuvchi bo‘ladi. Haqiqatan ham, har bir berilgan $x \in X$ uchun qator yaqinlashuvchiligining zaruriy sharti bajarilmaydi, ya’ni $u_n(x)$ X to‘plamda 0 ga tekis intilmaydi: $\lim\limits_{n\to\infty} u_n(x) \overset{X}{\not\rightrightarrows} 0.$

6.27- misol. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3}+\sqrt{x}}(1+\frac{x}{n})^n$ funksional qatorni X=[0; 1] da tekis yaqinlashuvchilikka tekshiring.

Yechilishi. Quyidagi belgilashlarni kiritamiz:

$$b_n(x) = \frac{(-1)^n}{\sqrt{n^3}+\sqrt{x}}, \quad a_n(x) = (1+\frac{x}{n})^n.$$

Veyershtrass alomatiga ko‘ra, $\sum\limits_{n=1}^{\infty} b_n(x)$ qator X da tekis yaqinlashuvchi, ya’ni $\forall x \in X$ va $\forall n \in N$ lar uchun $|b_n(x)| \le \frac{1}{n^{\frac{3}{2}}}$ tengsizlik o‘rinli va $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ sonli qator yaqinlashuvchi.

$\{a_n(x)\} = \left\{(1+\frac{x}{n})^n\right\}$ ketma-ketlik [0;1] da chegaralangan,

$$\left(1+\frac{x}{n}\right)^n \le (1+\frac{1}{n})^n < e$$

va $\varphi(t) = \left(1+\frac{x}{t}\right)^t$ funksiya barcha $t \ge 1$ va $\forall x \in X$ lar uchun o‘suvchi funksiyadir. Demak, berilgan funksional qator, Abel alomatiga ko‘ra, [0;1] da tekis yaqinlashuvchi bo‘ladi.

Mustaqil yechish uchun misollar

X da $\{f_n(x)\}$ funksional ketma-ketlikning tekis yaqinlashuvchiligini isbotlang:

6.1. $f_n(x)=e^{-nx^2}$, $X=[1;+\infty)$.

6.2. $f_n(x)=\dfrac{x^n}{1+x^n}$, $X=[0;1-\varepsilon], 0<\varepsilon<1$.

6.3. $f_n(x)=x^{4n}$, $X=[0;\varepsilon]$, $0<\varepsilon<1$

6.4. $f_n(x)=\ln(1+\dfrac{\cos nx}{\sqrt{n+x}})$, $X=[0;+\infty)$.

6.5. $f_n(x)=\sqrt{x^2+\dfrac{1}{n}}$, $X=(-\infty;+\infty)$.

6.6. $f_n(x)=e^{-(x-n)^2}$, $X=[-4;4]$.

6.7. $f_n(x)=n^{\frac{3}{4}}xe^{-\sqrt{nx}}$, $X=[0;+\infty)$.

6.8. $f_n(x)=\sqrt{n}\sin\dfrac{x}{n\sqrt{n}}$, $X=(-\infty;+\infty)$.

6.9. $f_n(x)=\dfrac{1}{x+n}$, $X=(0;+\infty)$.

6. 10. $f_n(x)=\dfrac{nx}{1+n+x}$, $X=[0;1]$.

6.11. $f_n(x)=\sqrt[n]{1+x^n}$, $X=[0;2]$.

6.12. $f_n(x)=n\sin\dfrac{1}{nx}$, $X=[1;+\infty)$.

6.13. $f_n(x)=\dfrac{n^2}{n^2+x^2}$, $X=[-1;1]$.

6.14 . $f_n(x)=\dfrac{n}{x^2+n^2}\operatorname{arctg}\sqrt{nx}$, $X=[0;+\infty)$.

6.15. $f_n(x)=x\sqrt{n}e^{-nx^2}$, $X=[\varepsilon;+\infty)$, $\varepsilon>0$.

X da $\{f_n(x)\}$ funksional ketma-ketlikni tekis hamda notekis yaqinlashuvchilikka tekshiring:

6.16. $f_n(x)=\frac{x^n}{1+x^n}$ $a) X=[1-\varepsilon;1+\varepsilon]$; $b) X=[1+\varepsilon;+\infty)$, $\varepsilon>0$.

6.17. $f_n(x)=e^{-(x-n)^2}$, $X=(-\infty;+\infty)$.

6.18. $f_n(x)=\frac{\ln nx}{nx^2}$, $X=[1;+\infty)$.

6.19. $f_n(x)=x^n+x^{2n}-2x^{3n}$, $X=[0;1]$.

6.20. $f_n(x)=\sin\frac{x}{n^a}$, $a>0$, $X=R$.

6.21. $f_n(x)=\frac{4n\sqrt{nx}}{3+4n^2x}$, $X=[\varepsilon;+\infty)$, $\varepsilon>0$.

6.22. $f_n(x)=nx(1-x)^n$, $X=[0;1]$.

6.23. $f_n(x)=\cos\left(\frac{\pi}{2}x^n\right)$, $X=(0;\frac{1}{2})$.

6.24. $f_n(x)=n(\sqrt[n]{x}-1)$, $X=[1;a]$, $1<a<+\infty$.

6.25 Agar $f(x)$ funksiya $[a;b]$ da aniqlangan ixtiyoriy funksiya bo‘lsa, $f_n(x)=\frac{[nf(x)]}{n}$ $(n=1,2,...)$ funksional ketma-ketlikning $(a;b)$ da $f(x)$ ga tekis yaqinlashishini isbotlang, bunda $[nf(x)]$ — ushbu $nf(x)$ ifodaning butun qismi.

6.26. Agar $f(x)$ funksiya [0;1] da aniqlangan va uzluksiz bo‘lsa,

$$f(x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)C_n^k x^k(1-x)^{n-k}$$

funksional ketma-ketlikning [0;1] da $f(x)$ ga tekis yaqinlashishini isbotlang.

6.27. Agar [0;1] da $f_0(x)=0$ bo‘lsa,

$$f_n(x)=\sqrt{xf_{n-1}(x)} \quad (n=1,2,...)$$

funksional ketma-ketlikning [0;1] da tekis yaqinlashuvchiligini isbotlang.

6.28. Agar $f(x)$ funksiya $(a;b)$ da uzluksiz $f'(x)$ hosilaga ega bo‘lsa,

$$f(x)=n\left[f\left(x+\frac{1}{n}\right)-f(x)\right]$$

funksional ketma-ketlikning $[\alpha;\beta]([\alpha;\beta]\subset(a;b)$ da $f'(x)$ funksiyaga tekis yaqinlashishini isbotlang.

6.29. $X=[0;1]$ da

$$f_n(x)=\begin{cases} n^2x, & 0\le x\le\frac{1}{n};\\ n^2(\frac{2}{n}-x)x, & \frac{1}{n}<x<\frac{2}{n};\\ 0, & x\ge\frac{2}{n};\end{cases}$$

funksional ketma-ketlikni tekis yaqinlashishga tekshiring.

6.30. $f_n(x)=\frac{\ln nx}{\sqrt{nx}}$ ketma-ketlikning $X=(0,1)$ da notekis yaqinlashuvchiligini isbotlang.

X_1 va X_2 to'plamlarda $\{f_n(x)\}$ funksional ketma-ketlikni tekis hamda notekis yaqinlashuvchilikka tekshiring:

6.31. $f_n(x)=e^{-(x-n)^2}$, $X_1=(-4;4)$ $X_2=(-\infty;+\infty)$.

6.32. $f_n(x)=\frac{nx^2}{n^3+x^3}$, $X_1=[0;\ 1]$, $X_2=[0;+\infty)$.

6.33. $f_n(x)=n\operatorname{arctg}\frac{1}{nx}$, $X_1=(0;2)$, $X_2=(2;+\infty)$.

6.34. $f_n(x)=\sqrt{n}\sin\frac{x}{\sqrt{n}}$, $X_1=[0;\pi]$, $X_2=[\pi;+\infty)$.

6.35. $f_n(x)=\frac{2nx}{1+n^2x^2}$, $X_1=[0;1]$, $X_2=(1;+\infty)$.

6.36. $f_n(x)=\frac{1}{x^2}\sqrt{1+\frac{x}{n}}$, $X_1=(0;1)$, $X_2=(1;+\infty)$.

6.37. $f_n(x)=n^2x^2e^{-nx}$, $X_1=[0;+\infty)$, $X_2=[\varepsilon;+\infty)$, $\varepsilon>0$.

6.38. $f_n(x)=\ln\left(x^2+\frac{1}{n}\right)$, $X_1=(0;+\infty)$, $X_2=(a;+\infty)$, $a>0$.

6.39. $f_n(x)=\operatorname{arctg}\frac{1-x^n}{1+x^n}$, $X_1=(0;\frac{1}{2})$, $X_2=(\frac{1}{2};1)$.

6.40. $f_n(x)=\frac{n^2x^3+nx+1}{n^2x^2+2}$, $X_1=(0;1)$, $X_2=(1;+\infty)$.

6.41. $f_n(x)=\ln(1+\sin\frac{x\sqrt{n}}{x^2+n})$, $X_1=(0;1)$, $X_2=(1;+\infty)$.

6.42. $f_n(x)=\left(1-\frac{x}{n}\right)^n$, $X_1=[-a;a]$, $a>0$, $X_2=(-\infty;+\infty)$.

6.43. $f_n(x)=\sqrt[n]{1+x^n}$, $X_1=[0;2]$, $X_2=[2;+\infty)$.

6.44. $f_n(x)=\sin\frac{x}{e^{-n}+e^nx^2}$, $X_1=(0;1)$, $X_2=(1;+\infty)$.

6.45. $f_n(x)=n(\sqrt[n]{x}-\sqrt[2n]{x})$, $X_1=(\frac{1}{2};1)$, $X_2=(1;+\infty)$.

Tekis yaqinlashish ta'rifidan foydalanib, X da berilgan funksional qatorni tekis yaqinlashuvchi ekanligini ko'rsating:

6.46. $\sum_{n=1}^{\infty}x^n$, $X=[-\frac{1}{3};\frac{1}{3}]$.

6.47. $\sum_{n=1}^{\infty}\left(\frac{x^{n-1}}{n}-\frac{x^n}{n+1}\right)$, $X=[-1;1]$.

6.48. $\sum_{n=1}^{\infty}(\frac{\sin nx}{\sqrt{n}}-\frac{\sin(n+1)x}{\sqrt{n+1}})$, $X=(-\infty;+\infty)$.

6.49. $\sum_{n=1}^{\infty}\left(\frac{1}{x^2+n\sqrt{n}}\right)$, $X=(-\infty;+\infty)$.

6.50. $\sum_{n=1}^{\infty}\frac{x}{3^n\sqrt{1+nx^2}}$, $X=[0;\ 2]$.

Funksional qatorlarning ko'rsatilgan oraliqlarda tekis yaqinlashuvchiligini, Veyershtrass alomatidan foydalanib, isbotlang:

6.51. $\sum_{n=1}^{\infty}\frac{1}{(n+x)^4}, \quad X=[0;+\infty)$.

6.52. $\sum_{n=1}^{\infty}\frac{x^4}{2+\sqrt[3]{n^4x^4}}, \quad X=(-\infty;+\infty)$.

6.53. $\sum_{n=1}^{\infty}\frac{\cos^2 2nx}{\sqrt[4]{n^5+x^4}}, \quad X=(-\infty;+\infty)$.

6.54. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^4+\ln^2 x}}, \quad X=(1;+\infty)$.

6.55. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^3+e^{-3x}}}, \quad X=[-3;3)$.

6.56. $\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{3^n+\cos x}, \quad X=(-\infty;+\infty)$.

6.57. $\sum_{n=1}^{\infty}\frac{\cos 3nx}{\sqrt{n^3+x^3}}, \quad X=[0;+\infty)$.

6.58. $\sum_{n=1}^{\infty}\frac{x}{4+n^4x^3}, \quad X=[0;+\infty)$.

6.59. $\sum_{n=1}^{\infty}\frac{x}{6+n^5x^2}, \quad X=[0;+\infty)$.

6.60. $\sum_{n=1}^{\infty}\ln\left(1+\frac{x\ln^3 n}{n^3}\right), \quad X=[0;100]$.

6.61. $\sum_{n=1}^{\infty}\sqrt[5]{x^n}e^{-\sqrt{nx}}, \quad X=[0;+\infty)$.

6.62. $\sum_{n=1}^{\infty}\frac{\cos nx}{n\sqrt{\ln^3 n+x^4}}, \quad X=(-\infty;+\infty)$.

6.63. $\sum_{n=1}^{\infty} n^4 \sin\frac{1}{n^4+4^n+4x}$, $X=[0;+\infty)$.

6.64. $\sum_{n=1}^{\infty} \frac{1}{3\sqrt[n]{1+(2n-1)x}}$, $X=[0;+\infty)$.

6.65. $\sum_{n=1}^{\infty} \frac{\sqrt{1-x^{2n}}}{2^n}$, $X=[-1;1]$.

6.66. $\sum_{n=1}^{\infty} \frac{\sin\sqrt{nx}}{n\sqrt[3]{\ln^4 n+x^2}}$, $X=(-\infty;+\infty)$.

6.67. $\sum_{n=1}^{\infty} 5^{2n}\sin^2\frac{1}{7^n x}$, $X=[a;+\infty),\ a>0$.

6.68. $\sum_{n=1}^{\infty} \frac{x}{n(7+n^7x^2)}$, $X=[0;+\infty)$.

6.69. $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+2^n}$, $X=(-2;+\infty)$.

6.70. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}(x^n-x^{-n}),\ \frac{1}{2}\le|x|\le 2$.

6.71. $\sum_{n=1}^{\infty} \frac{x^n}{[\frac{n}{2}]!}$, $X=(-a;a),\ a>0$.

6.72. $\sum_{n=1}^{\infty} x^2e^{-nx}$, $X=[0;+\infty)$.

6.73. $\sum_{n=1}^{\infty} \operatorname{arctg}\frac{2x}{x^2+n^3}$, $X=(-\infty;+\infty)$.

6.74. $\sum_{n=1}^{\infty} \frac{nx}{n^5x^2+1}$, $X=(-\infty;+\infty)$.

6.75. $\sum_{n=1}^{\infty} \frac{n\ln(1+nx)}{x^n}$, $1+\alpha\le x<+\infty,\ \alpha>0.$

X to‘plamda quyidagi funksional qatorlarning tekis yoki notekis yaqinlashuvchiligini aniqlang:

6.76. $\sum_{n=1}^{\infty} 3^n \sin\frac{1}{4^n x}, \quad X=(0;+\infty)$.

6.77. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+\sin x}, \quad X=[0;2\pi]$.

6.78. $\sum_{n=1}^{\infty} \left(\operatorname{arctg}\frac{x}{x^2+n^2}\right)^2, \quad X=[0;+\infty)$

6.79. $\sum_{n=1}^{\infty} \ln^2\left(1+\frac{x}{1+n^2x^2}\right), \quad X=[0;+\infty)$.

6.80. $\sum_{n=1}^{\infty} \frac{\cos\frac{2\pi n}{3}}{\sqrt{n^2+x^2}}, \quad X=(-\infty;+\infty)$

6.81. $\sum_{n=1}^{\infty} \frac{\sin x\cdot\sin nx}{\sqrt{n+x}}, \quad X=[0;+\infty)$.

6.82. $\sum_{n=1}^{\infty} \frac{x^2}{(1+nx)^4}, \quad X=[0;+\infty)$

6.83. $\sum_{n=1}^{\infty} e^{-nx}, \quad X=(0;+\infty)$.

6.84. $\sum_{n=1}^{\infty} \frac{\sin^2 x}{x+x^2n^3}, \quad X=(0;+\infty)$.

6.85. $\sum_{n=1}^{\infty} \left(\frac{x\sin x}{1+nx^3}\right)^2, \quad X=[0;+\infty)$.

6.86. $\sum_{n=1}^{\infty} e^{-n^2x^2}\sin nx, \quad X=[0;1]$.

6.87. $\sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad X=(0;+\infty)$.

6.88. $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+\sqrt{n}}$, $X=[0;+\infty)$.

6.89. $\sum_{n=1}^{\infty} (-1)^n n^{-x}$, $X=[\varepsilon;+\infty)$, $\varepsilon>0$.

6.90. $\sum_{n=1}^{\infty} \frac{x-1}{x^n}$, $X=[1;+\infty)$.

6.91. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \operatorname{arctg} x^n$, $X=[1;+\infty)$.

6.92. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, $X=[0;1]$.

6.93. $\sum_{n=1}^{\infty} \frac{n\sqrt{x}}{1+n^3x^3}$, $X=[0;+\infty)$.

6.94. $\sum_{n=1}^{\infty} x^n \sin\frac{1}{4^n}$, $X=(-4;4)$.

6.95. $\sum_{n=1}^{\infty} \frac{n}{x^2} e^{-\frac{n^2}{x}}$, $X=(0;+\infty)$.

Misollarning javoblari

6.16. a) $f(x)=0$ ga tekis yaqinlashadi. b) $f(x)=1$ ga tekis yaqinlashadi.

6.17. $f(x)=0$ ga notekis yaqinlashadi. **6.18.** $f(x)=0$ ga tekis yaqinlashadi.

6.19. $f(x)=0$ ga notekis yaqinlashadi. **6.20.** $f(x)=0$ ga notekis yaqinlashadi. **6.21.** $f(x)=0$ ga tekis yaqinlashadi. **6.22.** $f(x)=0$ ga notekis yaqinlashadi. **6.23.** $f(x)=1$ ga tekis yaqinlashadi. **6.24.** $f(x)=\ln x$ ga tekis yaqinlashadi. **6.29.** Notekis yaqinlashadi. **6.31.** Funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.32.** $f(x)=0$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi.

6.33. $f(x)=\frac{1}{x}$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.34.** $f(x)= x$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.35.** $f(x)=0$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.36.** $f(x)=\frac{1}{x^2}$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.37.** $f(x)= 0$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.38.** $f(x)=2\ln x$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.39.** $f(x)=\frac{\pi}{4}$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.40.** $f(x)= x$ funksiyaga X_2 da tekis yaqinlashadi, X_1 da esa notekis yaqinlashadi. **6.41.** $f(x)=0$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.42.** $f(x)=e^x$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.43.** $f(x)=\begin{cases}1, & x\in[0;1);\\ x, & x\in[1;\infty)\end{cases}$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa, notekis yaqinlashadi. **6.44.** $f(x)=0$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa, notekis yaqinlashadi. **6.45.** $f(x)=\frac{\ln x}{2}$ funksiyaga X_1 da tekis yaqinlashadi, X_2 da esa notekis yaqinlashadi. **6.76.** Notekis yaqinlashadi. **6.77.** Tekis yaqinlashadi. **6.78.** Tekis yaqinlashadi. **6.79.** Tekis yaqinlashadi. **6.80.**Tekis yaqinlashadi. **6.81.** Tekis yaqinlashadi. **6.82.** Tekis yaqinlashadi. **6.83.** Notekis yaqinlashadi. **6.84.** Tekis yaqinlashadi. **6.85.** Tekis yaqinlashadi. **6.86.** Notekis yaqinlashadi. **6.87.** Notekis yaqinlashadi. **6.90.** Notekis yaqinlashadi. **6.91.** Tekis yaqinlashadi. **6.92.** Tekis yaqinlashadi. **6.93.** Notekis yaqinlashadi. **6.94.** Notekis yaqinlashadi. **6.95.** Notekis yaqinlashadi.

7- §. Funksional qatorlar yig'indisining va funksional ketma-ketlik limit funksiyasining funksional xossalari

7.1. Funksional qator yig'indisining uzluksizligi. X to'plamda yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \qquad (7.1)$$

funksional qator berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin.

7.1- teorema. Agar (7.1) qatorning har bir $u_n(x)$ $(n=1,2,...)$ hadi X to'plamda uzluksiz bo'lib, qator X da tekis yaqinlashuvchi bo'lsa, u holda qatorning yig'indisi $S(x)$ ham X to'plamda uzluksiz bo'ladi.

7.1- eslatma. 7.1- teoremadagi (7.1) qatorning X da tekis yaqinlashuvchilik sharti funksional qator yig'indisi $S(x)$ning uzluksiz bo'lishi uchun yetarli shart bo'ladi, lekin zaruriy shart bo'la olmaydi.

7.1- misol. Ushbu

$$\sum_{n=1}^{\infty} [nxe^{-nx} - (n-1)xe^{-(n-1)x}]$$

funksional qatorning $X=[0;1]$ da notekis yaqinlashishini, lekin uning yig'indisi X da uzluksiz funksiya bo'lishini ko'rsating.

Yechilishi. Berilgan funksional qatorning n- qismiy yig'indisini topamiz:

$$S_n(x) = \sum_{k=1}^{n} [kxe^{-kx} - (k-1)xe^{-(k-1)x}] = xe^{-x} +$$
$$+2xe^{-2x} - xe^{-x} + 3xe^{-3x} - 2xe^{-2x} + \ldots +$$
$$+nxe^{-nx} - (n-1)xe^{-(n-1)x} = nxe^{-nx}.$$

Endi $n \to \infty$ da $S_n(x)$ funksiyaning limitini hisoblaymiz:

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx}} = 0 \ .$$

Demak, berilgan funksional qatorning yig'indisi $S(x)=0$ *qaralayotgan X to'plam* da uzluksiz funksiya ekan. Ammo berilgan qator X da tekis yaqinlashuvchi emas. Haqiqatan ham,

$$\sup_{x\in[0;1]} |S_n(X) - S(x)| = \sup_{x\in[0;1]} |nxe^{-nx}| = \frac{1}{e} \Rightarrow$$
$$\Rightarrow \lim_{n\to\infty} \sup_{x\in[0;1]} |r_n(x)| = e^{-1} \neq 0.$$

Shuning uchun, qator X da o'zining $S(x)$ yig'indisiga tekis yaqinlashmaydi.

7.2- misol. Ushbu

$$\sum_{n=1}^{\infty} x^2 e^{-nx}$$

funksional qatorning yig'indisini $X=[0;1]$ da uzluksizlikka tekshiring.

Yechilishi. Berilgan funksional qatorning har bir $u_n(x)=x^2e^{-nx}$ $(n=1,2...)$ hadi X da uzluksiz bo'lib, bu funksional qator X da, Veyershtrass alomatiga ko'ra, tekis yaqinlashuvchi bo'ladi. Haqiqatan ham, $\forall n\in N$ va $\forall x\in[0;1]$ lar uchun

$$|u_n(x)|=|x^2e^{-xe}|\le\frac{1}{e^n}=a_n$$

va $\sum_{n=1}^{\infty}\frac{1}{e^n}$ sonli qator yaqinlashuvchi. Haqiqatan ham, $S(x)=\frac{x^2}{e^x-1}, x>0$. $x=0$ bo'lganda $S(0)=0$. Demak, 7.1-teoremaga asosan, berilgan funksional qatorning yig'indisi X da uzluksiz funksiya bo'ladi.

7.3- misol. $f(x)=\sum_{n=1}^{\infty}\left(x+\frac{1}{n}\right)^n\frac{1}{2^n}$ funksiyaning aniqlanish sohasini toping va uzluksizlikka tekshiring.

Yechilishi. Koshi alomatiga asosan funksional qatorning yaqinlashish sohasini topamiz:

$$u_n(x)=\left(\frac{1}{2}\right)^n\cdot(x+\frac{1}{n})^n,$$

$$K_n^*=\sqrt[n]{\left(\frac{1}{2}\right)^n\left|(x+\frac{1}{n})^n\right|}=\left|x+\frac{1}{n}\right|\frac{1}{2},\ \lim_{n\to+\infty}K_n^*=\frac{|x|}{2}.$$

a) $|x|<2$ agar bo'lsa, funksional qator yaqinlashuvchi bo'ladi;

b) $|x|\ge 2$ agar bo'lsa, funksional qator uzoqlashuvchi bo'ladi, chunki qator yaqinlashuvchi bo'lishining zaruriy sharti bajarilmaydi, ya'ni qatorning umumiy hadi nolga intilmaydi.

Demak, $f(x)$ funksiyaning aniqlanish sohasi (– 2; 2) bo‘ladi.

Funksional qatorning har bir $u_n(x)=\left(\frac{1}{2}\right)^n\left(x+\frac{1}{n}\right)^n$ $(n\in N)$ hadi (–2; 2) da uzluksiz.

Endi (–2; 2) da $f(x)$ funksiyani uzluksizlikka tekshiramiz. Agar $|x|\le r<2$ bo‘lsa, u holda berilgan funksional qator [–*r*; *r*] da tekis yaqinlashuvchi bo‘ladi. Haqiqatan ham, Veyershtrass alomatiga asosan, barcha $x\in[-r;\, r]$ va $\forall n\in N$ lar uchun

$$|u_n(x)|=\left|\left(\frac{1}{2}\right)^n\cdot\left(x+\frac{1}{n}\right)^n\right|\le\left(\frac{1}{2}\right)^n\left(r+\frac{1}{n}\right)^n=c_n$$

tengsizlik o‘rinli bo‘ladi hamda $\sum_{n=1}^{\infty}\left(r+\frac{1}{n}\right)^n\frac{1}{2^n}$ sonli qator yaqinlashuvchi bo‘ladi. Demak, 7.1- teoremaning hamma shartlari bajarilayapti, u holda $f(x)$ funksiya $|x|\le r<2$ da uzluksiz bo‘ladi.

7.2. Funksional ketma-ketlik limit funksiyasining uzluksizligi. X $(X\subset R)$ to‘plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x), ..., f_n(x)... \qquad (7.2)$$

funksional ketma-ketlik berilgan bo‘lib, uning limit funksiyasi $f(x)$ bo‘lsin, ya’ni

$$\lim_{x\to\infty} f_n(x)=f(x).$$

7.2- teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi X to‘plamda uzluksiz bo‘lib, bu (7.2) funksional ketma-ketlik X to‘plamda $f(x)$ ga tekis yaqinlashuvchi bo‘lsa, u holda $f(x)$ limit funksiya ham X da uzluksiz bo‘ladi.

7.2- eslatma. 7.2- teoremaning shartlari bajarilganda

$$f(x_0)=\lim_{x\to x_0}(\lim_{n\to\infty} f_n(x))=\lim_{n\to\infty}(\lim_{x\to x_0} f_n(x)),\ (x_0\in X)$$

tenglik o‘rinli bo‘ladi.

7.4- misol. $\{f_n(x)\}=\frac{n^2x^2}{2+n^2+x^2}$ funksional ketma-ketlikning limit funksiyasini $X=[0;1]$ da uzluksizlikka tekshiring.

Yechilishi. $f_n(x)$ $(n=1,2,...)$ ketma-ketlikning har bir hadi $X=[0;1]$ da uzluksiz. Berilgan ketma-ketlikning limit funksiyasini topamiz:

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{n^2x^2}{2+n^2+x^2}=x^2;\ \ f(x)=x^2.$$

$$\sup_{x\in[0;1]}\left|f_n(x)-f(x)\right|=\sup_{x\in[0;1]}\left|\frac{n^2x^2}{2+n^2+x^2}-x^2\right|=\sup_{x\in[0;1]}\left|\frac{2x^2+x^4}{2+n^2+x^2}\right|\le\frac{3}{n^2}.$$

Bundan $\lim\limits_{n\to\infty}\sup\limits_{x\in[0;1]}\left|f_n(x)-f(x)\right|=0$ ekanligi kelib chiqadi, ya'ni

$f_n(x)\overset{[0;1]}{\rightrightarrows}x^2$. Demak, 7.2- teoremaga asosan, berilgan ketma-ketlikning limit funksiyasi $f(x)=x^2$ ham [0;1] da uzluksiz funksiya bo‘ladi. Bu yerda 7.2- eslatmadagi tenglikning o‘rinli ekanligiga ishonch hosil qilish qiyin emas, ya'ni

$$\lim_{x\to x_0}(\lim_{n\to\infty} f_n(x))=\lim_{n\to\infty}(\lim_{x\to x_0} f_n(x))=x_0^2,\ \ x_0\in[0;1].$$

7.3. Funksional qatorlarda hadma-had limitga o‘tish. Yaqinlashuvchi (7.1) funksional qator berilgan bo‘lib, uning yig‘indisi $S(x)$, x_0 nuqta esa X to‘plamning limit nuqtasi bo‘lsin.

7.3- teorema. Agar $x\to x_0$ da (7.1) funksional qatorning har bir $u_n(x)$ $(n=1,2,...)$ hadi chekli

$$\lim_{x\to x_0} u_n(x)=c_n\ \ (n=1,2,...)$$

limitga ega bo‘lib, berilgan qator X to‘plamda tekis yaqinlashuvchi bo‘lsa,

$$\sum_{n=1}^{\infty}c_n=c_1+c_2+...+c_n+...$$

qator ham yaqinlashuvchi, uning yig‘indisi C esa $S(x)$ ning $x\to x_0$ dagi limitiga teng bo‘ladi:

$$\lim_{x\to x_0} S(x)=C.$$

7.3- eslatma. 7.3- teoremaning shartlari bajarilganda

$$\lim_{x\to x_0}\sum_{n=1}^{\infty}u_n(x)=\sum_{n=1}^{\infty}\lim_{x\to x_0}u_n(x)$$

tenglik o‘rinli bo‘ladi.

7.5- misol. Ushbu limitni toping:

$$\lim_{x\to 1-0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\cdot\frac{x^n}{x^n+1}.$$

Yechilishi. Berilgan funksional qator $X=[1;+\infty)$ to‘plamda Abel alomatiga ko‘ra, tekis yaqinlashuvchi bo‘ladi (6.21- misolga q.), ya’ni

$$a)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\overset{X}{\rightrightarrows} S(x);$$

$b)$ barcha $x\in X$ va $\forall n\in N$ lar uchun $a_n(x)=1-\frac{1}{x^n+1}\le 1$ va $a_{n+1}(x)\le a_n(x)$ bo‘ladi. Bundan tashqari,

$$\lim_{x\to 1-0}u_n(x)=\lim_{x\to 1-0}\frac{(-1)^{n+1}}{x+n-1}\frac{x^n}{x^n+1}=\frac{(-1)^{n+1}}{2n}=c_n,\quad n\in N.$$

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}$$

yaqinlashuvchi va uning yig‘indisi $\frac{1}{2}\ln 2$ ga teng. 7.3- teoremaning shartlari bajarilayapti. Endi funksional qatorda hadma-had limitga o‘tish mumkin:

$$\lim_{x\to 1-0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x+n-1}\frac{x^n}{x^n+1}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\frac{1}{2}\ln 2.$$

7.4. Funksional ketma-ketlikda hadma-had limitga o‘tish. (7.2) ketma-ketlik X to‘plamda berilgan bo‘lib, uning limit funksiyasi $f(x)$, x_0 nuqta esa X to‘plamning limit nuqtasi bo‘lsin.

7.4- teorema. Agar $x\to x_0$ da $\{f_n(x)\}$ ketma-ketlikning har bir $f_n(x)$ $(n=1,2,...)$ hadi chekli

$$\lim_{x\to x_0}f_n(x)=a_n$$

limitga ega bo‘lib, bu ketma-ketlik X da tekis yaqinlashuvchi bo‘lsa, u holda $\{a_n\}$ ketma-ketlik ham yaqinlashuvchi bo‘ladi, uning $\lim\limits_{n\to\infty} a_n = a$ limiti esa $f(x)$ ning $x \to x_0$ dagi limitiga teng: $\lim\limits_{x\to x_0} f(x) = a$.

7.6- misol. $\{f_n(x)\} = \{\dfrac{1}{1+(1+\frac{x}{n})^n}\}$ funksional ketma-ketlik [0;1] da 7.4- teoremaning shartlarini qanoatlantirishini ko‘rsating.

Yechilishi. $\lim\limits_{x\to 1} f_n(x)$ ni topamiz:

$$\lim_{x\to 1} f_n(x) = \lim_{x\to 1} \frac{1}{1+(1+\frac{x}{n})^n} = \frac{1}{1+\left(1+\frac{1}{n}\right)^n} = a_n, \ \forall n \in N .$$

Berilgan funksional ketma-ketlikning [0;1] da tekis yaqinlashuvchiligini ko‘rsatamiz:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(1+\frac{x}{n})^n} = \frac{1}{1+e^x} = f(x);$$

$$|f_n(x) - f(x)| = \left| \frac{1}{1+(1+\frac{x}{n})^n} - \frac{1}{1+e^x} \right| = \frac{\left|e^x - (1+\frac{x}{n})^n\right|}{(1+e^x)(1+(1+\frac{x}{n})^n)} \le \left|e^x - (1+\frac{x}{n})^n\right| <$$

$$< \sup_{x\in[0;1]} \left|e^x - (1+\frac{x}{n})^n\right| \le \max\{(e-(1+\frac{1}{n})^n, \ \frac{1}{n}(1+\frac{1}{n})^{n-1}\} = \frac{1}{n}(1+\frac{1}{n})^{n-1} < \frac{3}{n} \to 0.$$

$$\forall x \in [0;1].$$

Demak, berilgan funksional ketma-ketlik [0;1] da tekis yaqinlashuvchi. $\{a_n\} = \{\dfrac{1}{1+\left(1+\frac{1}{n}\right)^n}\}$ yaqinlashuvchi va

$$\lim_{n\to\infty} a_n = \frac{1}{1+e}, \ \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{1+e^x} = \frac{1}{1+e} = a.$$

Shunday qilib, funksional kema-ketlik 7.4- teoremaning hamma shartlarini qanoatlantirar ekan.

7.5. Funksional qatorni hadma-had integrallash. Yaqinlashuvchi (7.1) funksional qator $X=[a;b]$ segmentda berilgan bo‘lib, uning yig‘indisi $S(x)$ bo‘lsin.

7.5- teorema. Agar (7.1) qatorning har bir $u_n(x)$ hadi $X=[a;b]$ segmentda uzluksiz bo‘lib, qatorning o‘zi shu segmentda tekis yaqinlashuvchi bo‘lsa, u holda

$$\int_a^b u_1(x)dx+\int_a^b u_2(x)dx+...+\int_a^b u_n(x)dx+...$$

qator ham yaqinlashuvchi bo‘ladi va

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)dx=\int_a^b S(x)dx$$

tenglik o‘rinli bo‘ladi.

7.4- eslatma. 7.5- teoremada qatorning tekis yaqinlashuvchiligi yetarli shart bo‘lib, lekin zaruriy shart bo‘la olmaydi, ya’ni ba’zan tekis yaqinlashuvchilik sharti bajarilmagan funksional qatorni ham hadma-had integrallash mumkin.

7.7- misol. $\sum_{n=1}^{\infty}(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}})$ funksional qatorni [0; 1] da hadma-had integrallash mumkinmi?

Yechilishi. Berilgan funksional qatorning n- qismiy yig‘indisini topamiz:

$$S_n(x)=(x^{\frac{1}{3}}-x)+(x^{\frac{1}{5}}-x^{\frac{1}{3}})+...+(x^{\frac{1}{2n-1}}-x^{\frac{1}{2n-3}})+$$
$$+(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}})=x^{\frac{1}{2n+1}}-x.$$

Endi $n\to\infty$ da limitga o‘tamiz:

$$S(x)=\lim_{n\to\infty}(x^{\frac{1}{2n+1}}-x)=\begin{cases}0, x=0, x=1 \text{ bo‘lganda,}\\ 1-x, 0<x<1 \text{ bo‘lganda.}\end{cases}$$

Demak, berilgan funksional qatorning yig‘indisi uzilishga ega bo‘lgan funksiyadir. Shu sababli berilgan funksional qator uchun [0;1] da tekis yaqinlashuvchilik sharti bajarilmaydi. Demak, 7.5-

teoremani qo‘llash huquqiga ega emasmiz. Lekin qator yig‘indisini va qatorni hadma-had integrallash mumkin:

$$\int_0^1 S(x)dx = \frac{1}{2}, \quad \sum_{n=1}^{\infty}\int_0^1 (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})dx = \sum_{n=1}^{\infty}(\frac{2n+1}{2(n+1)} - \frac{2n-1}{2n}) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{2}\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) = \frac{1}{2}.$$

Buni e’tiborga olsak,

$$\int_0^1 S(x)dx = \int_0^1 [\sum_{n=1}^{\infty}(x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})]dx = \sum_{n=1}^{\infty}\int_0^1 (x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}})dx$$

tenglik o‘rinli bo‘ladi.

7.8- misol. $f(x) = \sum_{n=1}^{\infty}\frac{\cos^2 nx}{n(n+1)}$ funksiyaning R da uzluksizligini isbotlang va $\int_0^{2\pi} f(x)dx$ integralni hisoblang.

Yechilishi. $u_n(x) = \frac{\cos^2 nx}{n(n+1)}$ $(n = 1,2,...)$ funksiyalar R da uzluksiz. $\forall x \in R$ va $\forall n \in N$ lar uchun $|u_n(x)| \le \frac{1}{n(n+1)}$ tengsizlik o‘rinli bo‘ladi. Veyershtrass alomatiga ko‘ra, berilgan funksional qator tekis yaqinlashuvchi. 7.1- teoremaga ko‘ra, $f(x)$ funksiya R da uzluksiz. Bu yerda 7.5- teoremaning ham hamma shartlari bajariladi. Shuning uchun berilgan funksional qatorni hadma-had integrallash mumkin:

$$\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{\cos^2 nx}{n(n+1)}dx = \int_0^{2\pi}\frac{\cos^2 x}{1\cdot 2}dx + \int_0^{2\pi}\frac{\cos^2 2x}{2\cdot 3}dx + ... + \int_0^{2\pi}\frac{\cos^2 nx}{n(n+1)}dx + ...$$

$$+ ... = \pi\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} + ...\right) = \pi.$$

Demak, $\int\limits_0^{2\pi} f(x)dx = \pi$.

7.6. Funksional ketma-ketliklarni hadma-had integrallash. (7.2) yaqinlashuvchi funksional ketma-ketlik $[a;b]$da berilgan bo'lib, $f(x)$uning limit funksiyasi bo'lsin.

7.6- teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,...)$ hadi $[a;b]$ segmentda uzluksiz bo'lib, funksional ketma-ketlik $[a;b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda

$$\int\limits_a^b f_1(x)dx,\ \int\limits_a^b f_2(x)dx,...,\ \int\limits_a^b f_n(x)dx,...$$

ketma-ketlik yaqinlashuvchi, uning limiti esa $\int\limits_a^b f(x)dx$ bo'ladi, ya'ni

$$\lim_{x\to\infty}\int\limits_a^b f_n(x)dx = \int\limits_a^b \lim_{x\to\infty} f_n(x)dx = \int\limits_a^b f(x)dx$$

tenglik o'rinli bo'ladi.

7.9- misol. $\{f_n(x)\} = \{n^{\frac{1}{2}}xe^{-nx}\}$ ketma-ketlik 7.6- teoremaning hamma shartlarini qanoatlantirishini ko'rsating, ya'ni

$$\lim_{n\to\infty}\int\limits_0^1 f_n(x)dx = \int\limits_0^1 (\lim_{n\to\infty} f_n(x))dx \qquad (*)$$

tenglik o'rinli ekanligini isbotlang.

Yechilishi. $f_n(x) = n^{\frac{1}{2}}xe^{-nx}$ $(n=1,2,...)$ funksiyalar $[0;1]$ segmentda uzluksiz hamda

$$\lim_{n\to\infty} f_n(x) = 0, \quad f(x) = 0,$$

$$\sup_{x\in[0;1]}|f_n(x) - f(x)| = \sup_{x\in[0;1]} n^{\frac{1}{2}}xe^{-n\cdot x} \le n^{\frac{1}{2}}\cdot\frac{1}{n}e^{-1} = \frac{e^{-1}}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0.$$

Demak, $\{f_n(x)\} = \{n^{\frac{1}{2}} x e^{-nx}\}$ ketma-ketlik [0;1] segmentda tekis yaqinlashuvchi. U holda, 7.6- teoremaga asosan, berilgan funksional ketma-ketlikni hadma-had integrallash mumkin:

$$\int_0^1 xe^{-x}dx,\ \int_0^1 xe^{-2x}dx,\ \int_0^1 xe^{-3x}dx, ...,\ \int_0^1 xe^{-nx}dx, ...$$

Bu ketma-ketlik yaqinlashuvchi va uning limiti 0 ga teng.

Shunday qilib, (*) tenglikning o'rinli ekanligiga ishonch hosil qilamiz.

7.7. Funksional qatorni hadma-had integrallash. Yaqinlashuvchi (7.1) funksional qator $[a;b]$ segmentda berilgan bo'lib, $S(x)$ uning yig'indisi bo'lsin.

7.7- teorema. Agar (7.1) qatorning har bir $u_n(x)$ hadi $[a;b]$ segmentda uzluksiz $u'_n(x)$ hosilaga ega bo'lib,

$$\sum_{n=1}^{\infty} u'_n(x) = u'_1(x) + u'_2(x) + ... + u'_n(x) + ...$$

qator $[a;b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda (7.1) qatorning $S(x)$ yig'indisi $[a;b]$ segmentda $S'(x)$ hosilaga ega va

$$S'(x) = \left(\sum_{n=1}^{\infty} u_n(x)\right)' = \sum_{n=1}^{\infty} u'_n(x)$$

bo'ladi.

7.5- eslatma. 7.7- teoremadagi funksional qatorning tekis yaqinlashuvchilik sharti yetarli bo'lib, lekin u zaruriy shart emas.

7.10- misol. $\sum_{n=1}^{\infty} \operatorname{arctg}\frac{x}{n^4}$ funksional qatorni $(-\infty;+\infty)$ da hadma-had differensiallash mumkinmi?

Yechilishi. Berilgan qatorning umumiy hadi $u_n(x) = \operatorname{arctg}\frac{x}{n^4}$ $(n \in N)$ $(-\infty;+\infty)$ da uzluksiz $u'_n(x) = \frac{n^4}{n^8 + x^2}$ hosilaga ega bo'ladi. $(-\infty;+\infty)$ da berilgan funksional qator taqqoslash alomatiga ko'ra ($n \to \infty$ da $\operatorname{arctg}\frac{x}{n^4} \sim \frac{x}{n^4}$) yaqinlashuvchi va $S(x)$ yig'indiga ega. Bundan tashqari, hosilalardan tuzilgan

$$\sum_{n=1}^{\infty} u'_n(x) = \sum_{n=1}^{\infty} \frac{n^4}{n^8 + x^2}$$

funksional qator $(-\infty;+\infty)$ da Veyershtrass alomatiga ko'ra tekis yaqinlashuvchi bo'ladi.

Demak, berilgan funksional qatorning yig'indisi $(-\infty;+\infty)$ da $S'(x)$ hosilaga ega va

$$S'(x)=\left(\sum_{n=1}^{\infty}\operatorname{arctg}\frac{x}{n^4}\right)'=\sum_{n=1}^{\infty}\frac{n^4}{n^8+x^2}.$$

7.11- misol. $f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^4}$ funksiyaning $(-\infty;+\infty)$ da uzluksiz va uzluksiz hosilaga ega ekanligini ko'rsating.

Yechilishi. $u_n(x)=\frac{\sin nx}{n^4}$ $(n=1,2,...)$ funksiyalar $(-\infty;+\infty)$ da uzluksiz va uzluksiz $u'_n(x)=\frac{\cos nx}{n^3}$ $(n=1,2,...)$ hosilalarga ega. $\sum_{n=1}^{\infty}\frac{\cos nx}{n^3}$ funksional qator, Veyershtrass alomatiga asosan, $(-\infty;+\infty)$ da tekis yaqinlashuvchi.

Demak, 7.7- teoremaga asosan, berilgan qatorni hadma-had differensiallash mumkin va

$$f'(x)=\left(\sum_{n=1}^{\infty}\frac{\sin nx}{n^4}\right)'=\sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^4}\right)'=\sum_{n=1}^{\infty}\frac{\cos nx}{n^3}.$$

Bundan tashqari, 7.2- teoremaga asosan, $f(x)$ va $f'(x)$ funksiyalar $(-\infty;+\infty)$ da uzluksiz bo'ladi.

7.8. Funksional ketma-ketliklarni hadma-had differensiallash. $[a;b]$ segmentda yaqinlashuvchi (7. 2) funksional ketma-ketlik berilgan bo'lib, uning limit funksiyasi $f(x)$ bo'lsin.

7.8- teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ hadi $[a;b]$ segmentda uzluksiz $f_n'(x)$ hosilaga ega bo'lib, bu hosilalardan tuzilgan

$$f_1'(x), f_2'(x), ..., f_n'(x), ...$$

funksional ketma-ketlik $[a;b]$ segmentda tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ limit funksiya shu $[a;b]$ segmentda $f'(x)$ hosilaga ega bo'lib, bu hosila $\{f_n'(x)\}$ ketma-ketlikning limitiga teng bo'ladi.

7.12- misol. $\{f_n(x)\} = \{\frac{1}{n}\text{arctg}x^n\}$ funksional ketma-ketlik $(-\infty;+\infty)$ da $f(x)$ limit funksiyaga tekis yaqinlashsa ham. Lekin

$$[\lim_{n\to\infty} f_n(x)]'\Big|_{x=1} \neq \lim_{n\to\infty} f_n'(1)$$

bo'lishini ko'rsating.

Yechilishi. Barcha $x \in (-\infty;+\infty)$ va $\forall n \in N$ lar uchun

$$f(x) = \lim_{n\to\infty} \frac{1}{n}\text{arctg}x^n = 0$$

bo'ladi. $\{f_n'(x)\}$ funksional ketma-ketlikning $(-\infty;+\infty)$ da tekis yaqinlashuvchi ekanligini ko'rsatamiz:

$$\sup_{x\in(-\infty;\infty)} |f_n(x) - f(x)| = \sup_{x\in R}\left|\frac{1}{n}\text{arctg}x^n\right| = \frac{\pi}{2n} = a_n,$$

$$\lim_{n\to\infty}\sup_{x\in R}|f_n(x) - f(x)| = \lim_{n\to\infty}\frac{\pi}{2n} = 0, \quad f_n \overset{R}{\underset{n\to}{\rightrightarrows}} 0.$$

Endi $f_n'(x) = (\frac{1}{n}(\text{arctg}x^n))' = \frac{nx^{n-1}}{n(1+x^{2n})}$, $f_n'(1) = \frac{1}{2}$ bo'ladi. Bu yerdan

$$[\lim_{n\to\infty} f_n(x)]'\Big|_{x=1} = [0]'\Big|_{x=1} = 0, \quad \lim_{n\to\infty} f_n'(1) = \frac{1}{2}; \quad 0 \neq \frac{1}{2}.$$

7.13- misol. $f_n(x) = nx(1-x)^n$ $(n = 1,2,...)$ ketma-ketlikning $[0;1]$ segmentda notekis yaqinlashuvchiligini hamda

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx \qquad (*)$$

tenglikning o'rinli ekanligini ko'rsating.

Yechilishi. $x = 0$ va $x = 1$ da $f_n(0) = 0$, $f_n(1) = 0$. $0<x<1$ da $\lim_{n\to\infty} nx(1-x)^n = 0$; $f(x) = 0$, $x \in [0;1]$,

$$\sup_{x\in[0;1]} |f_n(x) - f(x)| = \sup_{x\in[0;1]} |nx(1-x)^n| = \left(\frac{n-1}{n}\right)^n,$$

$$\lim_{n\to\infty}\sup_{x\in[0;1]} |f_n(x) - f(x)| = \lim_{n\to\infty}\sup_{x\in[0;1]} |nx(1-x)^n| = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = e^{-1} \neq 0.$$

Demak, $\{nx(1-x)^n\}$ berilgan ketma-ketlik $f(x)=0$ ga $[0;1]$ segmentda notekis yaqinlashuvchi. Endi (*) tenglikning o'rinli ekanligini ko'rsatamiz:

$$\lim_{n\to\infty}\int_0^1 nx(1-x)^n\,dx=\lim_{n\to\infty}\frac{n}{(n+1)\cdot(n+2)}=0,$$

$$\int_0^1 \lim_{n\to\infty} nx(1-x)^n\,dx=0.$$

Bu yerdan (*) tenglikning o'rinli ekanligi kelib chiqadi.

Mustaqil yechish uchun misollar

X to'plamda quyidagi funksional qatorlar yig'indisining uzluksiz ekanligini ko'rsating.

7.1. $\sum\limits_{n=1}^{\infty}\frac{\mathrm{arctg}\,nx}{\sqrt[3]{n^5+x}}$, $X=(-\infty;+\infty)$. **7.2.** $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n+x^2}$, $X=[4;\,8]$.

7.3. $\sum\limits_{n=1}^{\infty}xe^{-n^2\cdot x}$, $X=[0;+\infty)$. **7.4.** $\sum\limits_{n=1}^{\infty}\frac{\cos^2 nx}{n(n+1)}$, $X=(-\infty;+\infty)$.

7.5. $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}\cos nx$, $X=\left[\frac{\pi}{3};\frac{2\pi}{3}\right]$. **7.6.** $\sum\limits_{n=1}^{\infty}\frac{1}{n^4+n^2x^2}$, $X=(-\infty;+\infty)$.

7.7. $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt[3]{n+x^2}}\sin\frac{1}{\sqrt{nx}}\cdot\mathrm{arctg}\sqrt{\frac{x}{n}}$, $X=(0;+\infty)$.

$f(x)$ funksiyaning aniqlanish sohasini toping va uni uzluksizlikka tekshiring.

7.8. $f(x)=\sum\limits_{n=1}^{\infty}\frac{x}{(1+x^2)^n}$. **7.9.** $f(x)=\sum\limits_{n=1}^{\infty}\frac{\ln^n x}{n^2}$.

7.10. $f(x)=\sum\limits_{n=1}^{\infty}e^{-n^2x}\cos nx$. **7.11.** $f(x)=\sum\limits_{n=1}^{\infty}\frac{x+n(-1)^n}{x^2+n^2}$.

7.12. $f(x)=\sum\limits_{n=1}^{\infty}\frac{x^n}{2^n}$.

Quyidagi limitlarni toping.

7.13. $\lim\limits_{x\to1-0}\sum\limits_{n=1}^{\infty}(x^n-x^{n+1})$. **7.14.** $\lim\limits_{x\to1-0}\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{x^n}{1+x^n}$.

7.15. $\lim\limits_{x\to 1-0}\sum\limits_{n=0}^{\infty} x^2 e^{-n\cdot x}$. 7.16. $\lim\limits_{x\to 0-0}\sum\limits_{n=1}^{\infty}\frac{1}{2^n n^x}$.

7.17. $\lim\limits_{x\to+\infty}\sum\limits_{n=1}^{\infty}\frac{x^2}{1+n^2x^2}$. 7.18. $\lim\limits_{x\to\frac{\pi}{2}}\sum\limits_{n=1}^{\infty}\left(\frac{\sin nx}{\sqrt{n}}-\frac{\sin(n+1)x}{\sqrt{n+1}}\right)$.

7.19. $\lim\limits_{x\to\frac{1}{3}}\sum\limits_{n=0}^{\infty} x^n$.

Funksional qatorlarni X to'plamda hadma-had integrallash mumkinmi?

7.20. $\sum\limits_{n=1}^{\infty}(x^{2n+2}-x^{2n}),\quad X=[-1;1]$.

7.21. $\sum\limits_{n=1}^{\infty}\frac{x}{\left(1+x^2\right)^n}.\quad X=(0;+\infty)$.

7.22. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^4},\quad X=(-a;a)$.

7.23. $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+1}}x^{2n+1},\quad X=[-q;q],\ 0<q<1$.

7.24. $\sum\limits_{n=1}^{\infty}\frac{1}{e^{n\cdot x}},\quad X=(-\varepsilon;+\infty),\ \ \varepsilon>0$.

7.25. $\sum\limits_{n=1}^{\infty}\frac{\cos^n x}{n!},\quad X=[a;b]$.

7.26. $\sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^2},\quad X=(-a;a),\ \ a>0$.

Funksional qatorlarni X to'plamda hadma-had differensiallash mumkinmi?

7.27. $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{n^3},\quad X=(-\infty;+\infty)$. 7.28. $\sum\limits_{n=1}^{\infty}e^{-n^2x},\quad X=[\varepsilon;+\infty),\ \ \varepsilon>0$.

7.29. $\sum\limits_{n=1}^{\infty}\frac{\cos 4^n\pi x}{4^n},\quad X=(-\infty;+\infty)$. 7.30. $\sum\limits_{n=1}^{\infty}e^{-(x-n)^2},\quad X=[-1;1]$.

7.31. $\sum_{n=1}^{\infty}\frac{x^n}{n}$, $X=[-a;a]$, $0<a<1$.

7.32. $\sum_{n=1}^{\infty}\frac{1}{n^x}$, $X=[\delta;+\infty)$, $\delta>1$.

7.33. $\sum_{n=1}^{\infty}\frac{\cos(n+1)x}{(n+1)^2\ln^2(n+1)}$, $X=(-\infty;+\infty)$.

7.34. $\sum_{n=1}^{\infty}[e^{-(n-1)^2x^2}-e^{-n^2x^2}]$, $X=(-\infty;+\infty)$.

7.35. $\{f_n(x)\}=\{nxe^{-n\cdot x^2}\}$ funksional ketma-ketlik $[0;1]$ segmentda $f(x)$ limit funksiyaga tekis yaqinlashsa ham, lekin

$$\int_0^1[\lim_{n\to\infty}f_n(x)]dx\neq\lim_{n\to\infty}\int_0^1 f_n(x)dx$$

bo‘lishini ko‘rsating.

7.36. $\{f_n(x)\}=\{nx(1-x)^n\}$ funksional ketma-ketlik $[0;1]$ segmentda $f(x)$ limit funksiyaga tekis yaqinlashsa ham, lekin

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx=\int_0^1\lim_{n\to\infty}f_n(x)dx$$

bo‘lishini ko‘rsating.

7.37. Quyidagi integral belgisi ostidagi ifodada limitga o‘tish mumkinmi?

$$\lim_{n\to\infty}\int_0^1\frac{nx}{1+n^2x^4}dx=\int_0^1\lim_{n\to\infty}\frac{nx}{1+n^2x^4}dx.$$

7.38. $\{f_n(x)\}=\{x^n\}$ funksional ketma-ketlik $[0;1]$ segmentda

$$f(x)=\begin{cases}0, 0\le x<1 \text{ bo‘lganda}\\ 1, x=1 \text{ bo‘lganda}\end{cases}$$

limit funksiyaga notekis yaqinlashsa ham, lekin

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx=\int_0^1\lim_{n\to\infty}f_n(x)dx$$

bo‘lishini ko‘rsating.

7.39. $\{f_n(x)\}=\{x^2+\frac{1}{n}\sin n(x+\frac{\pi}{2})\}$ funksional ketma-ketlik $(-\infty;+\infty)$ da $f(x)$ limit funksiyaga tekis yaqinlashsa ham, lekin

$$[\lim_{n\to\infty} f_n(x)]' \neq \lim_{n\to\infty} f_n'(x)$$

bo'lishini ko'rsating.

Misollarning javoblari

7.8. $(-\infty;+\infty)$ da mavjud, $x=0$ uzilish nuqtasi. **7.9.** $[e^{-1};e]$. **7.10.** $(-\infty;+\infty)$. **7.11.** $(-\infty;+\infty)$. **7.12.** $[1-\varepsilon;1+\varepsilon]$, $0<\varepsilon<1$. **7.13.** 1. **7.14.** $\frac{1}{2}\ln 2$. **7.15.** $\frac{e}{e-1}$. **7.16.** 1. **7.17.** $\frac{\pi^2}{6}$. **7.18.** 1. **7.19.** $\frac{3}{2}$. **7.20.** Mumkin. **7.21.** Mumkin emas. **7.22.** Mumkin. **7.23.** Mumkin. **7.24.** Mumkin. **7.25.** Mumkin. **7.26.** Mumkin emas. **7.27.** Mumkin. **7.28.** Mumkin. **7.29.** Mumkin emas. **7.30.** Mumkin. **7.31.** Mumkin. **7.32.** Mumkin. **7.33.** Mumkin. **7.34.** Mumkin emas.

8- §. Darajali qatorlar

8.1. Darajali qator, uning yaqinlashish radiusi va yaqinlashish intervali. Ushbu

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \quad (8.1)$$

qator *darajali qator* deyiladi. Bunda $a_0, a_1, a_2, \ldots, a_n, \ldots$ o'zgarmas haqiqiy sonlar darajali qatorning koeffitsiyentlari deyiladi, x_0 esa ixtiyoriy o'zgarmas son.

(8.1) darajali qator $\sum_{n=0}^{\infty} u_n(x)$ funksional qatorning xususiy holi bo'lib hisoblanadi:

$$u_n(x) = a_n(x-x_0)^n, \quad n=0,1,2,\ldots .$$

$x-x_0=t$ belgilash yordamida (8.1) darajali qatorni

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots \quad (8.2)$$

ko'rinishga keltirish mumkin. Shuning uchun biz bundan keyin

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

ko‘rinishdagi qatorlarni o‘rganish bilan kifoyalanamiz.

8.1- teorema (Abel teoremasi). Agar (8.2) darajali qator x ning $x = x_0$ $(x_0 \neq 0)$ qiymatiga yaqinlashuvchi bo‘lsa, u holda x ning $|x| < |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida (8.2) darajali qator absolut yaqinlashuvchi bo‘ladi.

Natija. Agar (8.2) qator x ning $x = x_0$ qiymatida uzoqlashuvchi bo‘lsa, u x ning $|x| > |x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida uzoqlashuvchi bo‘ladi.

8.2- teorema. Har qanday darajali (8.2) qator uchun $\exists\rho$ $(\rho \geq 0$ son yoki $+\infty)$ son mavjud bo‘lib:

a) agar $\rho \neq 0$ va $\rho \neq +\infty$ bo‘lsa, u holda (8.2) qator $K = \{x : |x| < \rho\}$ intervalda absolut yaqinlashuvchi bo‘ladi va K intervalning tashqarisida uzoqlashuvchi bo‘ladi;

b) agar $\rho = 0$ bo‘lsa, (8.2) darajali qator faqat $x = 0$ nuqtada yaqinlashuvchi bo‘lib, sonlar o‘qining qolgan hamma nuqtalarida uzoqlashuvchi bo‘ladi;

d) agar $\rho = +\infty$ bo‘lsa, (8.2) darajali qator sonlar o‘qining hamma joyida yaqinlashuvchi bo‘ladi.

8.1- ta’rif. 8.2- teoremadagi ρ son (8.2) darajali qatorning *yaqinlashish radiusi*, $K = \{x \in R : |x| < \rho\}$ esa darajali qatorning *yaqinlashish intervali* deyiladi.

8.1- eslatma. K intervalning chegarasida, ya’ni $x = \pm\rho$ da (8.2) darajali qator yaqinlashuvchi bo‘lishi ham, uzoqlashuvchi bo‘lishi ham mumkin. K ga nisbatan kichik istalgan $K_1 = \{x : |x| \leq \rho_1 < \rho\}$ intervalda (8.2) qator absolut va tekis yaqinlashuvchi bo‘ladi.

8.3- teorema (Koshi — Adamar). Agar: 1) chekli yoki cheksiz $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ mavjud bo‘lsa, u holda (8.2) qatorning yaqinlashish radiusi ρ uchun

$$\frac{1}{\rho}=\lim_{n\to\infty}\sqrt[n]{|a_n|} \qquad (8.3)$$

formula o‘rinli;

2) chekli yoki cheksiz $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ mavjud bo‘lsa, u holda (8.2) darajali qatorning yaqinlashish radiusi r uchun

$$\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (8.4)$$

formula o‘rinli.

8.2- eslatma. Darajali qatorlarning har bir hadi $(-\infty;+\infty)$ da berilgan funksiya bo‘lsa ham, tabiiyki, darajali qatorlar ixtiyoriy nuqtada yaqinlashuvchi bo‘ladi, deb ayta olmaymiz.

8.3- eslatma. $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ darajali qatorning yaqinlashish intervali $(x_0-\rho;x_0+\rho)$ bo‘ladi, bunda ρ ushbu $\sum_{n=0}^{\infty}a_nx^n$ qatorning yaqinlashish radiusi.

8.4- eslatma. (8.3) — (8.4) limitlar mavjud bo‘lmasligi ham mumkin. Ammo (8.2) darajali qatorning yaqinlashish radiusini hisoblash uchun umumiy formula

$$\rho=\frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}} \qquad (8.5)$$

ga egamiz. (8.5) formula *Koshi — Adamar formulasi* deyiladi.

8.1- misol. $\sum_{n=0}^{\infty}\left(\frac{2+(-1)^n}{3}\right)^n x^n$ darajali qatorning yaqinlashish radiusini toping.

Yechilishi. Berilgan darajali qator umumiy hadining koeffitsiyenti $a_n=\left(\frac{2+(-1)^n}{3}\right)^n$. Endi ρ radiusni topish uchun $\{K_n\}=\{\sqrt[n]{|a_n|}\}$ ketma-ketlikning limitini topamiz:

$$\varlimsup_{n\to\infty} K_n = \varlimsup_{n\to\infty} \sqrt[n]{\left(\frac{2+(-1)^n}{3}\right)^n} = \varlimsup_{n\to\infty} \frac{2+(-1)^n}{3} = \begin{cases} 1, & n=2k,\ k\in N, \\ \frac{1}{3}, & n=2k-1,\ k\in N. \end{cases}$$

Ravshanki, $\{K_n\}$ ketma-ketlikning limiti mavjud emas. Berilgan $\{K_n\}$ ketma-ketlikning qismiy limitlari mavjud, ya'ni

$$\lim_{k\to\infty} K_{2k} = 1,\quad \lim_{k\to\infty} K_{2k-1} = \frac{1}{3}.$$

$\{K_n\}$ ketma-ketlik qismiy ketma-ketliklarining limitlari orasida eng kattasi 1, eng kichigi esa $\frac{1}{3}$ ga teng ekanligini ko'rish qiyin emas. Demak,

$$\overline{\lim_{n\to\infty}} K_n = \overline{\lim_{n\to\infty}} \frac{2+(-1)^n}{3} = 1,\quad \underline{\lim_{n\to\infty}} K_n = \underline{\lim_{n\to\infty}} \frac{2+(-1)^n}{3} = \frac{1}{3}.$$

8.4- eslatmaga ko'ra, berilgan darajali qatorning yaqinlashish radiusi

$$\rho = \overline{\lim_{n\to\infty}} \left[\left(\frac{2+(-1)^n}{3}\right)^n\right]^{-\frac{1}{n}} = 1$$

bo'ladi.

8.2- misol. $\sum_{n=0}^{\infty} \frac{b^n}{n!} x^n,\ b>0$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. Berilgan darajali qator n-hadining koeffitsiyenti $a_n = \frac{b^n}{n!}$. Qatorning yaqinlashish radiusini (8.4) formulaga asosan topamiz:

$$\lim_{n\to\infty} \left|\frac{a_n}{a_n+1}\right| = \lim_{n\to\infty} \frac{(n+1)!\, b^n}{b^{n+1}\, n!} = +\infty,$$

$\rho = +\infty$. Yaqinlashish intervali $(-\infty;+\infty)$ bo'ladi. Demak, berilgan darajali qatorning yaqinlashish sohasi ham $(-\infty;+\infty)$ dan iborat ekan.

8.3- misol. $\sum_{n=0}^{\infty}(\frac{2}{3})^n x^n$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. Bu yerda $a_n = \left(\frac{2}{3}\right)^n$. Qatorning yaqinlashish radiusini (8.3) formulaga asosan topamiz:

$$\rho = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{\left(\frac{2}{3}\right)^n}} = \frac{3}{2}.$$

Demak, berilgan darajali qatorning yaqinlashish radiusi $\rho = \frac{3}{2}$, yaqinlashish intervali esa $\left(-\frac{3}{2};\frac{3}{2}\right)$ bo'ladi. Endi $x = \pm\rho = \pm\frac{3}{2}$ da darajali qator mos ravishda

$$1-1+1-...+(-1)^n+..., \quad 1+1+1+...+1+...$$

sonli qatorlarga aylanadi. Bu qatorlarning uzoqlashuvchiligi ravshan. Demak, berilgan darajali qatorning yaqinlashish sohasi $\left(-\frac{3}{2};\frac{3}{2}\right)$ intervaldan iborat.

8.4- misol. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n3^n}$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. Berilgan darajali qator n - hadining koeffitsiyenti $a_n = \frac{1}{3^n n}$. Qatorning yaqinlashish radiusini (8.4) formulaga asosan topamiz:

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)3^{n+1}}{n3^n} = \lim_{n\to\infty}3\left(1+\frac{1}{n}\right) = 3, \quad \rho = 3.$$

8.3- eslatmaga ko'ra, $x_0 = 1$, $\rho = 3$, berilgan qatorning yaqinlashish intervali esa $(x_0 - \rho;\ x_0 + \rho) = (-2;4)$ bo'ladi.

Endi qatorning yaqinlashishini yaqinlashish intervalining chegaralarida tekshirib ko'ramiz. Agar:

a) $x = -2$ bo'lsa, $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n n}$ qator hosil bo'ladi. Bu qator Leybnis alomatiga ko'ra yaqinlashuvchi.

b) $x = 4$ bo'lsa, uzoqlashuvchi $\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator hosil bo'ladi.

Demak, berilgan darajali qatorning yaqinlashish sohasi $[-2;4)$ yarim intervaldan iborat.

8.5- misol. Ushbu $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1} \cdot \frac{x^{2n}}{3^n}$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. $x^2 = t$ deb belgilaymiz, u holda berilgan darajali qator

$$\sum_{n=1}^{\infty} \frac{n^3}{(n^3+1)3^n} t^n \qquad (*)$$

ko'rinishga ega bo'ladi, bu yerda $a_n = \frac{n^3}{(n^3+1)3^n}$. (8.4) formulaga asosan qatorning yaqinlashish radiusini topamiz:

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n^3 3^{n+1}((n+1)^3+1)}{(n+1)^3 3^n (n^3+1)} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{n^3}} \left(1 + \frac{1}{(n+1)^3} \right) = 3.$$

(*) qatorning yaqinlashish intervali $(-3;\ 3)$ bo'ladi, ya'ni $|t| < 3$. Endi berilgan darajali qatorning yaqinlashish intervalini topamiz:

$$|t| = |x^2| = x^2 < 3 \text{ yoki } |x| < \sqrt{3}.$$

Berilgan darajali qatorning yaqinlashish intervali $(-\sqrt{3};\sqrt{3})$ bo'ladi. Intervalning chegaralarida qatorni yaqinlashishga tekshiramiz: $x=\pm\sqrt{3}$ da $\sum_{n=1}^{\infty}\frac{n^3}{n^3+1}$ sonli qator hosil bo'lib, uning umumiy hadi uchun

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^3}{n^3+1}=1\neq 0$$

bo'lganligi (ya'ni sonli qator yaqinlashishining zaruriy sharti bajarilmaganligi) sababli, qator uzoqlashuvchi bo'ladi. Demak, berilgan darajali qatorning yaqinlashish sohasi ham $(-\sqrt{3};\sqrt{3})$ intervaldan iborat ekan.

8.6- misol. $\sum_{n=0}^{\infty}n!x^n$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. Bu yerda $a_n=n!$. Qatorning yaqinlashish radiusini (8.4) formulaga asosan topamiz:

$$\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n!}{(n+1)!}=0.$$

Yaqinlashish radiusi $\rho=0$, u holda berilgan qator faqat $x=0$ nuqtada yaqinlashuvchi, son o'qining qolgan nuqtalarida esa uzoqlashuvchi bo'ladi.

8.7- misol. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)^{n^3}x^n$ darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping.

Yechilishi. Berilgan darajali qatorda $a_n=\left(1-\frac{1}{n^2}\right)^{n^3}$. Darajali qatorning yaqinlashish radiusini (8.3) formulaga asosan topamiz:

$$\rho=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}}=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{(1-\frac{1}{n^2})^{n^3}}}=\frac{1}{\lim\limits_{n\to\infty}(1-\frac{1}{n^2})^{n^2}}=\frac{1}{e},\quad \rho=\frac{1}{e}.$$

Darajali qatorning yaqinlashish intervali $(-\frac{1}{e},\frac{1}{e})$ dan iborat. Berilgan darajali qator $|x|<\frac{1}{e}$ da absolut yaqinlashuvchi.

Yaqinlashish intervalining chegaralarida darajali qatorning yaqinlashish xususiyatini tekshiramiz: $x=\frac{1}{e}$ bo‘lganda $\sum_{n=1}^{\infty}(1-\frac{1}{n^2})^{n^3}\cdot\frac{1}{e^n}$ sonli qator hosil bo‘ladi. Koshi alomatiga ko‘ra

$$\lim_{n\to\infty}K_n=\lim_{n\to\infty}\sqrt[n]{(1-\frac{1}{n^2})^{n^3}\frac{1}{e^n}}=\lim_{n\to\infty}\frac{1}{e}(1-\frac{1}{n^2})^{n^2}=\frac{1}{e}\cdot\frac{1}{e}=\frac{1}{e^2}<1.$$

Demak, $x=\frac{1}{e}$ nuqtada darajali qator yaqinlashuvchi. Xuddi shunday, $x=-\frac{1}{e}$ bo‘lganda ham darajali qator yaqinlashuvchi bo‘ladi. Shunday qilib, berilgan darajali qatorning yaqinlashish sohasi $[-\frac{1}{e},\frac{1}{e}]$ segmentdan iborat ekan.

8.2. Darajali qatorlarning xossalari.

1- xossa. Agar (8.1) qatorning yaqinlashish radiusi ρ $(\rho>0)$ bo‘lsa, $0<r<\rho$ tengsizlikni qanoatlantiruvchi shunday ρ son topiladiki, (8.1) qator $[-r; r]$ da x ga nisbatan tekis yaqinlashuvchi bo‘ladi.

2- xossa. Agar (8.1) qatorning yaqinlashish radiusi ρ bo‘lsa, bu qatorning $S(x)=\sum_{n=0}^{\infty}a_nx^n$ yig’indisi $(-\rho;\rho)$ da uzluksiz funksiya bo‘ladi.

3- xossa. Agar (8.1) qatorning yaqinlashish radiusi ρ bo‘lib, bu qator $x=\rho$ $(x=-\rho)$ nuqtada yaqinlashuvchi (hech bo‘lmaganda shartli yaqinlashuvchi) bo‘lsa, qatorning $S(x)$ yig’indisi $x=\rho$ $(x=-\rho)$ nuqtada chapdan (o‘ngdan) uzluksiz bo‘ladi, ya’ni

$$S(\rho-0)=\lim_{x\to\rho-0}\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_n\rho^n$$

$$\left(S(-\rho+0)=\lim_{x\to-\rho+0}\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_n(-\rho)^n\right).$$

4- xossa. Agar (8.1) qatorning yaqinlashish radiusi ρ bo'lsa, bu qatorni $[a, b]$ $([a, b] \subset (-\rho;\rho))$ segmentda hadma-had integrallash mumkin, ya'ni

$$\int_a^b S(x)dx = \int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx.$$

5- xossa. Agar (8.1) qatorning yaqinlashish radiusi ρ bo'lsa, bu qatorni $(-\rho;\ \rho)$ da hadma-had differensiallash mumkin, ya'ni

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

6- xossa. Ushbu darajali qatorlar bir xil yaqinlashish radiusiga ega:

$$\sum_{n=0}^{\infty} a_n x^n,\quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1},\quad \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

8.8- misol. $x - 4x^2 + 9x^3 - 16x^4 + \dots$ darajali qatorning yig'indisini toping.

Yechilishi. Berilgan darajali qatorning yaqinlashish radiusini (8.4) formula orqali topamiz:

$$\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|, \text{ bunda } \quad a_n = (-1)^{n-1} n^2. \quad \rho = \lim_{n\to\infty}\left|\frac{(-1)^{n-1} n^2}{(-1)^n (n+1)^2}\right| = 1.$$

Darajali qator yig'indisini $S(x) = x - 4x^2 + 9x^3 - 16x^4 + \dots$ kabi belgilaymiz. Berilgan qator yig'indisi $S(x)$ ni x $(x \neq 0)$ ga bo'lib, uni darajali qatorlarning 5-xossasiga ko'ra, $|x| < 1$ intervalda hadma-had integrallaymiz:

$$\int \frac{S(x)}{x} dx = x - 2x^2 + 3x^3 - 14x^4 + \dots + C =$$

$$= (x^2 - x^3 + x^4 - \dots)' - x + x^2 - x^3 + \dots + C = \frac{x}{(1+x)^2} + C.$$

Bu tenglikni hadma-had $(|x| < 1, x \neq 0)$ differensiallab, $S(x) = \dfrac{x(1-x^2)}{(1+x)^3}$ ni topamiz $(|x| < 1,\ x \neq 0)$. Bu yerda $x \neq 0$ degan shartni olib tashlash mumkin, chunki $S(0) = 0$.

8.9- misol. Ushbu

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}=\frac{1}{\sqrt{1-x^2}}\quad(-1<x<1)$$

darajali qatorni hadma-had integrallashdan foydalanib,

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}=\frac{\pi}{2}$$

bo‘lishini ko‘rsating.

Yechilishi. Bu darajali qatorni, 4- xossaga asosan, $[0;x]$ $([0;x]\subset(-1;1),x>0)$ segmentda hadma-had integrallash mumkin. U holda arksinusning bizga ma’lum bo‘lgan yoyilmasiga ega bo‘lamiz:

$$\int_0^x\frac{dt}{\sqrt{1-t^2}}=\arcsin x=x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}.$$

Bu hosil bo‘lgan darajali qatorning yaqinlashish intervali $(-1;\ 1)$ bo‘ladi. $x=\pm1$ da

$$x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}.$$

darajali qator

$$1\pm\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}$$

sonli qatorga aylanadi. Bu qator, Raabe alomatiga ko‘ra, yaqinlashuvchi bo‘ladi. U holda x=1 da

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}=\arcsin 1=\frac{\pi}{2}$$

bo‘ladi.

8.10- misol. $\sum\limits_{n=1}^{\infty}(2n-1)x^{2n-1}$ funksional qatorning yig’indisini toping.

Yechilishi. Ma’lumki (1- bob, 1- § dagi (D) formulaga qarang),

$$x+x^3+...+x^{2n-1}+...$$

darajali qator (– 1; 1) da yaqinlashuvchi va uning yig'indisi $\frac{x}{1-x^2}$ ga teng:

$$\sum_{n=1}^{\infty} x^{2n-1} = \frac{x}{1-x^2}.$$

Bu qatorni 5-xossaga asosan, $(-1;1)$ da hadma-had differensiallash mumkin, ya'ni

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} x^{2n-1}\right) = \frac{d}{dt}\left(\frac{x}{1-x^2}\right) \Rightarrow \sum_{n=1}^{\infty}(2n-1)x^{2n-2} = \frac{1+x^2}{(1-x^2)^2}.$$

Oxirgi tenglikning chap va o'ng tomoniga $x \in (-1;1)$ ni ko'paytirib, berilgan qatorning yig'indisiga ega bo'lamiz:

$$\sum_{n=1}^{\infty}(2n-1)x^{2n-1} = \frac{x+x^3}{(1-x^2)^2}.$$

8.11- misol. Ushbu darajali qatorning yig'indisini toping:

$$2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots + \frac{2}{2n+1}x^{2n+1} + \ldots .$$

Yechilishi. Berilgan darajali qator umumiy hadining koeffitsiyenti $a_n = \frac{2}{2n+1}$. Darajali qatorning yaqinlashish radiusini (8.4) formulaga asosan topamiz:

$$\rho = \lim_{n\to\infty}\left|\frac{2}{2n+1} : \frac{2}{2n+3}\right| = \lim_{n\to\infty}\frac{2n+3}{2n+1} = 1, \quad \rho = 1.$$

Darajali qatorni $(-1;1)$ intervalda, 5- xossaga ko'ra, hadma-had differensiallash mumkin:

$$2\left(1+\frac{1}{3}x^2+\frac{1}{5}x^4+\ldots\right)+2x\left(\frac{2}{3}x+\frac{4}{5}x^3+\ldots+\frac{2n}{2n+1}x^{2n-1}+\ldots\right)=$$

$$=2\left(1+x^2+x^4+\ldots+x^{2n}+\ldots\right)=\frac{2}{1-x^2} \quad (|x|<1).$$

4- xossaga asosan, keyingi tenglikni $|x|<1$ da hadma-had integrallab,

$$2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+...+\frac{x^{2n+1}}{2n+1}+...\right)=\ln\frac{1+x}{1-x}+C$$

ifodaga ega bo‘lamiz. Bunda x=0 deb, C=0 ekanligini topamiz.

Shunday qilib,

$$2x\left(1+\frac{x^2}{3}+\frac{x^4}{5}+...+\frac{x^{2n}}{2n+1}+...\right)=\ln\frac{1+x}{1-x}\quad(|x|<1).$$

8.12- misol. Ushbu darajali qatorning yig‘indisini toping:

$$x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}\frac{x^{2n+1}}{(2n+1)}.$$

Yechilishi. (8.4) formulaga asosan:

$$\rho=\lim_{n\to\infty}\left|\frac{(2n-1)!!}{2^n n!(2n+1)}:\frac{(2n+1)!!}{2^{n+1}(n+1)!(2n+3)}\right|=\lim_{n\to\infty}\frac{2(n+1)(2n+3)}{(2n+1)(2n+1)}=1;$$

$\rho=1$. Darajali qatorning yaqinlashish intervali $(-1;1)$ dan iborat. 5- xossaga ko‘ra, darajali qatorni $(-1;1)$ intervalda hadma-had differensiallash natijasida ushbu

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n}\quad(|x|<1)$$

qatorni hosil qilamiz. Ma’lumki, $|t|<1$ da

$$(1+t)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)...(m-n+1)}{n!}t^n$$

yoyilma o‘rinli. Keyingi yoyilmada $t=-x^2$, $m=-\frac{1}{2}$ deyilsa,

$$(1-x^2)^{-\frac{1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}x^{2n}\quad(|x|<1)\qquad(1)$$

ga ega bo‘lamiz. (1) qatorni yaqinlashish intervali ichida hadma-had integrallasak,

$$\arcsin x=C+x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}\frac{x^{2n+1}}{2n+1}\quad(|x|<1)$$

bo‘ladi. Bunda x=0 deb va arcsin0=0 ekanligini e’tiborga olib, C=0 ekanligini topamiz.

Shunday qilib, berilgan darajali qatorning yig'indisi arcsin$x$ funksiyadan iborat bo'lar ekan, ya'ni

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!} \frac{x^{2n+1}}{2n+1}. \qquad (2)$$

Bu darajali qatorni $(-1;1)$ intervalning chegaralarida yaqinlashishga tekshirishda Raabe alomatini qo'llab,

$$\lim_{n\to\infty} n\left(\frac{2(n+1)(2n+3)}{(2n+1)^2}-1\right) = \lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1} = \frac{3}{2} > 1$$

ekanligini topamiz. Demak, $x=\pm1$ bo'lganda darajali qator absolut yaqinlashuvchi bo'lar ekan.

Shunday qilib, Abel teoremasiga asosan, (2) tenglik $\forall x \in [-1;1]$ lar uchun o'rinli bo'lar ekan.

Mustaqil yechish uchun misollar

Darajali qatorning yaqinlashish radiusi, yaqinlashish intervali va yaqinlashish sohasini toping:

8.1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-3)^{2n}}{n5^n}$. **8.2.** $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n3^n \ln n}$. **8.3.** $\sum_{n=1}^{\infty}\left(\frac{n}{3n-1}\right)^{3n+1} x^n$.

8.4. $\sum_{n=1}^{\infty}\ln^3\left(1+\frac{1}{\sqrt{n}}\right)x^n$. **8.5.** $\sum_{n=1}^{\infty}\frac{(x-1)^n}{2^n+3^n}$. **8.6.** $\sum_{n=1}^{\infty}\operatorname{tg}\frac{\pi}{3^n}x^{2n}$.

8.7. $\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{n-1}n\sqrt{n}}$. **8.8.** $\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n}x^{n+1}$. **8.9.** $\sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n}\right)^{n^2} x^n$.

8.10. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{n^2} x^n$. **8.11.** $\sum_{n=1}^{\infty}\frac{x^n}{a^n+b^n}$.

8.12. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{n}{e}\right)^n x^n$. **8.13.** $\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)x^n$.

8.14. $\sum_{n=1}^{\infty}\frac{[3+(-1)^n]^n}{n}x^n$. **8.15.** $\sum_{n=1}^{\infty}\frac{n!}{n^{\sqrt{n}}}x^n$.

8.16. $\sum_{n=1}^{\infty}\frac{n^{\sqrt{n}}}{\sqrt{n!}}x^n$. **8.17.** $\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^p x^n$. **8.18.** $\sum_{n=1}^{\infty}\left(\frac{x}{\sin x}\right)^n$.

8.19. $\sum_{n=1}^{\infty}\frac{5^n+(-3)^n}{n+1}x^n$. **8.20.** $\sum_{n=1}^{\infty}\frac{\sqrt[3]{2n+1}-\sqrt[3]{2n-1}}{\sqrt{n}}(x+3)^n$.

8.21. $\sum_{n=1}^{\infty}\frac{m(m-1)...(m-n+1)}{n!}x^n$.

8.22. $\sum_{n=1}^{\infty}\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n$. $(a>0, b>0)$. **8.23.** $\sum_{n=1}^{\infty}3^n(n^3+2)(x-1)^{2n}$.

8.24. $\sum_{n=1}^{\infty}\frac{n!}{a^{n^2}}x^n$. $(a>1)$. **8.25.** $\sum_{n=1}^{\infty}\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}x^n$.

Qatorlarning yaqinlashish sohalarini toping:

8.26. $\sum_{n=1}^{\infty}x^{2n+1}\sin\frac{\pi}{2^n}$. **8.27.** $\sum_{n=1}^{\infty}\frac{\lg^n x}{n}$.

8.28. $\sum_{n=1}^{\infty}(\sin(\sqrt{n+1}-\sqrt{n}))(x+1)^n$.

8.29. $\sum_{n=1}^{\infty}\left(\operatorname{arctg}\frac{1}{5^n}\right)(x-5)^n$. **8.30.** $\sum_{n=1}^{\infty}\frac{2^n n}{n^n}(x-1)^{2n}$.

8.31. $\sum_{n=1}^{\infty}(2-\sqrt{e})(2-\sqrt[3]{e})...(2-\sqrt[n]{e})x^n$. **8.32.** $\sum_{n=1}^{\infty}\frac{1}{x^n}\sin\frac{\pi}{2^n}$.

8.33. $\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{nx}{e}\right)^n$.

Darajali qatorlarning yig'indilarini hadma-had differensiallash yordamida toping:

8.34. $x+\frac{x^3}{3}+\frac{x^5}{5}+...$. **8.35.** $x-\frac{x^3}{3}+\frac{x^5}{5}-...$.

8.36. $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+...$. **8.37.** $1+\frac{x^2}{2!}+\frac{x^4}{4!}+...$.

8.38. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+...$.

Darajali qatorlarning yig'indilarini hadma-had integrallash yordamida toping:

8.39. $x+2x^2+3x^3+\dots$. **8.40.** $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\dots+\frac{1}{n+1}x^{n+1}+\dots$.

8.41. $1\cdot 2x+2\cdot 3x^2+3\cdot 4x^3+\dots$.

Misollarning javoblari

8.1. $\rho=\sqrt{5}$, $(3-\sqrt{5};\ 3+\sqrt{5})$, $[3-\sqrt{5};\ 3-\sqrt{5}]$.

8.2. $\rho=3$, $(-3;3)$, $[-3;3]$.

8.3. $\rho=27$, $(-27;27)$. **8.4.** $\rho=1, (-1;1), [-1;1]$.

8.5. $\rho=3$, $(-2;4)$. **8.6.** $\rho=\sqrt{3}, (-\sqrt{3};\sqrt{3})$.

8.7. $\rho=\sqrt{2}$, $(-\sqrt{2};\sqrt{2}), [-\sqrt{2};\sqrt{2}]$. **8.8.** $\rho=1, (-1;1)$, $[-1;1)$.

8.9. $\rho=1$, $(-1;1)$. **8.10.** $\rho=\frac{1}{e}, (-\frac{1}{e};\frac{1}{e})$.

8.11. $\rho=\max(a;b)$, $(-\rho;\rho)$. **8.12.** $\rho=1$, $(-1;1)$, $(-1;1]$.

8.13. $\rho=1$, $(-1;1)$. **8.14.** $\rho=\frac{1}{4}, (-\frac{1}{4};\frac{1}{4})$. **8.15.** $\rho=0, x=0$.

8.16. $\rho=+\infty$, $(-\infty;+\infty)$. **8.17.** $\rho=1$, $0<x<2$. Agar $\rho>1$ bo‘lsa, $x=0$ da absolut yaqinlashuvchi. Agar $0<\rho\le 1$ bo‘lsa, $x=0$ da shartli yaqinlashuvchi. **8.18.** $\rho=0, x=0$.

8.19. $\rho=\frac{1}{5}$, $(-\frac{1}{5};\frac{1}{5})$, $[-\frac{1}{5};\frac{1}{5})$. **8.20.** $\rho=1, (-4;-2), [-4;-2]$.

8.21. $\rho=1, (-1;1)$, agar $m\ge 0$ bo‘lsa, $x=-1$ da absolut yaqinlashuvchi, agar $m<0$ bo‘lsa, $x=-1$ da uzoqlashuvchi. Agar $m\ge 0$ bo‘lsa, $x=1$ da absolut yaqinlashuvchi, agar $-1<m<0$ bo‘lsa, $x=1$ da shartli yaqinlashadi. **8.22.** $\rho=\min\left(\frac{1}{a},\frac{1}{b}\right)$, $(-\rho;\rho)$. Agar $a<b$ bo‘lsa, $x=-\rho$ da absolut yaqinlashuvchi, agar $a\ge b$ bo‘lsa, $x=-\rho$ da shartli yaqinlashadi. Agar $a<b$ bo‘lsa, $x=\rho$ da absolut yaqinlashadi. Agar $a\ge b$ bo‘lsa, $x=\rho$ da uzoqlashadi.

8.23. $\rho = +\infty$, $(-\infty;+\infty)$. **8.24.** $\rho = \frac{1}{\sqrt{3}}$, $\left(1-\frac{1}{\sqrt{3}};1+\frac{1}{\sqrt{3}}\right)$.

8.25. $\rho = 1$, $(-1;1)$, $[-1;1)$. **8.26.** $(-\sqrt{2};\sqrt{2})$. **8.27.** $[-10^{-1};10]$.

8.28. $(-2;0)$. **8.29.** $(0;6)$. **8.30.** $\left(1-\sqrt{\frac{e}{2}};1+\sqrt{\frac{e}{2}}\right)$. **8.31.** $(-1;1)$.

8.32. $(-\infty;-\frac{1}{2})\cup(\frac{1}{2};+\infty)$. **8.33.** $(-1;1)$. **8.34.** $\frac{1}{2}\ln\frac{1+x}{1-x}$ $(|x|<1)$.

8.35. $\operatorname{arctg} x$ $(|x|\le 1)$. **8.36.** $\frac{1}{\sqrt{1-x}}$ $(-1\le x<1)$.

8.37. $\operatorname{ch} x$ $(|x|<+\infty)$. **8.38.** $1+\frac{1-x}{x}\ln(1-x)$ $(|x|\le 1)$.

8.39. $\frac{x}{(1-x)^2}$ $(|x|<1)$. **8.40.** $-\ln|1-x|$ $(|x|<1)$.

8.41. $\frac{2x}{(1-x)^3}$ $(|x|<1)$.

9- §. Teylor qatori

9.1. Funksiyalarni Teylor qatoriga yoyish. $f(x)$ funksiya x_0 $(x_0\in R)$ nuqtaning biror

$$U_\delta(x_0)=\{x\in R: x_0-\delta<x<x_0+\delta\} \quad (\delta>0)$$

atrofida berilgan bo‘lib, u shu atrofda istalgan tartibdagi hosilaga ega bo‘lsa, ushbu

$$f(x_0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (9.1)$$

darajali qator, u yaqinlashuvchi bo‘ladimi, yaqinlashuvchi bo‘lib, uning yig’indisi $f(x)$ funksiyaga teng bo‘ladimi yoki yo‘qmi, bundan qatiy nazar, $f(x)$ funksiyaning $x=x_0$ nuqtadagi *Teylor qatori* deyiladi. Bu qator (8.1) darajali qatorga o‘xshash bo‘lib, bunda

$$f(x_0)=a_0,\quad \frac{f'(x_0)}{1!}=a_1,\quad \frac{f''(x_0)}{2!}=a_2,$$

$$\frac{f'''(x_0)}{3!}=a_3,\ \dots,\ \frac{f^{(n)}(x_0)}{n!}=a_n,\dots$$

lar *Teylor koeffitsiyentlari* deyiladi.

Xususiy holda, ya'ni $x_0=0$ bo'lganda (9.1) Teylor qatori

$$f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

ko'rinishga keladi. Bu qator ko'p hollarda *Makloren qatori* deb yuritiladi.

9.1- teorema. $f(x)$ funksiya biror $U_\delta(x_0)$ to'plamda istalgan tartibdagi hosilaga ega bo'lib, (9.1) qator uning $x=x_0$ nuqtadagi Teylor qatori bo'lsin. Bu qator $U_\delta(x_0)$ da $f(x)$ ga yaqinlashishi uchun uning

$$f(x)=f(x_0)+\frac{f^{(1)}(x_0)}{1!}(x-x_0)+$$
$$+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x)$$

Teylor formulasi qoldiq hadi $\forall x\in U_\delta(x_0)$ da nolga intilishi, ya'ni $\lim\limits_{n\to\infty} r_n(x)=0$ bo'lishi zarur va yetarli.

Ma'lumki, Teylor formulasi qoldiq hadi:

a) integral ko'rinishda

$$r_n(x)=\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt;$$

b) Lagranj ko'rinishida

$$r_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1},$$

bunda $c=x_0+\theta(x-x_0),\ 0<\theta<1;$

d) Koshi ko'rinishida

$$r_n(x)=\frac{f^{(n+1)}(c)}{n!}(1-\theta)^n(x-x_0)^{n+1},\quad c=x_0+\theta(x-x_0),\quad 0,<\theta<1;$$

e) Peano ko'rinishida

$$r_n(x)=o((x-x_0)^n)$$

bo'ladi.

9.2- teorema. $f(x)$ funksiya $(x_0-\rho,x_0+\rho)$ $(\rho>0)$ oraliqda darajali qatorga yoyilgan, ya'ni:

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+...+a_n(x-x_0)^n+...$$

bo'lsa bu qator $f(x)$ funksiyaning Teylor qatori bo'ladi, bunda

$$a_0=f(x_0),\ a_1=\frac{f'(x_0)}{1!},\ a_2=\frac{f''(x_0)}{2!},\ a_3\frac{f'''(x_0)}{3!},...,a_n=\frac{f^{(n)}(x_0)}{n!},...$$

9.3- teorema. Agar $f(x)$ funksiya biror $(x_0-\rho,x_0+\rho)$ intervalda istalgan tartibdagi hosilaga ega bo'lib, shunday o'zgarmas $M>0$ son topilsaki, barcha $x\in(x_0-\rho,x_0+\rho)$ hamda barcha $n\in N$ lar uchun

$$\left|f^{(n)}(x)\right|\le M$$

bajarilsa, u holda $\left(x_o-\rho,x_0+\rho\right)$ intervalda $f(x)$ funksiya Teylor qatoriga yoyiladi, ya'ni

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

bo'ladi.

9.1- misol. $f(x)=\cos ax$ funksiyani $x=x_0$ nuqta atrofida Teylor qatoriga yoying.

Yechilishi. Bizga ma'lumki, berilgan $\cos ax$ funksiyaning n- tartibli $(n\in N)$ hosilasi quyidagicha bo'ladi:

$$f^{(n)}(x)=(\cos ax)^{(n)}=a^n\cos(ax+n\frac{\pi}{2})\ .$$

Bundan, $x=x_0$ da $f^{(n)}(x_0)=a^n\cos(ax_0+n\frac{\pi}{2})$ bo'lishi kelib chiqadi. Demak, berilgan funksiyaning Teylor qatori

$$\cos ax_0 - a\sin ax_0(x-x_0) - \frac{a^2\cos ax_0}{2!}(x-x_0)^2 +$$

$$+\frac{a^3\sin ax_0}{3!}(x-x_0)^3 - \ldots + \frac{a^n\cos(ax_0+n\frac{\pi}{2})}{n!}(x-x_0)^n + \ldots$$

bo'ladi. Bu qatorning yaqinlashish intervalini topamiz. Buning uchun

$$u_n(x) = \frac{a^n\cos(ax_0+n\frac{\pi}{2})}{n!}(x-x_0)^n,$$

$$u_{n+1}(x) = \frac{a^{n+1}\cos(ax_0+(n+1)\frac{\pi}{2})}{(n+1)!}(x-x_0)^{n+1}$$

bo'lganligini inobatga olib, Dalamber alomatiga ko'ra, ning har bir berilgan qiymatida quyidagi limitni hisoblaymiz:

$$D = \lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{a^{n+1}\cos(ax_0+n\frac{\pi}{2}+\frac{\pi}{2})n!(x-x_0)^{n+1}}{a^n\cos(ax_0+n\frac{\pi}{2})(n+1)!(x-x_0)^n}\right| =$$

$$= \begin{cases} 0, & n=2k,\ k\in N, \\ 0, & n=2k-1,\ k\in N. \end{cases}$$

$D = 0 < 1$ bo'lganligi uchun qaralayotgan qatorning yaqinlashish intervali $(-\infty;+\infty)$ bo'lishini aniqlaymiz. $f(x)=\cos ax$ funksiya Teylor formulasining Lagranj ko'rinishidagi qoldiq hadi

$$r_n(x) = \frac{a^{n+1}\cos(ac+(n+1)\frac{\pi}{2})}{(n+1)!}(x-x_0)^{n+1}$$

bo'ladi, bunda $c = x_0+\theta(x-x_0)$, $0<\theta<1$. Endi Teylor formulasidagi $r_n(x)$ qoldiq hadning nolga intilishini ko'rsatamiz. Har qanday $n\in N$ va $\forall c\in R$ uchun

$$\left|\cos(ac+(n+1)\frac{\pi}{2})\right| \le 1,\quad \lim_{n\to\infty}\frac{|a|^{n+1}}{(n+1)!} = 0$$

o'rinli. Demak, har qanday chekli $x\in R$ uchun $\lim_{n\to\infty} r_n(x) = 0$ bo'ladi.

Shunday qilib, berilgan $\cos ax$ funksiya istalgan tartibdagi hosilaga ega va $\lim\limits_{n\to\infty} r_n(x)=0$ bo'lgani uchun, 9.1-teoremaga asosan, bu funksiya $x-x_0$ ning darajalari bo'yicha Teylor qatoriga yoyiladi.

$$\cos ax=\cos ax_0-\frac{a\sin ax_0}{1!}(x-x_0)-\frac{a^2\cos ax_0}{2!}(x-x_0)^2+$$

$$+\frac{a^3\sin ax_0}{3!}(x-x_0)^3-\ldots+\frac{a^n\cos(ax_0+n\frac{\pi}{2})}{n!}(x-x_0)^n+\ldots .$$

9.2- misol. $f(x)=\frac{1}{1+x}$ funksiyani x=2 nuqta atrofida Teylor qatoriga yoying.

Yechilishi. Berilgan $f(x)=\frac{1}{1+x}$ funksiyaning istalgan tartibli hosilalarini topamiz, so'ngra uning va hosilalarining $x_0=2$ nuqtadagi qiymatlarini hisoblaymiz:

$$f(x)=\frac{1}{x+1}=(x+1)^{-1},\quad f(2)=\frac{1}{3},$$

$$f^{(1)}(x)=\left(\frac{1}{1+x}\right)'=-\frac{1}{(x+1)^2},\quad f^{(1)}(2)=-\frac{1!}{3^2},$$

$$f^{(2)}(x)=\frac{2!}{(x+1)^3},\quad f^{(2)}(2)=\frac{2!}{3^3},$$

....,

$$f^{(n)}(x)=\frac{(-1)^n n!}{(x+1)^{n+1}},\quad f^{(n)}(2)=\frac{(-1)^n n!}{3^{n+1}},$$

....

Topilganlarni (9.1) formulaga qo'ysak, $f(x)=\frac{1}{1+x}$ funksiyaning Teylor qatori

$$\frac{1}{3}-\frac{1!}{3^2}(x-2)+\frac{2!}{3^3}\frac{(x-2)^2}{2!}+\ldots+\frac{(-1)^n n!(x-2)^n}{3^{n+1}n!}+ \qquad (**)$$
$$+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n$$

ko'rinishda bo'ladi. Bu qatorning yaqinlashish intervalini Koshi alomatidan foydalanib topamiz:

$$u_n(x)=\frac{(-1)^n}{3^{n+1}}(x-2)^n,\quad \lim_{n\to\infty}\sqrt[n]{|u_n(x)|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{3^{n+1}}(x-2)^n\right|}=\frac{|x-2|}{3}<1.$$

Endi $|x-2|<3$ tengsizlikni yechamiz: $-3<x-2<3\Rightarrow$ $\Rightarrow-1<x<5$. (**) qatorning yaqinlashish intervali $(-1;\,5)$ bo'ladi. Intervalning chegaralarida, ya'ni $x=-1$ va $x=5$ nuqtalarda mos ravishda

$$1+1+1+\ldots \text{ va } 1-1+1-1+\ldots$$

uzoqlashuvchi qatorlar hosil bo'ladi.

Demak, (**) qatorning yaqinlashish sohasi $(-1;\,5)$ intervaldan iborat bo'ladi. Bu (**) qator $(-1;\,5)$ intervalda $f(x)=\dfrac{1}{1+x}$ funksiyaga yaqinlashishi uchun $f(x)$ funksiya Teylor formulasining qoldiq hadi barcha $x\in(-1;5)$ da nolga intilishi zarur. Haqiqatan ham,

$$|r_n(x)|=\left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(3+\theta(x-2))^{n+2}}\right|\le\frac{1}{3\left(1+\theta\dfrac{(x-2)}{3}\right)^{n+2}}\underset{n\to\infty}{\longrightarrow}0,$$

bunda $0<\theta<1$. Shunday qilib, 9.1- teoremaga asosan, (**)qatorning yig' indisi $\dfrac{1}{1+x}$ ga teng bo'ladi, ya'ni

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-2)^n.$$

9.3- misol. $f(x)=e^{-x}$ funksiyani $[-\rho;\rho]$, $\rho>0$ da Makloren qatoriga yoying.

Yechilishi. Berilgan funksiya $[-\rho;\rho]$ da istalgan tartibli hosilaga ega, ya'ni

$$f^{(n)}(x)=(e^{-x})^{(n)}=(-1)^n e^{-x},\ n\in N$$

va shunday o'zgarmas $M>0$ son topiladiki, barcha $x\in[-\rho;\rho]$ hamda barcha $n\in N$ lar uchun

$$\left|f^{(n)}(x)\right|=\left|(-1)^n e^{-x}\right|\le M$$

tengsizlik o'rinli, u holda 9.3- teoremaga asosan, $f(x)=e^{-x}$ funksiya $[-\rho;\rho]$ da Makloren qatoriga yoyiladi , ya'ni

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-...+(-1)^n\frac{x^n}{n!}+\dots .$$

9.2. Elementar funksiyalarning Teylor qatorlari. (9.1′) formulada $x_0=0$ deb, amaliyotda ko'p uchraydigan ba'zi elementar funksiyalarning darajali qatorlarga yoyilmalarini keltiramiz:

1. Ko'rsatkichli funksiyalar:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad(-\infty<x<+\infty,\quad \rho=\infty). \tag{9.2}$$

$$a^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ln^n a\ ,\ a>0,\ a\neq 1\ (-\infty<x<+\infty,\quad \rho=\infty). \tag{9.3}$$

2. Trigonometrik funksiyalar:

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\quad(-\infty<x<+\infty,\quad \rho=\infty). \tag{9.4}$$

$$\sin x=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}\ (-\infty<x<+\infty,\quad \rho=\infty). \tag{9.5}$$

3. Darajali funksiyalar:

$$(x+1)^\alpha=1+\sum_{n=1}^{\infty}C_\alpha^n x^n, \tag{9.6}$$

bunda $C_\alpha^n=\dfrac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-(n-1))}{n!}$.

Agar $\alpha \neq 0, \alpha \neq n, (n \in N)$ bo'lsa, (9.6) qatorning yaqinlashish radiusi 1 ga teng bo'ladi.

(9.6) formulaning muhim hususiy hollari:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (|x| < 1, \quad \rho = 1); \tag{9.7}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad (|x| < 1, \quad \rho = 1); \tag{9.8}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad (|x| < 1, \quad \rho = 1). \tag{9.8'}$$

4. Logarifmik funksiyalar:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \leq 1, \ \rho = 1); \tag{9.9}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad (-1 \leq x < 1, \ \rho = 1). \tag{9.10}$$

5. Giperbolik funksiyalar:

$$\text{ch}x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad (-\infty < x < +\infty, \ \rho = \infty); \tag{9.11}$$

$$\text{sh}x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty < x < +\infty, \ \rho = \infty). \tag{9.12}$$

6. Teskari trigonometrik funksiyalar:

$$\text{arctg}x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}, \quad (|x| \leq 1, \quad \rho = 1); \tag{9.13}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n+1)!!\,x^{2n+1}}{(2n)!!(2n+1)}, \quad (|x| < 1, \rho = 1); \tag{9.14}$$

$$\arccos x = \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{(2n-1)!!\,x^{2n+1}}{(2n)!!(2n+1)}, \quad (|x| < 1, \ \rho = 1); \tag{9.14'}$$

$$\text{arcctg}x = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n+1}, \quad (|x| < 1, \rho = 1). \tag{9.14''}$$

9.4-misol. $f(x)=\cos^4 x$ funksiyani $x_0=\frac{\pi}{8}$ nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping.

Yechilishi. Bizga ma'lumki, $\cos^2 x=\frac{1}{2}(1+\cos 2x)$, u holda

$$f(x)=\left[\frac{1}{2}(1+\cos 2x)\right]^2=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

bo'ladi. So'ngra $x=t+\frac{\pi}{8}$ almashtirishni bajaramiz:

$$f(x)=f(t+\frac{\pi}{8})=g(t)=\frac{3}{8}+\frac{\sqrt{2}}{4}(\cos 2t-\sin 2t)-\frac{1}{8}\sin 4t. \quad (9.15)$$

Endi (9.4) va (9.5) yoyilmalardan foydalanib, cos2*t*, sin 2*t* va sin 4*t* uchun

$$\cos 2t=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!}t^{2n}, \quad \sin 2t=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}t^{2n+1},$$

$$\sin 4t=\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n+1}t^{2n+1}}{(2n+1)!}$$

tengliklarga ega bo'lamiz. $\cos 2t$, $\sin 2t$ va $\sin 4t$ funksiyalarning yoyilmalarini (9.15) ga keltirib qo'ysak,

$$g(t)=\frac{3}{8}+\frac{\sqrt{2}}{4}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}t^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}t^{2n+1}}{(2n+1)!}\right)-$$
$$-\frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n 4^{2n+1}t^{2n+1}}{(2n+1)!}$$

ifodaga ega bo'lamiz. Endi $t=x-\frac{\pi}{8}$ ekanligini hisobga olsak, natijada

$$f(x)=\frac{3}{8}+\sqrt{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n-2}}{(2n)!}(x-\frac{\pi}{8})^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n-1}}{(2n+1)!}(x-\frac{\pi}{8})^{2n+1}\right)-$$
$$-\sum_{n=0}^{\infty}\frac{(-1)^n 2^{4n-1}}{(2n+1)!}(x-\frac{\pi}{8})^{2n+1}$$

bo'ladi. Hosil bo'lgan qatorning yaqinlashish radiusi $\rho=\infty$.

9.5- misol. Ushbu funksiyani Makloren qatoriga yoying va yaqinlashish radusini toping:

$$f(x)=\frac{3x+5}{(3x-5)(9x^2+25)}.$$

Yechilishi. Berilgan to‘g‘ ri kasrni, aniqmas koeffitsiyentlar usulidan foydalanib, oddiy kasrlarga ajratamiz:

$$\frac{3x+5}{(3x-5)(9x^2+25)}=\frac{A}{(3x-5)}+\frac{Bx+C}{(9x^2+25)}. \qquad (9.16)$$

Bu yerdan quyidagi ayniyatga ega bo‘lamiz:

$$3x+5=A(9x^2+25)+(Bx+C)(3x-5).$$

x ning bir xil darajalari oldidagi koeffitsientlarini tenglashtirib, so‘ngra hosil bo‘lgan tenglamalar sistemasini yechib, $A=\frac{1}{5}, B=-\frac{3}{5}, C=0$ bo‘lishini topamiz. Topilgan koeffitsiyentlarni (9.16) ga keltirib qo‘ysak,

$$f(x)=\frac{1}{5(3x-5)}+\frac{-3x}{5(9x^2+25)}=\frac{-1}{25(1-\frac{3}{5}x)}-\frac{3x}{125(1+(\frac{3x}{5})^2)}$$

bo‘ladi. Endi (9.7), (9.8) yoyilmalardan foydalanib, quyidagiga ega bo‘lamiz:

$$f(x)=-\frac{1}{25}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^2 x^n+\frac{3x}{125}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{3}{5}\right)^{2n}x^{2n}=$$

$$=-\frac{1}{25}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n x^n-\frac{3}{125}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{5}\right)^{2n}x^{2n+1}.$$

Bu qatorning yaqinlashish radiusi $\rho=\frac{5}{3}$ bo‘ladi.

9.6- misol. $f(x)=\ln(6x-5-x^2)$ funksiyani $x_0=3$ nuqta atrofida Teylor qatoriga yoying va yaqinlashish radiusini toping.

Yechilishi. Kvadrat uchhadni quyidagi ko‘rinishda ifodalaymiz: $6x-5-x^2=(5-x)(x-1)$. U holda $f(x)=\ln(5-x)+\ln(x-1)$ bo‘ladi. Endi $t=x-3$ almashtirishni bajaramiz:

$$f(x)=f(t+3)=g(t)=\ln(2-t)+\ln(t+2)=\ln 4+\ln(1-\frac{t}{2})+\ln(1+\frac{t}{2}).$$

(9.9), (9.10) formulalardan foydalanib,

$$g(t)=\ln 4-\sum_{n=1}^{\infty}\frac{t^n}{n2^n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}t^n}{n2^n}$$

ifodaga ega bo'lamiz. Endi $t=x-3$ ekanligini hisobga olsak,

$$f(x)=\ln 4-\sum_{n=1}^{\infty}\frac{1}{n2^n}((-1)^n+1)(x-3)^n=\ln 4-\sum_{k=1}^{\infty}\frac{1}{k2^{2k}}(x-3)^{2k}$$

bo'ladi. Hosil bo'lgan qatorning yaqinlashish radiusi 2 ga teng, yaqinlashish intervali esa [1; 5] bo'ladi (o'ylab ko'ring!)

9.7- misol. $f(x)=(\mathrm{sh}x-\mathrm{ch}x)^2$ funksiyani Makloren qatoriga yoying.

Yechilishi. Berilgan funksiyani quyidagi ko'rinishda ifodalaymiz:

$$f(x)=\mathrm{sh}^2x-2\mathrm{sh}x\mathrm{ch}x+\mathrm{ch}^2x=\mathrm{ch}2x-\mathrm{sh}2x.$$

(9.11) va (9.12) yoyilmalardan foydalanib, ushbu yoyilmaga ega bo'lamiz:

$$f(x)=(\mathrm{sh}x-\mathrm{ch}x)^2=\sum_{n=0}^{\infty}\frac{2^{2n}x^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{2^{2n+1}x^{2n+1}}{(2n+1)!}=$$

$$=\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}\left[1-\frac{2x}{2n+1}\right]\cdot x^{2n}.$$

9.3. Taqribiy hisoblashlarda qatorlarning tadbig'i. Taqribiy hisoblashlarda sonli va funksional qatorlar keng qo'llaniladi. Taqribiy hisoblashlarda qatorlar qo'llanishlarining ba'zi muhim hollarini keltiramiz.

a) **Funksiyalarning qiymatini qatorlar yordamida hisoblash.** $f(x)$ funksiyaning $x=x_0$ nuqtadagi qiymatini biror berilgan aniqlikda hisoblash talab qilingan bo'lsin.

Faraz qilaylik, x_0 nuqtani ichiga olgan $(a-\rho;a+\rho)$ intervalda berilgan funksiya

$$f(x)=a_0+a_1(x-a)+...+a_n(x-a)^n+...$$

darajali qatorga yoyilgan bo'lsin. U holda

$$f(x_0)=a_0+a_1(x_0-a)+...+a_n(x_0-a)^n+...$$

bo'ladi. Bu sonli qatorning birinchi n ta hadini olib,

$$f(x_0)\approx S_n(x_0)=a_0+a_1(x_0-a)+...+a_n(x_0-a)^n$$

taqribiy tenglikka ega bo'lamiz. n ning o'sib borishi bilan bu tenglikning aniqligi oshadi. Bu taqribiy tenglikning absolut xatoligi, ya'ni $|f(x_0)-S_n(x_0)|$ qator qoldig' ining moduliga teng bo'ladi:

$$|f(x_0)-S_n(x_0)|=|r_n(x_0)|,$$

bunda

$$r_n(x_0)=a_{n+1}(x_0-a)^{n+1}+a_{n+2}(x_0-a)^{n+2}+....$$

Bizga $f(x)$ funksiyaning $x=x_0$ nuqtadagi qiymatini ε>0 aniqlik bilan hisoblash talab qilinganda, qatorning shunday n ta hadlar yig'indisini olish kerakki, natijada $|f(x_0)-S_n(x_0)|=|r_n(x_0)|<\varepsilon$ bo'lsin. Ma'lumki, funksiya darajali qatorga yoyilganda, bu yoyilma $f(x)$ funksiyaning Teylor qatoridan iborat bo'lib,

$$a_n=\frac{f^{(n)}(x_0)}{n!}\quad (n=0,1,...)$$

bo'ladi. Teylor qatorining qoldig'i $r_n(x_0)$ ni berilgan aniqlikda baholashda Teylor (yoki Makloren) formulasi qoldiq hadi formulalarining biri ishlatiladi.

9.8- misol. $f(x) = e^x$ funksiyaning x=1 nuqtadagi qiymatini $\varepsilon=0,001$ aniqlikda hisoblang.

Yechilishi. Ma'lumki, e^x funksiyaning x bo'yicha yoyilmasi

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

ko'rinishda bo'ladi. Bundan x=1 bo'lganda

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}+...$$

munosabatga ega bo'lamiz. Bu qatorning $n+1$ ta hadini olib,

$$e\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$$

taqribiy tenglikni hosil qilamiz.

$f^{(n+1)}(x)=e^x$ ekanligini e'tiborga olib, Makloren qatorining $r_n(x)$ qoldiq hadini Lagranj ko'rinishida baholaymiz:

$$r_n(x)=\frac{e^c}{(n+1)!}x^{n+1},\ 0<c<x.$$

x=1 bo‘lganda, $r_n(1)=\frac{e^c}{(n+1)!}$, $0<c<1$. Ravshanki, $e^c<e^1<3$ tengsizlikni e'tiborga olsak, $r_n(1)<\frac{3}{(n+1)!}$ tengsizlikka ega bo‘lamiz.

$n=5$ bo‘lganda $\frac{3}{(n+1)!}=\frac{3}{6!}=\frac{1}{240}>0{,}001$;

$n=6$ bo‘lganda esa $\frac{3}{(n+1)!}=\frac{3}{7!}=\frac{1}{1680}<0{,}001$ bo‘ladi.

Demak, e^x funksiyaning x=1 nuqtadagi qiymatini $\varepsilon=0{,}001$ aniqlikda hisoblash uchun qatorning n=6 ta hadini olish yetarli.

$$e\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}$$

yoki

$$e\approx 1{,}0000+1{,}0000+0{,}5000+0{,}1667+0{,}0417+$$
$$+0{,}0083+0{,}0014=2{,}7181.$$

Shunday qilib, e sonining 0,001 aniqlikdagi qiymati e=2,7181 bo‘lar ekan.

b) **Integrallarni qatorlar yordamida hisoblash.**

9.9- misol. $\int_0^1\frac{\sin x}{x}dx$ integralni 0,001 aniqlikda hisoblang.

Yechilishi. sin$x$ funksiyaning x bo‘yicha (9.5) yoyilmasini, ya'ni

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-...$$

munosabatni e'tiborga olib, berilgan integralni quyidagicha yozib olamiz:

$$\int_0^1\frac{\sin x}{x}dx=\int_0^1\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-...}{x}dx=\int_0^1(1-\frac{x^2}{3!}+\frac{x^4}{5!}-...)dx=$$
$$=(x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-...)\Big|_0^1=1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}-\frac{1}{7\cdot 7!}+... . \quad (*)$$

Hosil bo‘lgan tenglikning o‘ng tomonidagi qator ishorasi almashinuvchi qator bo‘lib, u Leybnis teoremasiga ko‘ra, yaqinlashuvchi. Shuning uchun

$$|S-S_n|=|r_n|\le C_{n+1},$$

bunda $C_{n+1}=\dfrac{1}{(2n+1)(2n+1)!}$. (*) qator yig‘indisini 0,001 aniqlikda topish uchun

$$|r_n|\le a_{n+1}=\frac{1}{(2n+1)(2n+1)!}<0,001$$

tengsizlikni qanoatlantiradigan n ni topamiz:

$n=2$ bo‘lganda $\dfrac{1}{5\cdot 5!}=\dfrac{1}{600}>0,001$,

$n=3$ bo‘lganda $\dfrac{1}{7\cdot 7!}=\dfrac{1}{35280}<0,001$ bo‘ladi.

Demak, $\int\limits_0^1 \dfrac{\sin x}{x}dx$ integralni 0,001 aniqlikda hisoblash uchun (*) qatorning uchta hadini olamiz:

$$\int\limits_0^1 \frac{\sin x}{x}dx\approx 1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}=0,946.$$

Shunday qilib, berilgan integralning 0,001 aniqlikdagi taqribiy qiymati, $\int\limits_0^1 \dfrac{\sin x}{x}dx\approx 0,946$ bo‘ladi.

d) **Limitlarni qatorlar yordamida hisoblash.**

9.10- misol. Ushbu limitni toping:

$$\lim_{x\to 0}\frac{\sin x-\operatorname{arctg}x}{\arcsin x}.$$

Yechilishi. $\sin x$, $\operatorname{arctg}x$, $\arcsin x$ funksiyalarning x bo‘yicha (9.5), (9.13), (9.14) yoyilmalaridan foydalanib, limitni topamiz:

$$\lim_{x\to 0}\frac{\sin x-\operatorname{arctg}x}{\arcsin x}=\lim_{x\to 0}\frac{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dots-x+\dfrac{x^3}{3}-\dfrac{x^5}{5}+\dots}{x+\dfrac{1}{2\cdot 3}x^3+\dfrac{1\cdot 3}{2\cdot 4\cdot 5}x^5+\dots}=$$

$$= \lim_{x\to 0} \frac{\frac{1}{6}x^2 - \frac{23}{120}x^4 + \ldots}{1 + \frac{x^2}{2\cdot 3} + \frac{1\cdot 3\cdot x^4}{2\cdot 4\cdot 5} + \ldots} = 0.$$

e) **Differensial tenglamalarni qatorlar yordamida yechish.**

9.11- misol. $y'' = y\cos x + x$ differensial tenglamaning $y(0)=1,\ y'(0)=0$ boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimining darajali qatorga yoyilmasidagi birinchi (noldan farqli) uchta hadni toping.

Yechilishi. Berilgan differensial tenglamaning yechimini ushbu

$$y = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y'''(0)\frac{x^3}{3!} + \ldots \qquad (*)$$

Makloren qatori ko'rinishida izlaymiz.

$x = 0$ da $y = 1$ bo'lishini e'tiborga olib, berilgan differensial tenglamadan $y''(0)$ ni topamiz:

$$y''(0) = 1\cos 0 + 0 = 1.$$

$y'''(0)$ ni topish uchun berilgan differensial tenglamaning har ikkala tomonini differensiallaymiz:

$$y''' = y'\cos x - y\sin x + 1.$$

Bundan

$$y'''(0) = y'(0)\cos 0 - y(0)\sin 0 + 1 = 1,$$

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 1$$

larni (*) qatorga qo'yib, berilgan differensial tenglamaning taqribiy xususiy yechimi

$$y(x) \approx 1 + \frac{x^2}{2!} + \frac{x^3}{3!}$$

ekanligini hosil qilamiz.

Mustaqil yechish uchun misollar

Quyidagi funksiyalarni x ning darajalari bo'yicha darajali qatorga yoying:

9.1. $y=\sin^3 x$. **9.2.** $y=\dfrac{x}{6-x-x^2}$. **9.3.** $y=\ln(1+x+x^2+x^3)$.

9.4. $y=\dfrac{1}{(1-x^3)^2}$. **9.5.** $y=\cos^2 x$. **9.6.** $y=\arcsin x^3$.

9.7. $y=\dfrac{1}{1+x+x^2+x^3+x^4}$. **9.8.** $y=e^x(1-x)+e^{-x}(1+x)$.

9.9. $y=\ln(x+\sqrt{1+x^2})$. **9.10.** $y=\arccos(1-2x^2)$.

9.11. $y=\dfrac{x}{\sqrt[3]{1+x^2}}$. **9.12.** $y=e^{-x^2}$. **9.13.** $y=\dfrac{x\cos\alpha-x^2}{1-2x\cos\alpha+x^2}$.

9.14. $y=e^{x\cos\alpha}\cos(x\sin\alpha)$. **9.15.** $y=\dfrac{1}{(1-x^2)\sqrt{1-x^2}}$.

9.16. $y=4^x$.

Funksiyalarni ko'rsatilgan nuqta atrofida Teylor qatoriga yoying va bu qatorlarning yaqinlashish radiusini toping:

9.17. $f(x)=\sin\dfrac{2\pi x}{3}$, $x_0=3$. **9.18.** $f(x)=\dfrac{1}{x^2+4x+7}$, $x_0=-2$.

9.19. $f(x)=e^x$, $x_0=3$. **9.20.** $f(x)=\sqrt{x}$, $x_0=1$.

9.21. $f(x)=\cos\dfrac{\pi x}{2}$, $x_0=1$. **9.22.** $f(x)=\dfrac{x}{x^2-5x+6}$, $x_0=5$.

9.23. $f(x)=2^x$, $x_0=1$. **9.24.** $f(x)=\ln(x^2+2x+2)$, $x_0=-1$.

9.25. $f(x)=\dfrac{1}{x^2-5x+6}$, $x_0=1$.

9.26. $f(x)=\dfrac{1}{\sqrt{x^2-12x+40}}$, $x_0=6$.

9.27. $f(x)=\dfrac{1}{\sqrt[3]{x^2-6x+36}}$, $x_0=3$.

9.28. $f(x)=\dfrac{1}{\left(x^2-2x+3\right)^2}$, $x_0=1$.

9.29. $f(x)=\dfrac{1}{\sqrt{x^2-4x+8}}$, $x_0=2$.

9.30. $f(x)=\cos^4 x$, $x_0=-\dfrac{\pi}{2}$.

Quyida keltirilgan sonlarni ko‘rsatilgan aniqlikda hisoblang:

9.31. $\cos 18°$, 0,0001. **9.32.** $e^{\frac{1}{2}}$, 0,00001.

9.33. $\ln 0{,}98$, 0,0001. **9.34.** $\sqrt{27}$, 0,001.

9.35. $\mathrm{ch}\,0{,}3$, 0,0001. **9.36.** $\sqrt[5]{1{,}1}$, 0,0001.

Integral ostidagi funksiyani darajali qatorga yoyish yordamida, berilgan integralni ko‘rsatilgan aniqlikda hisoblang:

9.37. $\int_0^{\frac{1}{4}} e^{-x^2}dx$, 0,001. **9.38.** $\int_0^{0,5} \dfrac{\sin 2x}{x}dx$, 0,001.

9.39. $\int_0^{0,1} \dfrac{\ln(1+x)}{x}dx$, 0,001. **9.40.** $\int_0^{0,1} \dfrac{e^x-1}{x}dx$, 0,001.

9.41. $\int_0^{\frac{1}{3}} \dfrac{dx}{\sqrt[3]{1-x^2}}$, 0,001. **9.42.** $\int_0^{\frac{1}{2}} \dfrac{\arcsin x}{x}dx$, 0,001.

Limitlarni darajali qatorlar yordamida hisoblang:

9.43. $\lim\limits_{x\to 0}\dfrac{x-\mathrm{arctg}x}{x^3}$. **9.44.** $\lim\limits_{x\to 0}\dfrac{1-\cos x}{e^x-1-x}$.

9.45. $\lim\limits_{x\to 0}\dfrac{\mathrm{ch}x-\cos x}{x^2}$. **9.46.** $\lim\limits_{x\to 0}\dfrac{\arcsin 2x-2\arcsin x}{x^3}$.

9.47. $\lim\limits_{x\to 0}\dfrac{2^x-2^{\sin x}}{x^3}$. **9.48.** $\lim\limits_{x\to 0}\dfrac{x\cdot \mathrm{ctg}x-1}{x^2}$.

Differensial tenglamaning berilgan boshlang‘ich shartlarni qanoatlantiruvchi yechimini darajali qatorlar yordamida toping:

9.49. $y'-y=0$, $y(0)=1$. **9.50.** $(1+x^2)y'-1=0$, $y(0)=0$

9.51. $y''+xy=0$, $y(0)=1$, $y'(0)=0$

Differensial tenglamaning berilgan boshlang'ich shartlarni qanoatlantiruvchi xususiy yechimini ko'rsatilgan sondagi noldan farqli hadlarini darajali qatorlar yordamida toping:

9.52. $y''=x+y^2,\quad y(0)=0,\ y'(0)=1$. Birinchi to'rtta hadini toping.

9.53. $y'=x^2y+y^3,\quad y(0)=1$. Birinchi to'rtta hadini toping.

9.54. $y''=\frac{y'}{y}-\frac{1}{x},\quad y(1)=1,\ y'(1)=0$. Birinchi oltita hadini toping.

Misollarning javoblari

9.1. $\frac{3}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1},\ (|x|<+\infty)$.

9.2. $\left(-\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^n-\sum_{n=0}^{\infty}\frac{1}{2^n}x^n\right),(-2<x<2)$.

9.3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+\left[1+(-1)^n\right](-1)^{\frac{n}{2}+1}}{n}x^n,\ (-1<x\le 1)$.

9.4. $\sum_{n=1}^{\infty}(n+1)\cdot x^{3n},\ (|x|<1)$. **9.5.** $1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}}{(2n)!}x^{2n},\ (|x|<+\infty)$.

9.6. $x^3+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!(2n+1)}x^{6n+3},\ (|x|\le 1)$.

9.7. $\sum_{n=0}^{\infty}x^{5n}-\sum_{n=0}^{\infty}x^{5n+1},\ (|x|<1)$.

9.8. $2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}-2\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(2n+1)!},\ (|x|<+\infty)$.

9.9. $x+\sum_{n=1}^{\infty}\left\{(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}\right\},\ (|x|\le 1)$.

9.10. $2|x|\left\{1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n}}{2n+1}\right\},\left(|x|\le 1\right).$

9.11. $x+\sum_{n=1}^{\infty}\frac{(-1)^n\frac{1}{3}\frac{4}{3}\cdots(\frac{4}{3}+n)}{n!}x^{2n+1},\left(|x|<1\right).$

9.12. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!},\left(|x|<+\infty\right).$ **9.13.** $\sum_{n=1}^{\infty}x^n\cos n\alpha,\left(|x|<1\right).$

9.14. $\sum_{n=0}^{\infty}\frac{\cos n\alpha}{n!}x^n,\left(|x|<+\infty\right).$ **9.15.** $\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n)!!}x^{2n},\left(|x|<1\right).$

9.16. $\sum_{n=0}^{\infty}\frac{\ln^n 4}{n!}x^n,\left(|x|<+\infty\right).$ **9.17.** $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-3)^{2n+1},\rho=+\infty.$

9.18. $\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x+2)^{2n},\rho=\sqrt{3}.$ **9.19.** $e^3\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n,\rho=+\infty.$

9.20. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{n!2^n}(x-1)^n,\ \rho=\frac{5}{4}.$

9.21. $\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}(x-1)^n,\ \rho=+\infty.$

9.22. $x\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-5)^n-x\sum_{n=1}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-5)^n,\quad \rho=2.$

9.23. $2\sum_{n=0}^{\infty}\frac{\ln^n 2}{n!}(x-1)^n,\ \rho=+\infty.$

9.24. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x+1)^{2n},\ R=1.$ **9.25.** $\sum_{n=0}^{\infty}(1-2^{-n-1})(x-1)^n,\quad \rho=1.$

9.26. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{n!2^{3n+1}}(x-6)^{2n},\quad \rho=2.$

9.27. $\frac{1}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 4\cdots(3n-2)}{n!3^{4n+1}}(x-3)^{2n},\ \rho=3\sqrt{3}.$

9.28. $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{2^{n+2}}(x-1)^{2n},\ \rho=\sqrt{2}.$

9.29. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!!}{n!2^{3n+1}}(x-2)^{2n},\ \rho=2.$

9.30. $\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}}{(2n)!}(2^{2n-2}-1)(x+\frac{\pi}{2})^{2n},\ \rho=+\infty.$

9.31. 0,9551. **9.32**. 1,648719. **9.33**. $\ln 0,98\approx -0,0202$.

9.34. $\sqrt{27}\approx 5,196$. **9.35**. $\mathrm{ch}0,3\approx 1,0453$.

9.36. $\sqrt[5]{1,1}\approx 1,0192$. **9.37**. 0,245. **9.38**. 0,946. **9.39**. 0,098.

9.40. 0,102. **9.41**. 0,337. **9.42**. 0,507. **9.43**. $\frac{1}{3}$. **9.44**. 1. **9.45**.1.

9.46.1. **9.47**. $\frac{1}{6}\ln 2$. **9.48**. $-\frac{1}{3}$. **9.49**. $y=\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$.

9.50. $y=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=\mathrm{arctg}x$.

9.51. $y=1-\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}-\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\ldots$.

9.52. $y=x+\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^6}{45}+\ldots$. **9.53**. $y=1+\frac{x}{1!}+\frac{3x^2}{2!}+\frac{17x^3}{3!}+\ldots$.

9.54. $y=1-\frac{(x-1)^2}{2!}-\frac{2(x-1)^4}{4!}+\frac{3(x-1)^5}{5!}+\ldots$.

III BOB

XOSMAS INTEGRALLAR

10- §. Chegaralari cheksiz xosmas integrallar

10.1. Chegaralari cheksiz xosmas integrallar. Biror $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo‘lib, bu oraliqning istalgan $[a,t]$ $(a<t<+\infty)$ qismida (Riman ma’nosida) integrallanuvchi bo‘lsin, ya’ni $\forall t\,(t>a)$ uchun ushbu integral mavjud bo‘lsin:

$$F(t)=\int_a^t f(x)dx\,. \tag{10.1}$$

10.1- ta’rif. $F(t)$ funksiyaning $t\to+\infty$ dagi chekli yoki cheksiz limiti $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo‘yicha *birinchi tur xosmas integrali* deyiladi va u $\int_a^{+\infty} f(x)dx$ kabi belgilanadi.

Demak, $\int_a^{+\infty} f(x)dx=\lim\limits_{t\to+\infty}F(t)=\lim\limits_{t\to+\infty}\int_a^t f(x)dx\,.$

10.2- ta’rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud bo‘lib, u chekli bo‘lsa, (10.1) xosmas integral *yaqinlashuvchi* deyiladi, $f(x)$ esa $[a;+\infty)$ oraliqda *integrallanuvchi funksiya* deb ataladi.

10.3- ta’rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo‘lmasa, (10.1) xosmas integral *uzoqlashuvchi* deyiladi.

$f(x)$ funksiyaning $(-\infty;a]$ va $(-\infty;+\infty)$ oraliqlar bo‘yicha *xosmas integrallari*, ularning *yaqinlashuvchiligi*, *uzoqlashuvchiligi* ham yuqoridagi kabi ta’riflanadi:

$$\int_{-\infty}^{a} f(x)dx=\lim_{\tau\to-\infty}\Phi(\tau)=\lim_{\tau\to-\infty}\int_\tau^a f(x)dx\,,$$

$$\int_{-\infty}^{+\infty} f(x)dx=\lim_{\substack{t\to+\infty\\ \tau\to-\infty}}\Psi(t;\tau)=\lim_{\substack{t\to+\infty\\ \tau\to-\infty}}\int_\tau^t f(x)dx\,. \tag{10.2}$$

10.1- eslatma. (10.2) da chekli limit t va t larning mos ravishda $+\infty$ va $-\infty$ ga qanday intilishiga bog'liq bo'lmasligi kerak. Boshqacha aytganda, integral, faqat va faqat

$$\lim_{t\to+\infty}\int_a^t f(x)dx=\lim_{t\to+\infty}F(t)=I_1 \text{ va } \lim_{\tau\to-\infty}\int_\tau^a f(x)dx=\lim_{\tau\to-\infty}\Phi(\tau)=I_2$$

limitlar chekli bo'lgandagina, yaqinlashuvchi bo'ladi, bunda $a\in R$. Bu holda u, xosmas integralning ta'rifiga ko'ra, I_1+I_2 ga teng bo'ladi, ya'ni

$$\int_{-\infty}^{+\infty} f(x)dx=\int_{-\infty}^{a} f(x)dx+\int_{a}^{+\infty} f(x)dx.$$

10.1- misol. $I=\int_0^{+\infty}\frac{dx}{4+x^2}$ xosmas integralning yaqinlashuvchiligini ko'rsating va qiymatini toping:

Yechilishi. Ta'rifga ko'ra,

$$I=\int_0^{+\infty}\frac{dx}{4+x^2}=\lim_{t\to+\infty}\int_0^t\frac{dx}{4+x^2}=\lim_{t\to+\infty}F(t)$$

bo'ladi. Bunda

$$F(t)=\int_0^t\frac{dx}{4+x^2}=\frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_0^t=\frac{1}{2}\operatorname{arctg}\frac{t}{2}.$$

Endi $t\to+\infty$ da $F(t)$ funksiyaning limitini topamiz:

$$I=\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}\frac{1}{2}\operatorname{arctg}\frac{t}{2}=\frac{\pi}{4}$$

limit mavjud va chekli. Demak, berilgan xosmas integral yaqinlashuvchi va uning qiymati $I=\frac{\pi}{4}$.

10.2- misol. $\int_{-\infty}^{0}\frac{x+1}{x^2+1}dx$ xosmas integralning uzoqlashuvchi ekanligini ko'rsating.

Yechilishi. Xosmas integralning ta'rifiga ko'ra

$$\int_{-\infty}^{0}\frac{x+1}{x^2+1}dx=\lim_{\tau\to-\infty}\int_{\tau}^{0}\frac{x+1}{x^2+1}dx=\lim_{\tau\to-\infty}\left(\frac{1}{2}\int_{\tau}^{0}\frac{d(x^2+1)}{x^2+1}+\int_{\tau}^{0}\frac{dx}{x^2+1}\right)=$$

$$=\lim_{\tau\to-\infty}\left(\frac{1}{2}\ln(x^2+1)\Big|_{\tau}^{0}+\text{arctg}x\Big|_{\tau}^{0}\right)=\lim_{\tau\to-\infty}\left(-\frac{1}{2}\ln(\tau^2+1)-\text{arctg}\tau\right)=+\infty.$$

Demak, 10.3- ta'rifga asosan, berilgan xosmas integral uzoqlashuvchi bo'ladi.

10.3 - misol. $\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+14}$ xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Yuqoridagi ta'rifga ko'ra

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+14}=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\int_{\tau}^{t}\frac{dx}{(x+3)^2+5}=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\left(\int_{\tau}^{-3}\frac{dx}{(x+3)^2+5}+\right.$$

$$\left.+\int_{-3}^{t}\frac{dx}{(x+3)^2+5}\right)=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\left(\frac{1}{\sqrt{5}}\text{arctg}\frac{x+3}{\sqrt{5}}\Big|_{\tau}^{-3}+\frac{1}{\sqrt{5}}\text{arctg}\frac{x+3}{\sqrt{5}}\Big|_{-3}^{t}\right)=$$

$$=\lim_{t\to+\infty}\frac{1}{\sqrt{5}}\text{arctg}\frac{t+3}{\sqrt{5}}-\lim_{\tau\to-\infty}\frac{1}{\sqrt{5}}\text{arctg}\frac{\tau+3}{\sqrt{5}}=\frac{1}{\sqrt{5}}\frac{\pi}{2}+\frac{\pi}{2\sqrt{5}}=\frac{\pi}{\sqrt{5}},$$

ya'ni chekli. Demak, berilgan xosmas integral yaqinlashuvchi bo'ladi.

10.4- misol. $I=\int_{0}^{+\infty}\sin x dx$ xosmas integralni yaqinlashishga tekshiring.

Yechilishi. Ta'rifga ko'ra

$$I=\int_{0}^{+\infty}\sin x dx=\lim_{t\to+\infty}\int_{0}^{t}\sin x dx=\lim_{t\to+\infty}F(t).$$

Bunda $F(t)=\int_{0}^{t}\sin x dx=-\cos x\Big|_{0}^{t}=1-\cos t$.

$t\to+\infty$ da $\cos t$ funksiya hech qanday limitga intilmaydi. Demak, berilgan xosmas integral uzoqlashuvchi va uning qiymati mavjud emas.

10.5- misol. $\int_a^{+\infty} \frac{dx}{(x-c)^\alpha}$ $(a>c)$ xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Ta'rifga ko'ra

$$I = \int_a^{\infty} \frac{dx}{(x-c)^\alpha} = \lim_{t\to+\infty} \int_a^t \frac{dx}{(x-c)^\alpha} = \lim_{t\to+\infty} \left(\frac{(x-c)^{1-\alpha}}{1-\alpha} \bigg|_a^t \right) =$$

$$= \lim_{t\to+\infty} \left(\frac{(t-c)^{1-\alpha}}{1-\alpha} - \frac{(a-c)^{1-\alpha}}{1-\alpha} \right) = \lim_{t\to+\infty} F(t).$$

$\alpha > 1$ bo'lganda, $\lim_{t\to+\infty} F(t) = -\frac{(a-c)^{1-\alpha}}{1-\alpha}$ mavjud va chekli bo'lgani uchun, 10.2- ta'rifga asosan, berilgan xosmas integral yaqinlashuvchi.

$\alpha < 1$ bo'lganda, $\lim_{t\to+\infty} F(t) = +\infty$. Bu holda, 10.3- ta'rifga ko'ra, berilgan xosmas integral uzoqlashuvchi.

$\alpha = 1$ bo'lsa, $\int_a^{+\infty} \frac{dx}{x-c} = \ln(x-c)\Big|_a^{+\infty} = +\infty$.

Demak, $\alpha > 1$ bo'lganda, xosmas integral yaqinlashuvchi, $\alpha \le 1$ bo'lganda esa, xosmas integral uzoqlashuvchi bo'ladi.

10.2. Yaqinlashuvchi xosmas integrallarning xossalari. Asosiy formulalar.

1- xossa. Agar $\int_a^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo'lsa, $\int_t^{+\infty} f(x)dx$ $(t>a)$ integral ham yaqinlashuvchi bo'ladi va, aksincha, ya'ni $\int_t^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashuvchi bo'ladi, shu bilan birga

$$\int_a^{+\infty} f(x)dx = \int_a^t f(x)dx + \int_t^{+\infty} f(x)dx$$

tenglik o'rinli.

2- xossa. $\int_a^{+\infty} f(x)dx$ integralning yaqinlashuvchi bo'lishidan

$\lim_{t \to +\infty} \int_t^{+\infty} f(x)dx = 0$ bo'lishi kelib chiqadi.

3- xossa. Agar $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi

bo'lsa, barcha $\alpha, \beta \in R$ sonlar uchun, $\int_a^{+\infty} (\alpha\, f(x) \pm \beta\, g(x))dx$ integral

ham yaqinlashuvchi bo'ladi va

$$\int_a^{+\infty} (\alpha\, f(x) \pm \beta\, g(x))dx = \alpha \int_a^{+\infty} f(x)dx \pm \beta \int_a^{+\infty} g(x)dx$$

tenglik o'rinli.

4- xossa. Agar $\forall x \in [a;+\infty)$ uchun $f(x) \le g(x)$ bo'lib, $\int_a^{+\infty} f(x)dx$

va $\int_a^{+\infty} g(x)dx$ integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

tengsizlik o'rinli bo'ladi.

5- xossa (Nyuton — Leybnis formulasi). $f(x)$ funksiya $[a;+\infty)$ oraliqda uzluksiz bo'lib, $F(x)$ uning shu oraliqdagi boshlang'ich funksiyasi bo'lsin. U holda

$$\int_a^{+\infty} f(x)dx = F(x)\Big|_a^{+\infty} = F(+\infty) - F(a) \qquad (10.3)$$

bo'ladi, bunda $F(+\infty) = \lim_{t \to +\infty} F(t)$.

Odatda (10.3) ham *Nyuton — Leybnis formulasi* deyiladi.

6- xossa (o'zgaruvchilarni almashtirish formulasi). $f(x)$ funksiya $[a;+\infty)$ oraliqda uzluksiz, $\varphi(t)$ funksiya esa $[\alpha;\beta)$ da uzluksiz differensiallanuvchi funksiya va $a = \varphi(\alpha) \le \varphi(t) < \lim_{t \to \beta - 0} \varphi(t) = +\infty$

bo'lsa, $\int_a^{\infty} f(x)dx$, $\int_a^{\beta} f(\varphi(t))\cdot\varphi'(t)dt$ integrallardan birining yaqinlashuvchiligidan, ikkinchisining ham yaqinlashuvchiligi kelib chiqadi va ushbu o'zgaruvchilarni almashtirish formulasi o'rinli:

$$\int_a^{\infty} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)dt\,. \qquad (10.4)$$

7- xossa (bo'laklab integrallash formulasi). Agar $u=u(x)$ va $v=v(x)$ funksiyalar $[a;+\infty)$ da uzluksiz differensiallanuvchi bo'lib, $\lim\limits_{x\to+\infty}(uv)$ mavjud bo'lsa, $\int_a^{+\infty} udv$, $\int_a^{+\infty} vdu$ integrallardan birining yaqinlashuvchiligidan ikkinchisining ham yaqinlashuvchiligi kelib chiqadi va ushbu bo'laklab integrallash formulasi o'rinli bo'ladi:

$$\int_a^{+\infty} udv = (uv)\Big|_a^{+\infty} - \int_a^{+\infty} vdu\text{ , bu yerda } (uv)\Big|_a^{+\infty} = \lim_{x\to+\infty}(uv) - u(a)v(a)\,.$$

10.6- misol. Ushbu integralni hisoblang:

$$\int_5^{+\infty}\left(\frac{4}{x^2-4}+\frac{5}{(x+3)^3}\right)dx\,.$$

Yechilishi. $f(x)=\dfrac{1}{x^2-4}$, $g(x)=\dfrac{1}{(x+3)^3}$ deb belgilaymiz. $f(x)$ va $g(x)$ funksiyalar uchun, mos ravishda,

$$F(x)=\frac{1}{4}\ln\frac{x-2}{x+2},\ G(x)\frac{1}{2(x+3)^2}$$

boshlang'ich funksiyalar mavjud bo'ladi. (10.3) Nyuton — Leybnis formulasi bo'yicha quyidagilarga ega bo'lamiz:

$$\int_5^{+\infty}\frac{dx}{x^2-4}=\frac{1}{4}\ln\frac{x-2}{x+2}\Big|_5^{+\infty}=-\frac{1}{4}\ln\frac{3}{7}=\frac{1}{4}\ln\frac{7}{3},$$

$$\int_5^{+\infty}\frac{dx}{(x+3)^3}=-\frac{1}{2(x+3)^2}\Big|_5^{+\infty}=\frac{1}{128}\,.$$

Demak, xosmas integralning 3- xossasiga asosan, berilgan integralning qiymati:

$$\int_5^{+\infty}\left(\frac{4}{x^2-4}+\frac{5}{(x+3)^3}\right)dx=4\int_5^{+\infty}\frac{dx}{x^2-4}+5\int_5^{+\infty}\frac{dx}{(x+3)^3}=\ln\frac{7}{3}+\frac{5}{128}.$$

10.7- misol. $\int_{\sqrt[4]{2}}^{\infty}\frac{x^3dx}{(x^4+4)^2}$ integralni hisoblang.

Yechilishi. Bunda $f(x)=\frac{x^3}{(x^4+4)^2}$ $[\sqrt[4]{2};+\infty)$ da uzluksiz funksiya, $x=\sqrt[4]{t}$ funksiya esa $[2;+\infty)$ da monoton o‘suvchi va differensiallanuvchi bo‘lgani uchun (10.2) xosmas integralda o‘zgaruvchilarni almashtirish formulasidan foydalanamiz: $x=\sqrt[4]{t}$ deb belgilab olganimiz uchun, $x^3dx=\frac{1}{4}dt$ va yangi chegaralar $\alpha=2,\quad \beta=+\infty$ bo‘ladi. Demak,

$$\int_{\sqrt[4]{2}}^{+\infty}\frac{x^3dx}{(x^4+4)^2}=\frac{1}{4}\int_2^{+\infty}\frac{dt}{(t+2)^2}=-\frac{1}{4}\frac{1}{t+2}\Bigg|_2^{+\infty}=\frac{1}{16}.$$

10.8- misol. $\int_2^{+\infty}\frac{\ln x}{x^3}dx$ integralni hisoblang.

Yechilishi. Bunda $u(x)=\ln x,\quad v(x)=-\frac{1}{2x^2}$ $[2;+\infty)$ da uzluksiz va differensiallanuvchi funksiyalar bo‘lgani uchun xosmas integrallarda bo‘laklab integrallash formulasidan foydalanamiz. Agar $u=\ln x,\ dv=\frac{dx}{x^3}$ deyilsa, $du=\frac{dx}{x},\quad v=-\frac{1}{2x^2}$ bo‘lib, quyidagiga ega bo‘lamiz:

$$\int_2^{+\infty}\frac{\ln x}{x^3}dx=-\frac{\ln x}{2x^2}\Bigg|_2^{+\infty}+\frac{1}{2}\int_2^{+\infty}\frac{dx}{x^3}.$$

Bunda

$$\left(-\frac{\ln x}{2x^2}\right)\Bigg|_2^{+\infty}=\lim_{x\to+\infty}\left(\frac{\ln 2}{8}-\frac{\ln x}{2x^2}\right)=\frac{\ln 2}{8}, \quad \int_2^{+\infty}\frac{dx}{x^3}=\frac{1}{8}$$

bo'lishini topamiz. Demak, $\int_2^{+\infty}\frac{\ln x}{x^3}dx=\frac{1}{16}\ln 4e$.

10.9- misol. Ushbu tengsizlikni isbotlang:

$$0<\int_{\sqrt{2}}^{+\infty}\frac{x}{x^{12}+64}dx<\frac{\pi}{16}.$$

Isboti. Ravshanki, $[\sqrt{2};+\infty)$ dagi barcha x lar uchun

$$0<\frac{x}{x^{12}+64}=\frac{x}{(x^4+4)(x^8-4x^4+16)}<\frac{x}{x^4+4}.$$

Bunda $f(x)=\frac{x}{x^{12}+64}$, $g(x)=\frac{x}{x^4+4}$ bo'ladi. Keyingi tengsizlikni $[\sqrt{2};+\infty)$ da integrallaymiz:

$$0<\int_{\sqrt{2}}^{+\infty}\frac{x}{x^{12}+64}dx<\int_{\sqrt{2}}^{+\infty}\frac{xdx}{x^4+4}.$$

Agar

$$\int_{\sqrt{2}}^{+\infty}\frac{xdx}{x^4+4}=\frac{1}{2}\int_{\sqrt{2}}^{+\infty}\frac{d(x^2)}{(x^2)^2+2^2}=\frac{1}{4}\operatorname{arctg}\frac{x^2}{2}\Bigg|_{\sqrt{2}}^{+\infty}=\frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{16}$$

bo'lishini e'tiborga olsak, $0<\int_{\sqrt{2}}^{+\infty}\frac{xdx}{x^{12}+64}<\frac{\pi}{16}$ bo'ladi.

10.10- misol. Ushbu tengsizlikni isbotlang:

$$\frac{1}{29}<\int_1^{\infty}\frac{x^{30}+1}{x^{60}+1}dx<\frac{1}{29}+\frac{1}{59}.$$

Yechilishi. $\forall x\in[1,+\infty)$ lar uchun

$$\frac{1}{x^{30}}<\frac{x^{30}+1}{1+x^{60}}<\frac{x^{30}+1}{x^{60}}=\frac{1}{x^{30}}+\frac{1}{x^{60}}$$

tengsizlik o'rinli bo'lganligi uchun, 3-, 4- xossalarga asosan,

$$\int_1^{+\infty}\frac{1}{x^{30}}dx<\int_1^{+\infty}\left(\frac{1}{x^{30}}+\frac{1}{x^{60}}\right)dx=\int_1^{+\infty}\frac{dx}{x^{30}}+\int_1^{+\infty}\frac{dx}{x^{60}}. \qquad (*)$$

Bundan

$$\int_1^{+\infty}\frac{dx}{x^{30}}=-\frac{x^{-29}}{29}\Big|_1^{+\infty}=\frac{1}{29},\quad \int_1^{+\infty}\frac{dx}{x^{60}}=-\frac{x^{-59}}{59}\Big|_1^{+\infty}=\frac{1}{59}.$$

Bularni e'tiborga olsak, (*) tengsizlikdan isbot qilinishi talab qilingan tengsizlik kelib chiqadi.

10.3. Chegarasi cheksiz bo'lgan xosmas integrallarning ba'zi bir tatbiqlari.

10.3.1. Xosmas integrallar yordamida yuzni hisoblash. $f(x)$ funksiya $[a,+\infty)$ da aniqlangan, uzluksiz va $\forall x\in[a,+\infty)$ uchun $f(x)\geq 0$ bo'lsin. Unda $D=\{(x,y): a\leq x<+\infty,\ 0\leq y\leq f(x)\}$ sohaning yuzi ushbu $S=\int_a^{+\infty}f(x)dx$ xosmas integral orqali ifoda qilinadi.

10.11- misol. $y=xe^{-x^2}, 0\leq x<+\infty$ funksiyaning grafigi hamda Ox o'qning musbat qismi bilan chegaralangan egri chiziqli trapetsiyaning yuzini hisoblang.

Yechilishi. Izlanayotgan trapetsiyaning yuzi quyidagi xosmas integral orqali hisoblanadi:

$$S=\int_0^{+\infty}xe^{-x^2}dx.$$

Bu integralni xosmas integralning ta'rifiga asosan hisoblaymiz:

$$S=\lim_{t\to+\infty}\int_0^t xe^{-x^2}dx=-\frac{1}{2}\lim_{t\to+\infty}\int_0^t e^{-x^2}d(-x^2)=-\frac{1}{2}\lim_{t\to+\infty}e^{-x^2}\Big|_0^t=\frac{1}{2}\ \text{(kv.birl.)}.$$

10.3.2. Xosmas integral yordamida aylanma jism hajmini hisoblash. Ushbu $D=\{(x,y): a\leq x<+\infty,\ 0\leq y\leq f(x)\}$ egri chiziqli trapetsiyani Ox va Oy o'qlar atrofida aylantirish natijasida hosil bo'lgan aylanma jismning hajmi mos ravishda quyidagi formulalar yordamida hisoblanadi.

$$V_x=\pi\int_a^{+\infty} f^2(x)dx,\quad V_y=2\pi\int_a^{+\infty} xydx.$$

10.12- misol. Ushbu chiziq bilan chegaralangan shaklni Ox o'q atrofida aylantirishdan hosil bo'lgan jismning hajmini toping.

$$y=\frac{1}{\sqrt{4+x^2}},\quad x\in[0;+\infty).$$

Yechilishi. Aylanma jismning hajmini topish uchun

$$V=\pi\int_0^{+\infty}\left(\frac{1}{\sqrt{4+x^2}}\right)^2dx$$

integralni hisoblaymiz. Bunda

$$V=\pi\int_0^{+\infty}\frac{dx}{4+x^2}=\frac{\pi}{2}\operatorname{arctg}\frac{x}{2}\Big|_0^{+\infty}=\frac{\pi}{2}\operatorname{arctg}(+\infty)=\frac{\pi}{4}.$$

10.3.3. Xosmas integrallar yordamida aylanma sirtning yuzini topish. $f(x)$ funksiya $[a,+\infty)$ da aniqlangan, uzluksiz va uzluksiz $f'(x)$ ga ega bo'lib, u $f(x)\geq 0$ bo'lsin. $f(x)$ funksiya grafigini Ox o'q atrofida aylantirish natijasida hosil bo'lgan aylanma sirt yuzi ushbu formula yordamida hisoblanadi:

$$S=2\pi\int_a^{+\infty} f(x)\cdot\sqrt{1+[f'(x)]^2}dx \qquad (*)$$

10.12- misol. $y=e^{-x}$, $x\in[0;+\infty)$ chiziqni Ox o'q atrofida aylantirishdan hosil bo'lgan aylanma sirtning yuzini hisoblang.

Yechilishi. Berilgan chiziqni ifodalovchi funksiyadan hosila olamiz. $y'=-e^{-x}$ ekanligini hisobga olib va (*) formuladan foydalanib, aylanma sirt yuzini hisoblaymiz:

$$S=2\pi\int_0^{+\infty} e^{-x}\sqrt{1+e^{-2x}}dx=-2\pi\int_0^{+\infty}\sqrt{1+e^{-2x}}d(e^{-x})=$$

$$=-2\pi\left(\frac{e^{-x}}{2}\sqrt{1+e^{-2x}}+\frac{1}{2}\ln\left|e^{-x}+\sqrt{1+e^{-2x}}\right|\right)\Bigg|_0^{+\infty}=\pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right),$$

$$S=\pi\left(\sqrt{2}+\ln(1+\sqrt{2})\right)\ \text{(kv.birl.)}.$$

10.4. Xosmas integrallarning yaqinlashuvchiligi haqida taqqoslash teoremalari. Integralning absolut yaqinlashuvchiligi. $f(x)$ funksiya $[a;+\infty)$ oraliqda berilgan bo'lib, ixtiyoriy $x\in[a;+\infty)$ da $f(x)\geq 0$ bo'lsin.

10.1- teorema. $f(x)$ funksiyaning $\int_a^{+\infty} f(x)dx$ xosmas integrali yaqinlashuvchi bo'lishi uchun, $\forall t\in[a;+\infty)$ da

$$\{F(t)\}=\left\{\int_a^t f(x)dx\right\}$$

to'plamning yuqoridan chegaralangan bo'lishi zarur va yetarli.

Agar bu to'plam chegaralanmagan bo'lsa, $\int_a^{+\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'ladi.

10.2- teorema. $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo'lib $\forall x\in[a,+\infty)$ larda $0\leq f(x)\leq g(x)$ munosabat o'rinli va $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo'ladi va agar $\int_a^{+\infty} f(x)dx$ uzoqlashuvchi bo'lsa, $\int_a^{+\infty} g(x)dx$ ham uzoqlashuvchi bo'ladi.

10.3- teorema. $[a,+\infty)$ oraliqda manfiy bo'lmagan $f(x)$ va $g(x)$ funksiyalar berilgan bo'lib, $x\to+\infty$ da $\frac{f(x)}{g(x)}$ nisbatning limiti mavjud va u biror k ga teng bo'lsin:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k\quad(0\leq k\leq+\infty)\ .$$

Bunda:

a) $k<+\infty$ bo'lib, $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo'ladi;

b) $k>0$ bo'lib, $\int_a^{+\infty} g(x)dx$ uzoqlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ham uzoqlashuvchi bo'ladi.

10.2- natija. 10.3- teoremaning shartlarida agar $0<k<+\infty$ bo'lsa, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar bir vaqtda yaqinlashadi yoki bir vaqtda uzoqlashadi.

Xususiy holda, agar $x\to\infty$ da $f\sim g$ bo'lsa, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar bir vaqtda yaqinlashadi yoki uzoqlashadi.

Yuqoridagi teoremada taqqoslanayotgan funksiyalarning o'rniga aniq funksiyalar olib, amaliyotda ko'p qo'llaniladigan alomatlarni keltiramiz.

10.4- teorema. $f(x)$ funksiya λ ning istalgancha katta qiymatlarida

$$f(x)=\frac{\varphi(x)}{x^{\lambda}} \quad (\lambda>0)$$

shaklda tasvirlangan bo'lsin. U holda:

a) agar $\lambda>1$ va $\forall x>x_0$ uchun $\varphi(x)\le C<+\infty$ bo'lsa, $\int_a^{+\infty} f(x)dx$ yaqinlashuvchi bo'ladi;

b) agar $\lambda\le 1$ va $\varphi(x)\ge C>0$ bo'lsa, $\int_a^{+\infty} f(x)dx$ uzoqlashuvchi bo'ladi.

10.5- teorema. Agar $x \to +\infty$ da $f(x)$ funksiya $\frac{1}{x}$ ga nisbatan α $(\alpha > 0)$ tartibli cheksiz bo'lsa, $\int_a^{+\infty} f(x)dx$ integral $\alpha > 1$ bo'lganda yaqinlashuvchi, $\alpha \le 1$ bo'lganda esa uzoqlashuvchi bo'ladi.

10.13- misol. Ushbu $\int_1^{+\infty} \frac{\ln x}{x^{\frac{1}{2}}} dx$ integralning yaqinlashuvchiligini ko'rsating.

Yechilishi. Bunda $f(x) = \frac{\ln x}{x^{\frac{1}{2}}}$, $\varphi(x) = \ln x \ge 0$, $\alpha = \frac{1}{2} < 1$.

Demak, 10.4-teoremaning *b)* bandiga asosan, xosmas integral uzoqlashuvchi.

10.14- misol. $\int_1^{+\infty} \frac{1}{x^2} e^{-\frac{1}{x}} dx$ integralning yaqinlashuvchiligini ko'rsating.

Yechilishi. Bu integral uchun

$$F(t) = \int_1^t \frac{1}{x^2} e^{-\frac{1}{x}} dx = e^{-\frac{1}{t}} - \frac{1}{e}$$

bo'lib, $\forall t \in [1;+\infty)$ da $F(t) = e^{\frac{1}{t}} - e < 1$ bo'ladi. Unda, 10.1-teoremaga asosan, berilgan integral yaqinlashuvchi bo'ladi.

10.15- misol. $\int_0^{+\infty} 2^x x dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. $F(t) = \int_0^t 2^x x dx = \frac{2^t}{\ln 2}(t - \frac{1}{\ln 2}) + \frac{1}{(\ln 2)^2}$. Bu funksiya $t \to +\infty$ da yuqoridan chegaralanmagan. Shuning uchun, 10.1-natijaga ko'ra, berilgan integral uzoqlashuvchi bo'ladi.

10.16- misol. $\int_1^{+\infty}\frac{dx}{\sqrt[8]{9x^3+\lg x}}$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. $\int_1^{+\infty}\frac{dx}{\sqrt[8]{x^3}}$ xosmas integralni qaraymiz. Bu integral uzoqlashuvchi. Endi

$$\lim_{x\to+\infty}\frac{\dfrac{1}{\sqrt[8]{9x^3+\lg x}}}{\dfrac{1}{\sqrt[8]{x^3}}}=\lim_{x\to+\infty}\sqrt[8]{\frac{1}{9+\dfrac{\lg x}{x^3}}}=\frac{1}{\sqrt[4]{3}}$$

bo'lishini e'tiborga olsak, 10.3- teoremaga asosan, berilgan xosmas integral uzoqlashuvchi bo'ladi.

10.17- misol. $\int_1^{+\infty}\frac{\operatorname{arctg}x}{4+x^2}dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Ravshanki, $\forall x\geq 1$ uchun $\operatorname{arctg}x\leq\frac{\pi}{2}$ va $\frac{\operatorname{arctg}x}{4+x^2}\leq\frac{\pi}{2}\frac{1}{4+x^2}$ tengsizliklar o'rinli. Unda $\int_1^{+\infty}\frac{dx}{4+x^2}$ integralning yaqinlashuvchiligini e'tiborga olsak, 10.2- teoremaga asosan, berilgan integral yaqinlashuvchi bo'ladi.

10.18- misol. $\int_1^{+\infty}\frac{\cos^2 3x}{\sqrt[5]{x^7+1}}dx$ integralning yaqinlashuvchiligini ko'rsating.

Yechilishi. Ravshanki, barcha $x\in[1;+\infty)$ lar uchun

$$f(x)=\frac{\cos^2 3x}{\sqrt[5]{x^7+1}}=\frac{\varphi(x)}{x^{\frac{7}{5}}},\ \text{bunda}\ \varphi(x)=\frac{\cos^2 3x}{\sqrt[5]{\dfrac{1}{x^7}+1}}\leq C<+\infty,\quad \lambda=\frac{7}{5}>1$$

bo‘lgani uchun, 10.4- teoremaning bandiga asosan, berilgan xosmas integral yaqinlashuvchi bo‘ladi.

10.19- misol. $\int_1^{+\infty} \frac{xdx}{x^3+1}$ integralning yaqinlashuvchiligini ko‘rsating.

Yechilishi. $\int_1^{+\infty} \frac{dx}{x^2+1}$ xosmas integralni qaraymiz. Bu integralning yaqinlashuvchi ekanligi ravshan, ya’ni

$$\int_1^{+\infty} \frac{dx}{x^2+1} = \operatorname{arctg} x \Big|_1^{+\infty} = \operatorname{arctg}(+\infty) - \operatorname{arctg} 1 = \frac{\pi}{4}.$$

Endi

$$\lim_{x\to+\infty} \frac{\frac{x}{x^3+1}}{\frac{1}{x^2+1}} = \lim_{x\to+\infty} \frac{x^3+x}{x^3+1} = \lim_{x\to+\infty} \frac{1+\frac{1}{x^2}}{1+\frac{1}{x^3}} = 1$$

bo‘lishini e’tiborga olsak, 10.2- natijaga ko‘ra, berilgan xosmas integralning yaqinlashuvchi bo‘lishi kelib chiqadi.

10.5. Ixtiyoriy funksiya xosmas integralining yaqinlashuvchiligi.

10.6- teorema (Koshi teoremasi). Ushbu $\int_a^{+\infty} f(x)dx$ xosmas integralning yaqinlashuvchi bo‘lishi uchun, $\forall\varepsilon > 0$ son olinganda ham, shunday $t_0 = t_0(\varepsilon)$ $(t_0 \geq a)$ son topilib, $t' > t_0$, $t'' > t_0$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ lar uchun

$$\left|F(t'') - F(t')\right| = \left|\int_{t'}^{t''} f(x)dx\right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Xosmas integrallarning uzoqlashuvchiligini isbotlash uchun ko‘pincha quyidagi tasdiqdan foydalaniladi: agar shunday $\varepsilon_0 > 0$

son topilib va barcha $t \geq a$ lar uchun $\exists t' > t, t'' > t$ mavjud bo'lib,

$$\left| \int_{t'}^{t''} f(x)dx \right| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\int_a^{+\infty} f(x)dx$ integral uzoqlashuvchi bo'ladi.

10.20- misol. $\int_1^{+\infty} \frac{\cos 2^x}{x^2} dx$ xosmas integralning yaqinlashuvchi ekanligini Koshi kriteriysidan foydalanib ko'rsating.

Yechilishi. $\forall \varepsilon > 0$ berilgan songa ko'ra, $\exists t_0 = t_0(\varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}}$ $(t_0 > 1)$ son topilib, $t' = 2\pi n > t_0$, $t'' = 2\pi n + 2\pi > t_0$ $(n \in N)$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ lar uchun

$$|F(t'') - F(t')| = \left| \int_{t'}^{t''} \frac{\cos 2^x}{x^2} dx \right| = \left| \int_{2\pi \cdot n}^{2\pi \cdot n + 2\pi} \frac{\cos 2^x dx}{x^2} \right| \leq \int_{2\pi \cdot n}^{2\pi \cdot n + 2\pi} \frac{1}{x^2} dx =$$

$$= -\frac{1}{x}\Bigg|_{2\pi \cdot n}^{2\pi \cdot n + 2\pi} = \frac{1}{2\pi n} - \frac{1}{2\pi n + 2\pi} = \frac{2\pi}{2\pi n \cdot (2\pi n + 2\pi)} =$$

$$= \frac{1}{n 2\pi (n+1)} < \frac{1}{2\pi n^2} < \varepsilon$$

tengsizlik o'rinli bo'ladi. Demak, Koshi kriteriysiga asosan, berilgan xosmas integral yaqinlashuvchi bo'ladi.

10.21- misol. $\alpha \leq 1$ uchun $\int_1^{+\infty} \frac{\sin^2 x}{x^\alpha} dx$ integralning uzoqlashuvchi ekanligini isbotlang.

Isboti. $t \in (1; +\infty)$ bo'lsin. n natural sonni shunday tanlaymizki, $\pi \cdot n > t$ tengsizlik o'rinli bo'lsin, $t_1 = \pi \cdot n$ va $t_2 = 2\pi \cdot n$ deb olaylik. U holda

$$\left|\int_{t_1}^{t_2}\frac{\sin^2 xdx}{x^\alpha}\right| = \int_{\pi n}^{2\pi n}\frac{\sin^2 xdx}{x^\alpha} \geq$$

$$\geq \int_{\pi\cdot n}^{2\pi\cdot n}\frac{\sin^2 x}{x}dx \geq \frac{1}{2n\pi}\int_{\pi\cdot n}^{2\pi\cdot n}\sin^2 xdx = \frac{1}{2n\pi}\int_{\pi\cdot n}^{2\pi\cdot n}\frac{1-\cos 2x}{2} = \frac{1}{4}.$$

Shunday qilib, shunday $\varepsilon_0 = \frac{1}{4}$ son mavjud bo'lib, barcha $t > 1$ lar uchun hamda $t_1 = n\pi > t$ va $t_2 = 2n\pi > t$ sonlar uchun

$$\left|\int_{t_1}^{t_2}\frac{\sin^2 x}{x^\alpha}dx\right| \geq \varepsilon_0$$

bo'ladi. Demak, $\alpha \leq 1$ uchun berilgan integral uzoqlashuvchi.

10.6. Absolut va shartli yaqinlashuvchi xosmas integrallar.

10.7- teorema. Agar $\int_a^{+\infty}|f(x)|dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^{+\infty}f(x)dx$ integral ham yaqinlashuvchi bo'ladi va $\left|\int_a^{+\infty}f(x)dx\right| \leq \int_a^{+\infty}|f(x)|dx$ tengsizlik o'rinli.

10.1- eslatma. Ushbu $\int_a^{+\infty}f(x)dx$ integralning yaqinlashuvchiligidan har doim ham $\int_a^{+\infty}|f(x)|dx$ integralning yaqinlashuvchiligi kelib chiqavermaydi.

10.4- ta'rif. Agar $\int_a^{+\infty}|f(x)|dx$ yaqinlashuvchi bo'lsa, $\int_a^{+\infty}f(x)dx$ *absolut yaqinlashuvchi integral*, $f(x)$ funksiya esa $[a;+\infty)$ da *absolut integrallanuvchi funksiya* deyiladi.

10.5- ta'rif. Agar $\int_a^{+\infty} f(x)dx$ yaqinlashuvchi bo'lib, $\int_a^{+\infty} |f(x)|dx$ uzoqlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ *shartli yaqinlashuvchi integral* deyiladi.

10.22- misol. Ushbu integralni absolut va shartli yaqinlashishga tekshiring:

$$\int_1^{+\infty} \frac{\cos x}{x^\alpha} dx .$$

Yechilishi. *a)* $\alpha > 1$ bo'lsin, u holda $\forall x \in [1;+\infty)$ uchun $\left|\frac{\cos x}{x^\alpha}\right| \le \frac{1}{x^\alpha}$ bo'ladi. Ravshanki, $\int_1^{+\infty} \frac{dx}{x^\alpha}$ yaqinlashuvchi integraldir. Unda taqqoslash haqidagi 3.2-teoremaga asosan,

$$\int_1^{+\infty} \frac{|\cos x|}{x^\alpha} dx$$

integral yaqinlashuvchi. 3.7- teoremadan esa

$$\int_1^{+\infty} \frac{\cos x}{x^\alpha} dx$$

integralning yaqinlashuvchiligi kelib chiqadi. Demak, berilgan integral absolut yaqinlashuvchi.

b) $0 < \alpha \le 1$ bo'lsin, u holda $\int_1^{+\infty} \frac{\cos x}{x^\alpha} dx$ integralni bo'laklab integrallaymiz:

$$\int_1^{+\infty} \frac{\cos x}{x^\alpha} dx = -\frac{\sin x}{x^\alpha}\Bigg|_1^{+\infty} + \alpha \int_1^{+\infty} \frac{\sin x}{x^{\alpha+1}} dx$$

bunda $\lim\limits_{x \to +\infty} \frac{\sin x}{x^\alpha} = 0$, $\int_1^{+\infty} \frac{\sin x}{x^{\alpha+1}} dx$ integral esa absolut yaqinlashuvchidir. Demak, berilgan integral yaqinlashuvchi.

Endi $\int\limits_1^{+\infty}\left|\frac{\cos x}{x^\alpha}\right|dx=\int\limits_1^{+\infty}\frac{|\cos x|}{x^\alpha}dx$ integralni qaraymiz. Ravshanki,

$|\cos x|\geq\cos^2 x=\frac{1}{2}(1+\cos 2x)$, unda ixtiyoriy $t>1$ lar uchun

$$\int\limits_1^t\left|\frac{\cos x}{x^\alpha}\right|dx\geq\frac{1}{2}\int\limits_1^t\frac{dx}{x^\alpha}+\frac{1}{2}\int\limits_1^t\frac{\cos 2x}{x^\alpha}dx\,. \qquad (*)$$

Ma'lumki,

$$\lim_{t\to+\infty}\int\limits_1^t\frac{dx}{x^\alpha}=\int\limits_1^{+\infty}\frac{dx}{x^\alpha}=+\infty,$$

$$\lim_{t\to+\infty}\int\limits_1^t\frac{\cos 2x}{x^\alpha}dx=\lim_{t\to+\infty}\left(\frac{1}{2}\frac{\sin 2t}{t^\alpha}-\frac{1\sin 2}{2}\right)+\frac{\alpha}{2}\int\limits_1^{+\infty}\frac{\sin 2x}{x^{\alpha+1}}dx\,.$$

Agar $\int\limits_1^{+\infty}\frac{dx}{x^\alpha}$ ning uzoqlashuvchiligini, $\int\limits_1^{+\infty}\frac{\cos 2x}{x^\alpha}dx$ ning esa yaqinlashuvchiligini e'tiborga olsak, u holda (*) tenglikda $x\to+\infty$ da limitga o'tib, $\int\limits_1^{+\infty}\left|\frac{\cos x}{x^\alpha}\right|dx$ xosmas integralning uzoqlashuvchiligini topamiz.

Demak, berilgan integral shartli yaqinlashuvchi.

c) $\alpha\leq 0$ bo'lsin. Koshi teoremasidan foydalanib, berilgan xosmas integralning uzoqlashuvchiligini ko'rsatamiz. $t>1$ uchun $n\in N$ son shunday tanlansinki, $2\pi n>t$ tengsizlik bajarilsin. $t'=2\pi n+\frac{\pi}{6}$, $t''=2\pi\cdot n+\frac{\pi}{3}$ larni tanlaymiz. Shartga ko'ra $x\in[t',t'']$ lar uchun $\cos x\geq\frac{1}{2}$ va $\frac{1}{x^\alpha}\geq 1$ $(x\geq 1$ va $\alpha<0)$, tengsizlik o'rinli bo'ladi, u holda

$$\left|\int\limits_{t'}^{t''}\frac{\cos x}{x^\alpha}dx\right|=\int\limits_{t'}^{t''}\frac{\cos x}{x^\alpha}dx\geq\frac{1}{2}\int\limits_{2\pi n+\frac{\pi}{6}}^{2\pi n+\frac{\pi}{3}}dx=\frac{\pi}{12}\,.$$

Demak, shunday $\varepsilon_0 = \frac{\pi}{12}$ son mavjudki, barcha $t > 1$ lar uchun

$t' = 2\pi n + \frac{\pi}{6}$, $t'' = 2\pi \cdot n + \frac{\pi}{3}$ sonlar mavjud bo'lib, quyidagi

$\left| \int_{t'}^{t''} \frac{\cos x}{x^\alpha} dx \right| \geq \varepsilon_0$ tengsizlik o'rinli bo'ladi.

Shunday qilib, berilgan integral $\alpha > 1$ da absolut yaqinlashuvchi, $0 < \alpha \leq 1$ da shartli yaqinlashuvchi, $\alpha \leq 0$ da esa uzoqlashuvchi.

10.23- misol. $\int_1^{+\infty} \frac{\cos x}{x^2 + 9} dx$ integralning absolut yaqinlashuvchiligini ko'rsating.

Yechilishi. Integral ostidagi funksiya uchun ixtiyoriy $x \in [1; +\infty)$ da

$$\left| \frac{\cos x}{x^2 + 9} \right| \leq \frac{1}{x^2 + 9}$$

tengsizlik o'rinli bo'ladi. Ravshanki, $\int_1^{+\infty} \frac{dx}{x^2 + 9}$ yaqinlashuvchi integraldir. Unda, 10.2- teoremaga asosan, $\int_1^{+\infty} \left| \frac{\cos x}{x^2 + 9} \right| dx$ integral ham yaqinlashuvchi bo'ladi. 10.7- teoremadan esa $\int_1^{+\infty} \frac{\cos x}{x^2 + 9} dx$ integralning yaqinlashuvchiligi kelib chiqadi. Demak, berilgan integral absolut yaqinlashuvchi.

10.7. Xosmas integrallar yaqinlashuvchiligining yetarli shartlari.

10.8- teorema (Dirixle teoremasi). $f(x)$ va $g(x)$ funksiyalar $[a; +\infty)$ da berilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a; +\infty)$ da uzluksiz va uning shu oraliqdagi boshlang'ich funksiyasi $F(x)$ $(F'(x) = f(x))$ chegaralangan;

2) $g(x)$ funksiya $[a; +\infty)$ da uzluksiz $g'(x)$ hosilaga ega;

3) $g(x)$ funksiya $[a;+\infty)$ da monoton;

4) $\lim\limits_{x\to+\infty} g(x)=0$.

U holda, $\int\limits_a^{+\infty} f(x)g(x)dx$ integral yaqinlashuvchi bo'ladi.

10.9- teorema (Abel teoremasi). $f(x)$ va $g(x)$ funksiyalar $[a;+\infty)$ da berilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a;+\infty)$ da uzluksiz va $\int\limits_a^{+\infty} f(x)dx$ integral yaqinlashuvchi;

2) $g(x)$ funksiya $[a;+\infty)$ da chegaralangan;

3) $g(x)$ funksiya uzluksiz differensiallanuvchi va $[a;+\infty)$ da monoton bo'lsin.

U holda $\int\limits_a^{+\infty} f(x)g(x)dx$ integral yaqinlashuvchi bo'ladi.

10.24- misol. $\int\limits_1^{+\infty} \sin x^2 dx$ integralni yaqinlashishga tekshiring.

Yechilishi. $x^2=t$ deb belgilaymiz, u holda $\int\limits_1^{+\infty} \sin x^2 dx = \int\limits_1^{+\infty} \frac{\sin t}{2\sqrt{t}} dt$ bo'ladi. Integral ostidagi funksiyani $\frac{\sin t}{2\sqrt{t}} = \sin t \cdot \frac{1}{2\sqrt{t}} = f(t)g(t)$ ko'rinishda yozamiz. Bu yerda $f(t)=\sin t$, $g(t)=\frac{1}{2\sqrt{t}}$. $f(t)$ va $g(t)$ funksiyalar Dirixle teoremasining barcha shartlarini qanoatlantiradi:

1) $f(t)=\sin t$ funksiya $[1;+\infty)$ da uzluksiz va uning $F(t)=-\cos t$ boshlang'ich funksiyasi chegaralangan;

2) $g(t)=\frac{1}{2\sqrt{t}}$ funksiya $[1;+\infty)$ da uzluksiz $g'(t)=-\frac{1}{4\sqrt{t^3}}$ hosilaga ega;

3) $g(t)=\frac{1}{2\sqrt{t}}$ funksiya $[1;+\infty)$ da kamayuvchi;

4) $\lim\limits_{t\to+\infty} g(t)=\lim\limits_{t\to+\infty}\frac{1}{2\sqrt{t}}=0$.

Demak, 10.8- teoremaga ko'ra, berilgan xosmas integral yaqinlashuvchi.

10.25- misol. $\int\limits_{1}^{+\infty}\frac{\cos x}{x^{\alpha}}\operatorname{arctg}x\,dx$ xosmas integralning $\alpha>0$ da yaqinlashuvchiligini ko'rsating.

Yechilishi. Berilgan xosmas integralning yaqinlashuvchiligini Abel teoremasidan foydalanib ko'rsatamiz. $f(x)=\frac{\cos x}{x^{\alpha}}$, $g(x)=\operatorname{arctg}x$ deb belgilaymiz.

1) $f(x)=\frac{\cos x}{x^{\alpha}}$ funksiya $[1;+\infty)$ da uzluksiz va $\alpha>0$ bo'lganda yuqoridagi 10.22- misolga asosan, $\int\limits_{1}^{+\infty}\frac{\cos x}{x^{\alpha}}dx$ integral yaqinlashuvchi;

2) $g(x)=\operatorname{arctg}x$ funksiya $[1;+\infty)$ da chegaralangan, ya'ni $|\operatorname{arctg}x|<\frac{\pi}{2}$;

3) $\operatorname{arctg}x$ funksiya $[1;+\infty)$ da uzluksiz differensiallanuvchi $\left(g'(x)=\frac{1}{1+x^2}>0\right)$ va monoton o'suvchi.

Demak, berilgan xosmas integral Abel teoremasining hamma shartlarini qanoatlantirgani uchun u yaqinlashuvchi bo'ladi.

10.8. Xosmas integrallarni yaqinlashishga tekshirishda funksiyaning bosh qismini ajratish usuli. Bosh qismni ajratish usuli quyidagicha ifodalanadi: agar integral ostidagi $f(x)$ funksiyani $x\to+\infty$ da $f(x)=g(x)+R(x)$ ko'rinishda tasvirlash mumkin bo'lsa, bunda $R(x)$ – absolut integrallanuvchi funksiya bo'lsa, u holda $f(x)$ va $g(x)$ funksiyalar bir vaqtda yoki absolut integrallanuvchi, yoki shartli integrallanuvchi bo'ladi, yoki integrallanuvchi bo'lmaydi.

10.26-misol. $\int\limits_{1}^{+\infty} \operatorname{arctg}\frac{\cos^2 x}{\sqrt[3]{x^2}}dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiyani

$$\operatorname{arctg}\frac{\cos^2 x}{\sqrt[3]{x^2}} = \frac{1}{2}\frac{1}{\sqrt[3]{x^2}} + \frac{\cos^2 x}{2\sqrt[3]{x^2}} + R(x), \quad x \to +\infty$$

ko'rinishda tasvirlaymiz, bunda $|R(x)| \le \frac{1}{3x^2}$, $R(x)$ funksiya absolut integrallanuvchidir. Berilgan integralning yaqinlashish xarakteri

$$\int\limits_{1}^{+\infty}\left(\frac{1}{2\sqrt[3]{x^2}} + \frac{\cos 2x}{2\sqrt[3]{x^2}}\right)dx$$

integralga bog'liq. Ravshanki, oxirgi integral uzoqlashuvchi, chunki $\frac{\cos 2x}{\sqrt[3]{x^2}}$ funksiya integrallanuvchi, $\frac{1}{\sqrt[3]{x^2}}$ funksiya esa integrallanuvchi emas. Demak, berilgan integral uzoqlashuvchi ekan.

10.26- misol. $\int\limits_{1}^{+\infty} \sin\left(\frac{\cos x}{\sqrt{x^5}}\right)dx$ integralni absolut va shartli yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiyani $x \to +\infty$ da

$$\sin\left(\frac{\cos x}{\sqrt{x^5}}\right) = \frac{\cos x}{\sqrt{x^5}} + R(x)$$

ko'rinishda tasvirlaymiz, bunda barcha $x > 1$ lar uchun $|R(x)| \le \frac{1}{3!x^5\sqrt{x^5}}$ tengsizlik o'rinli bo'ladi. Demak, $\int\limits_{1}^{+\infty} R(x)dx$ absolut yaqinlashuvchi. Yuqoridagi 10.3- misolga ko'ra, ($\alpha = \frac{5}{2}$) $\int\limits_{1}^{+\infty} \frac{\cos x}{\sqrt{x^5}}dx$ integral absolut yaqinlashuvchi. Shuning uchun, berilgan integral ham absolut yaqinlashuvchi.

10.27- misol. Ushbu xosmas integralni yaqinlashishga tekshiring:

$$\int_0^{+\infty} x^2 \sin\left(\frac{\cos x^3}{1+x}\right)dx .$$

Yechilishi. Berilgan integralda $t = x^3$ almashtirishni bajaramiz:

$$x = t^{\frac{1}{3}}, \quad dx = \frac{1}{3}t^{-\frac{2}{3}}dt; \; x \to 0 \text{ da } t \to 0 \text{ va } x \to +\infty \text{ da } t \to +\infty,$$

$$\int_0^{+\infty} x^2 \sin\left(\frac{\cos x^3}{1+x}\right)dx = \frac{1}{3}\int_0^{+\infty} \sin\left(\frac{\cos t}{1+t^{1/3}}\right)dt .$$

Integral ostidagi funksiyani $t \to +\infty$ da

$$\sin\left(\frac{\cos t}{1+t^{1/3}}\right) = \frac{\cos t}{1+t^{1/3}} + R(t)$$

ko‘rinishda tasvirlaymiz, bu yerda $R(t) = -\frac{\cos^3 t}{\left(1+t^{1/3}\right)^3}$. Barcha $t \in [0,+\infty)$ uchun $|R(t)| \leq \frac{1}{(1+t^{1/3})^3}$. $\int_0^{+\infty} \frac{dt}{(1+t^{1/3})^3}$ integral absolut yaqinlashuvchi.

Demak, $R(x)$ funksiya absolut integrallanuvchi ekan. Berilgan integralning yaqinlashish xarakteri $\int_0^{+\infty} \frac{\cos t}{1+t^{1/3}}dt$ integralning yaqinlashish xarakteriga bog‘liq. Lekin $\int_0^{+\infty} |\cos t| \frac{dt}{1+t^{1/3}}$ integral uzoqlashuvchi, $\int_0^{+\infty} \frac{\cos t}{1+t^{1/3}}dt$ integral esa Dirixle alomatiga asosan, yaqinlashuvchi. Shunday qilib, berilgan xosmas integral shartli yaqinlashuvchi bo‘ladi.

Mustaqil yechish uchun misollar

Xosmas integrallarning yaqinlashuvchi ekanligini ko‘rsating va qiymatini toping:

10.1. $\int_{1}^{+\infty}\frac{dx}{\sqrt[3]{x^5}}$. **10.2.** $\int_{0}^{+\infty}e^{-5x}dx$. **10.3.** $\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}$.

10.4. $\int_{0}^{+\infty}xe^{-x^2}dx$. **10.5** $\int_{1}^{+\infty}\frac{\mathrm{arctg}x}{1+x^2}dx$. **10.6.** $\int_{1}^{+\infty}\frac{dx}{(x+2)\ln^2(x+2)}$.

10.7. $\int_{0}^{+\infty}\frac{2xdx}{(x^2+1)^3}$. **10.8.** $\int_{-\infty}^{0}\frac{dx}{(x+1)^3}$. **10.9.** $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+x+1)^2}$.

10.10. $\int_{-\infty}^{0}a^x dx, \quad (a>1)$.

Xosmas integrallarning uzoqlashuvchi ekanligini isbotlang:

10.11. $\int_{1}^{+\infty}\frac{dx}{\sqrt[3]{x}}$. **10.12.** $\int_{0}^{+\infty}\frac{xdx}{x^2+5}$. **10.13.** $\int_{0}^{+\infty}\cos xdx$.

10.14. $\int_{1}^{+\infty}\frac{dx}{\sqrt{16+x^2}}$. **10.15.** $\int_{e}^{+\infty}\frac{dx}{(x+1)\ln(x+1)}$. **10.16.** $\int_{1}^{+\infty}4^x dx$.

10.17. $\int_{-\infty}^{0}\frac{x+1}{x^2+1}dx$. **10.18.** $\int_{1}^{+\infty}xe^{x^2}dx$.

Xosmas integrallarni hisoblang:

10.19. $\int_{\sqrt{2}}^{+\infty}\frac{xdx}{(x^2+1)^3}$. **10.20.** $\int_{1}^{+\infty}\frac{dx}{(1+x)\sqrt{x}}$. **10.21.** $\int_{0}^{+\infty}\frac{dx}{e^x+\sqrt{e^x}}$.

10.22. $\int_{\sqrt{2}}^{+\infty}\frac{dx}{(x-1)\sqrt{x^2-2}}$. **10.23.** $\int_{0}^{+\infty}\frac{x^2+1}{x^4+1}dx$. **10.24.** $\int_{0}^{+\infty}x^{10}e^{-x}dx$.

10.25. $\int_{0}^{+\infty}\frac{\ln x}{1+x^2}dx$. **10.26.** $\int_{1}^{+\infty}\frac{\mathrm{arctg}x}{x^2}dx$.

10.27. $\int_0^{+\infty} e^{-ax}\sin bxdx,\quad a>0$. **10.28.** $\int_0^{+\infty} e^{-ax}\cos bxdx,\quad a>0$.

10.29. $\int_1^{+\infty} x^n e^{-x}dx$.

Funksiyalarning grafiklari va abssissalar o'qi bilan chegaralangan shakllarning yuzini hisoblang:

10.30. $f(x)=\dfrac{1}{4+x^2},\quad -\infty<x<+\infty$.

10.31. $f(x)=x^2e^{-x^3},\quad 0\le x<+\infty$.

10.32. $f(x)=\dfrac{\sqrt{x}}{(1+x)^2},\quad 1\le x<+\infty$.

10.33. $f(x)=\dfrac{1}{\sqrt{1+e^x}},\quad 0\le x<+\infty$.

10.34. $f(x)=\dfrac{x\sqrt{x}}{1+x^5},\quad 0\le x<+\infty$.

10.35. $f(x)=|\sin x|e^{-x},\quad 0\le x<+\infty$.

Chiziqlar bilan chegaralangan shakllarni Ox o'q atrofida aylantirish natijasida hosil bo'lgan jismning hajmini toping:

10.36. $y=\dfrac{1}{x},\quad x\in[1;+\infty)$. **10.37.** $y=xe^{-x},\quad x\in[0;+\infty)$.

10.38. $y=e^{-x}\sqrt{\sin x},\quad x\in[0;+\infty)$.

10.39. $y=x^2e^{-x^2},\quad x\in(-\infty,+\infty)$.

10.40. $y=2\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)\quad x\in[1;+\infty)$.

Tengsizliklarni isbotlang:

10.41. $0<\int_{10}^{+\infty}\dfrac{x^2dx}{x^4+x+1}<0,1$. **10.42.** $0,25<\int_1^{+\infty}\dfrac{x^6+1}{x^{11}+1}dx<0,35$.

10.43. $\left|\int\limits_0^{+\infty} \frac{\cos 4x}{x^2+4}dx\right| < \frac{\pi}{4}$. **10.44.** $0 < \int\limits_{10}^{+\infty} \frac{x^{20}+1}{x^{40}+1}dx - \frac{20}{19} < 0{,}05$.

10.45. $0 < \int\limits_2^{+\infty} e^{-x}dx < \frac{1}{4e^4}$.

10.46. $0 < \int\limits_0^{+\infty} e^{-x}dx - \int\limits_0^3 e^{-x}dx - < 0{,}5^{11}$.

Integrallarning yaqinlashuvchiligini isbotlang.

10.47. $\int\limits_0^{+\infty} \frac{x^3}{x^5+1}dx$. **10.48.** $\int\limits_0^{+\infty} \frac{x}{\sqrt[3]{1+x^7}}dx$. **10.49.** $\int\limits_2^{+\infty} (\cos\frac{x}{2} - 1)dx$.

10.50. $\int\limits_0^{+\infty} \left(e^{-\frac{1}{x^2}} - e^{-\frac{4}{x^2}} \right)dx$. **10.51.** $\int\limits_e^{+\infty} \frac{dx}{x \cdot \ln^5 x}$.

10.52. $\int\limits_0^{+\infty} \frac{\ln(1+x+x\sqrt{x})}{\sqrt{x^3}}dx$. **10.53.** $\int\limits_1^{+\infty} \frac{\sin^4 x}{\sqrt{x^3+1}}dx$.

10.54. $\int\limits_0^{+\infty} \frac{x^3+7}{x^5-x^2+2}dx$. **10.55.** $\int\limits_0^{+\infty} \frac{\sin^2 x}{x^2}dx$.

10.56. $\int\limits_0^{+\infty} \left(\frac{1}{x\text{sh}x} - \frac{1}{x} \right)dx$. **10.57.** $\int\limits_0^{+\infty} \frac{\text{arctg}2x}{x\sqrt{x}}dx$.

Integrallarning uzoqlashuvchiligini isbotlang.

10.58. $\int\limits_0^{+\infty} \frac{x^3+1}{x^4}dx$. **10.59.** $\int\limits_0^{+\infty} \frac{xdx}{\sqrt[3]{x^5+2}}$. **10.60.** $\int\limits_0^{+\infty} \frac{\sin^2 x}{x}dx$.

10.61. $\int\limits_0^{+\infty} \frac{\sin\frac{1}{x}}{\left(x - \cos\frac{\pi}{x}\right)^2}dx$. **10.62.** $\int\limits_0^{+\infty} \frac{xdx}{1+x^2\sin^2 x}$.

10.63. $\int\limits_0^{+\infty} \frac{1}{\sqrt{x}} \text{arctg}\frac{x}{2+x}dx$.

Integrallarni absolut va shartli yaqinlashuvchilikka tekshiring:

10.64. $\int_0^{+\infty} x\cos x^4 dx$. **10.65.** $\int_0^{+\infty} \frac{\sin(\ln x)}{\sqrt{x}} dx$. **10.66.** $\int_0^{+\infty} \frac{x\cos 7x}{x^2+2x+2} dx$.

10.67. $\int_0^{+\infty} \cos^3(x^2+2x)dx$. **10.68.** $\int_1^{+\infty} \operatorname{arctg}\frac{\cos x}{\sqrt[3]{x^2}} dx$.

10.69. $\int_1^{+\infty} \frac{\sin x}{x^4} dx$. **10.70.** $\int_1^{+\infty} \frac{\sqrt{x}\cos x}{x+100} dx$. **10.71.** $\int_0^{+\infty} \frac{\sin(\sin x)}{\sqrt{x}} dx$.

10.72. $\int_1^{+\infty} \ln^2\left(1+\frac{1}{x}\right)\sin x dx$. **10.73.** $\int_1^{+\infty} \frac{\cos(1+2x)}{\left(\sqrt{x}-\ln x\right)^3} dx$.

10.74. $\int_1^{+\infty} \frac{\cos x dx}{x^\alpha+\ln x}$. **10.75.** $\int_1^{+\infty} \frac{\sin(x+x^2)}{x^\alpha} dx$.

10.76. $\int_1^{+\infty} \sin(x+\frac{1}{x})\frac{dx}{x^\alpha}$. **10.77.** $\int_1^{+\infty} \frac{\cos\sqrt{x}}{x^\alpha \ln x} dx$.

10.78. $\int_1^{+\infty} \frac{\sin(\ln x)}{x^\alpha}\sin x dx$.

10.79. k ning qanday qiymatlarida $\int_0^{+\infty} x^k dk$ integral yaqinlashuvchi bo'lishi mumkin?

10.80. k va t ning qanday qiymatlarida $\int_0^{+\infty} \frac{x^k}{1+x^t} dx$ integral yaqinlashuvchi bo'ladi?

10.81. k ning qanday qiymatida: 1) $\int_2^{+\infty} \frac{dx}{x^k \ln x}$, 2) $\int_2^{+\infty} \frac{dx}{x(\ln x)^k}$ integrallar yaqinlashuvchi bo'ladi?

Misollarning javoblari

10.1. $\frac{3}{2}$. **10.2.** $\frac{1}{5}$. **10.3.** π. **10.4.** $\frac{1}{2}$. **10.5.** $\frac{3\pi^2}{32}$. **10.6.** $\frac{1}{\ln 3}$. **10.7.** $\frac{1}{2}$. **10.8.** $-\frac{1}{2}$. **10.9.** $\frac{4\pi}{3\sqrt{3}}$. **10.10.** $\frac{1}{\ln a}$. **10.19.** $\frac{1}{36}$. **10.20.** $\frac{\pi}{2}$. **10.21.** $2(1-\ln 2)$. **10.22.** $\frac{3\pi}{4}$. **10.23.** $\frac{\pi\sqrt{2}}{4}$. **10.24.** 10!. **10.25.** 0. **10.26.** $\frac{\pi}{4}+\frac{\ln 2}{2}$. **10.27.** $\frac{b}{a^2+b^2}$. **10.28.** $\frac{a}{a^2+b^2}$. **10.29.** $n!$ **10.30.** $\frac{\pi}{2}$. **10.31.** $\frac{1}{3}$. **10.32.** $\frac{1}{2}+\frac{\pi}{4}$. **10.33.** $2\ln(1+\sqrt{2})$. **10.34.** $\frac{\pi}{5}$. **10.35.** $\frac{e^{\pi}+1}{2(e^{\pi}-1)}$. **10.36.** π. **10.37.** $\frac{\pi}{4}$. **10.38.** $\frac{\pi}{5(1-e^{-2\pi})}$. **10.39.** $\frac{3\pi\sqrt{2\pi}}{32}$. **10.40.** $\frac{4\pi}{3}$. **10.64.** Shartli yaqinlashuvchi. **10.65.** Shartli yaqinlashuvchi. **10.66.** Shartli yaqinlashuvchi. **10.67.** Shartli yaqinlashuvchi. **10.68.** Absolut yaqinlashuvchi. **10.69.** Absolut yaqinlashuvchi. **10.70.** Shartli yaqinlashuvchi. **10.71.** Shartli yaqinlashuvchi. **10.72.** Absolut yaqinlashuvchi. **10.73.** Absolut yaqinlashuvchi. **10.74.** $\alpha>1$ da absolut yaqinlashuvchi, $\alpha\le 1$ da shartli yaqinlashuvchi. **10.75.** $\alpha>1$ da absolut yaqinlashuvchi, $-1<\alpha\le 1$ da shartli yaqinlashuvchi. **10.76.** $\alpha>1$ da absolut yaqinlashuvchi, $0<\alpha\le 1$ da shartli yaqinlashuvchi. **10.77.** $\alpha>1$ da absolut yaqinlashuvchi, $\frac{1}{2}\le\alpha\le 1$ da shartli yaqinlashuvchi. **10.78.** $\alpha>1$ da absolut yaqinlashuvchi, $0<\alpha\le 1$ da shartli yaqinlashuvchi. **10.79.** k ning har qanday qiymatida uzoqlashuvchi. **10.80.** $k>-1$ va $t>k+1$ bo'lganda yaqinlashuvchi. **10.81.** 1) $k>1$ da yaqinlashuvchi, $k\le 1$ da uzoqlashuvchi; 2) agar $k\le 1$ bo'lsa, $I=\frac{1}{(k-1)(\ln 2)^{k-1}}$; $k\ge 1$ da uzoqlashuvchi.

11- §. Chegaralanmagan funksiyaning xosmas integrallari

11.1. Chegaralanmagan funksiyaning xosmas integrallari tushunchasi

11.1- ta'rif. $f(x)$ funksiya chekli $[a;b]$ oraliqda berilgan bo'lib, $[a;c)$, $(c;b]$ oraliqlarda chegaralanmagan bo'lsin. Bu holda c nuqta $f(x)$ funksiya uchun *maxsus nuqta* deyiladi.

$f(x)$ funksiya $[a;b)$ oraliqda berilgan bo'lib, u $[a;b-\eta]$ $(0<\eta<b-a)$ oraliqda xos ma'noda (Riman ma'nosida) integrallanuvchi, ya'ni $F(\eta)=\int\limits_a^{b-\eta} f(x)dx$ integral mavjud bo'lsin, $[b-\eta;\, b)$ da esa integrallanuvchi bo'lmasin, ya'ni $\forall\eta>0$ uchun $f(x)$ chegaralanmagan bo'lsin.

11.2- ta'rif. Agar $\eta\to 0$ da $F(\eta)$ funksiyaning (chekli yoki cheksiz) $\lim\limits_{\eta\to 0} F(\eta)$ limiti mavjud bo'lsa, bu limit $f(x)$ funksiyaning $[a;b)$ oraliqda olingan *2- tur xosmas integrali* deyiladi va u

$$\int\limits_a^b f(x)dx \tag{11.1}$$

kabi belgilanadi:

$$\int\limits_a^b f(x)dx=\lim_{\eta\to 0} F(\eta)=\lim_{\eta\to 0}\int\limits_a^{b-\eta} f(x)dx.$$

11.3- ta'rif. Agar $\eta\to 0$ da $F(\eta)$ funksiyaning limiti mavjud bo'lib, u chekli bo'lsa, (11.1) xosmas integral *yaqinlashuvchi,* $f(x)$ esa $[a;b)$ da *xosmas ma'noda integrallanuvchi funksiya* deyiladi.

Agar $\eta\to 0$ da $F(\eta)$ funksiyaning limiti cheksiz bo'lsa, u holda (11.1) xosmas integral *uzoqlashuvchi* deyiladi.

11.1- eslatma. $\eta\to 0$ da $F(\eta)$ funksiyaning limiti mavjud bo'lmaganda ham (11.1) integral uzoqlashuvchi bo'ladi, deb kelishib olamiz.

Shuningdek, a nuqta $f(x)$ funksiyaning maxsus nuqtasi bo'lganda ham $(a,b]$ oraliq bo'yicha olingan xosmas integral yuqoridagidek ta'riflanadi. a va b nuqtalar bir vaqtda berilgan funksiyaning maxsus nuqtalari bo'lganda $(a;b)$ oraliq bo'yicha xosmas integral quyidagicha ta'riflanadi:

$$\int_a^b f(x)dx = \lim_{\substack{\varepsilon \to 0 \\ \eta \to 0}} \int_{a+\varepsilon}^{b-\eta} f(x)dx .$$

11.1- misol. $I = \int_0^1 \frac{dx}{x^\alpha}$ integralni yaqinlashishga tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=0$ nuqta maxsus nuqtadan iborat. Ushbu $F(\mu) = \int_\mu^1 \frac{dx}{x^\alpha}$ $(0 < \mu < 1)$ integralni qaraymiz:

$$F(\mu) = \int_\mu^1 \frac{dx}{x^\alpha} = \begin{cases} \frac{1}{1-\alpha}(1-\mu^{1-\alpha}), & \alpha \neq 1 \text{ bo'lganda}, \\ -\ln|\mu|, & \alpha = 1 \text{ bo'lganda}. \end{cases}$$

Agar $\alpha < 1$ bo'lsa, $\lim_{\mu \to 0} F(\mu) = \frac{1}{1-\alpha}$ bo'ladi.

Agar $\alpha \geq 1$ bo'lsa, $\lim_{\mu \to 0} F(\mu) = -\infty$ bo'ladi.

Shunday qilib, yuqoridagi ta'rifga asosan, $\alpha < 1$ bo'lganda I integral yaqinlashuvchi, $\alpha \geq 1$ bo'lganda esa I integral uzoqlashuvchi bo'ladi.

c $(a<c<b)$ nuqta $f(x)$ funksiya uchun maxsus nuqta bo'lsin. Agar $f(x)$ funksiya $[a;c)$ va $(c;b]$ oraliqlarda xosmas ma'noda integrallanuvchi bo'lsa, $f(x)$ funksiya $[a;b]$ da xosmas ma'noda integrallanuvchi deyiladi. Bu holda xosmas integral quyidagi tenglik bilan aniqlanadi:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\substack{\varepsilon \to 0 \\ \mu \to 0}} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\mu}^b f(x)dx \right].$$

11.2- misol. $\int_0^2 \frac{dx}{\ln x}$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=1$ nuqta maxsus nuqtadan iborat. U holda, yuqoridagi ta'rifga asosan, quyidagiga ega bo'lamiz:

$$\int_0^2 \frac{dx}{\ln x} = \int_0^1 \frac{dx}{\ln x} + \int_1^2 \frac{dx}{\ln x} = \lim_{\substack{\varepsilon\to 0\\ \mu\to 0}} \left(\int_0^{1-\varepsilon} \frac{dx}{\ln x} + \int_{1+\mu}^2 \frac{dx}{\ln x} \right) = \lim_{\substack{\varepsilon\to 0\\ \mu\to 0}} \left[I_1(\varepsilon) + I_2(\mu) \right]. \quad (1)$$

$\varepsilon > 0, \quad \mu > 0$ va $0 < x < 2$ bo'lsin.

$$\frac{1}{\ln x} = \frac{1}{\ln(1+(x-1))} = \left[\ln(1+(x-1))\right]^{-1} =$$

$$= \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots \right]^{-1} = \frac{1}{x-1} + \frac{1}{2} + o(x-1),$$

$$I_1(\varepsilon) = \int_0^{1-\varepsilon} \frac{dx}{\ln x} = \ln|x-1| \Big|_0^{1-\varepsilon} + \frac{1}{2}(1-\varepsilon) + o(\varepsilon^2) =$$

$$= \ln\varepsilon - \frac{\varepsilon}{2} + o(\varepsilon^2) + C_1, \quad (2)$$

$$I_2(\mu) = \int_{1+\mu}^2 \frac{dx}{\ln x} = \ln|x-1| \Big|_{1+\mu}^2 + \frac{1-\mu}{2} + o(\mu^2) =$$

$$= -\ln\mu - \frac{\mu}{2} + o(\mu^2) + C_2. \quad (3)$$

(2) va (3) ni (1) ga qo'yamiz:

$$\int_0^2 \frac{dx}{\ln x} = \lim_{\substack{\varepsilon\to 0\\ \mu\to 0}} \left[\ln\frac{\varepsilon}{\mu} - \frac{1}{2}(\varepsilon+\mu) + o(\mu^2) + o(\varepsilon^2) + C_1 + C_2 \right].$$

Oxirgi tenglikda o'rta qavs ichidagi ifodaning limiti mavjud emas. Demak, berilgan 2- tur xosmas integral uzoqlashuvchi bo'ladi.

11.3- misol. $\int_{1/e}^{1} \frac{dx}{x\ln^2 x}$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya $x=1$ nuqtaning atrofida chegaralanmagan. Ta'rifga ko'ra

$$\int_{\frac{1}{e}}^{1} \frac{dx}{x\ln^2 x} = \lim_{\varepsilon\to 0} \int_{\frac{1}{e}}^{1-\varepsilon} \frac{dx}{x\ln^2 x} = \lim_{\varepsilon\to 0} \frac{-1}{\ln x}\Big|_{\frac{1}{e}}^{1-\varepsilon} = \lim_{\varepsilon\to 0} \left(\frac{-1}{\ln(1-\varepsilon)} - 1 \right) = -\infty.$$

Demak, berilgan integral uzoqlashuvchi bo‘ladi.

11.4- misol. $\int_0^{\pi} \frac{|\cos x|}{\sqrt{\sin x}} dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi $f(x) = \frac{|\cos x|}{\sqrt{\sin x}}$ funksiya x=0 va $x=\pi$ nuqtalarning atrofida chegaralanmagan. Ta’rifga ko‘ra

$$\int_0^{\pi} \frac{|\cos x|}{\sqrt{\sin x}} dx = \lim_{\substack{\varepsilon\to 0\\ \mu\to 0}} \int_{\mu}^{\pi-\varepsilon} \frac{|\cos x|}{\sqrt{\sin x}} dx = -\lim_{\varepsilon\to 0} \int_{\frac{\pi}{2}}^{\pi-\varepsilon} \frac{\cos x}{\sqrt{\sin x}} dx +$$

$$+\lim_{\mu\to 0} \int_{\mu}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx = \lim_{\varepsilon\to 0} I_1(\varepsilon) + \lim_{\mu\to 0} I_1(\mu). \tag{1}$$

Bunda

$$I_1(\varepsilon) = -\int_{\frac{\pi}{2}}^{\pi-\varepsilon} \frac{d(\sin x)}{\sqrt{\sin x}} = -2\sqrt{\sin x}\Bigg|_{\frac{\pi}{2}}^{\pi-\varepsilon} = -2\sqrt{\sin(\pi-\varepsilon)} + 2, \tag{2}$$

$$I_2(\mu) = \int_{-\mu}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}} dx = 2\sqrt{\sin x}\Bigg|_{\mu}^{\frac{\pi}{2}} = 2\left(1-\sqrt{\sin\mu}\right). \tag{3}$$

(2), (3) larni (1) ga qo‘yib, limitni hisoblaymiz:

$$\int_0^{\pi} \frac{|\cos x|}{\sqrt{\sin x}} dx = 4.$$

Demak, berilgan xosmas integral yaqinlashuvchi.

11.2. Yaqinlashuvchi xosmas integrallarning xossalari. Asosiy formulalar. $f(x)$ va $g(x)$ funksiyalar $[a; b)$ da berilgan bo‘lib, b nuqta shu funksiyalarning maxsus nuqtasi bo‘lsin.

1- xossa (integralning chiziqliligi). Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ xosmas integrallar yaqinlashuvchi bo‘lsa, barcha α, $\beta \in R$ sonlar uchun $\int_a^b [\alpha f(x) \pm \beta g(x)]dx$ xosmas integral ham yaqinlashuvchi bo‘lib,

$$\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha\int_a^b f(x)dx \pm \beta\int_a^b g(x)dx$$

tenglik o‘rinli bo‘ladi. Bu yerda tenglikning o‘ng tomonidagi integrallarning mavjudligi muhim. Aks holda, chap tomondagi integralning mavjudligidan o‘ng tomondagi integrallarning mavjudligi har doim ham kelib chiqavermaydi. Masalan, x^3 funksiyani $x^3 = \left(x^3 - \frac{1}{x^2}\right) + \frac{1}{x^2}$ ko‘rinishda tasvirlash mumkin. Ravshanki, $\int_0^1 x^3 dx -$ integral yaqinlashuvchi, lekin $\int_0^1 \left(x^3 - \frac{1}{x^2}\right)dx$, $\int_0^1 \frac{1}{x^2}dx$ integrallar uzoqlashuvchi.

2- xossa (integrallash tengsizligi). Agar $\forall x \in [a;b)$ lar uchun $f(x) \le g(x)$ bo‘lib, $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo‘lsa,

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo‘ladi.

2- xossadagi $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni ham qanoatlantirsin:

1) $f(x)$ funksiya $[a;b)$ da chegaralangan, ya’ni shunday m va M o‘zgarmas sonlar mavjudki, $\forall x \in [a;b)$ da $m \le f(x) \le M$;

2) $g(x)$ funksiya $[a;b)$ da o‘z ishorasini o‘zgartirmasin, ya’ni barcha x ($x \in [a; b)$) larda $g(x) \ge 0$ yoki $g(x) \le 0$ bo‘lsin. U holda o‘rta qiymat haqidagi teorema o‘rinli.

3- xossa (o'rta qiymat haqidagi teorema). Agar $\int_a^b f(x)g(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo'lsa, shunday o'zgarmas μ $(m \le \mu \le M)$ son topiladiki,

$$\int_a^b f(x)g(x)dx = \mu \cdot \int_a^b g(x)dx$$

tenglik o'rinli bo'ladi.

4- xossa (Nyuton — Leybnis formulasi). Agar $f(x)$ funksiya $[a;b)$ da uzluksiz bo'lib, $F(x)$ esa uning shu oraliqdagi boshlang'ich funksiyasi $(F'(x) = f(x))$ bo'lsa,

$$\int_a^b f(x)dx = F(x)\Big|_a^{b-0} = F(b-0) - F(a) \qquad (*)$$

bo'ladi, bunda $F(b-0) = \lim\limits_{t \to b-0} F(t)$.

(*) formula *Nyuton – Leybnis formulasi* deyiladi.

5- xossa (o'zgaruvchini almashtirish formulasi). $f(x)$ funksiya $[a;b)$ da $\varphi(x)$ uzluksiz, funksiya esa $[\alpha;\beta)$ da uzluksiz differensiallanuvchi funksiya bo'lib, $a = \varphi(\alpha) \le \varphi(t) < \lim\limits_{t \to b-0} \varphi(t) = b$ bo'lsa, $\int_a^b f(x)dx$, $\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$ integrallarning biri yaqinlashuvchi bo'lsa, ikkinchisi ham yaqinlashuvchi va

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

tenglik o'rinli bo'ladi.

6- xossa (bo'laklab integrallash formulasi). Agar $u=u(x)$ va $v=v(x)$ funksiyalar $[a;b)$ da uzluksiz, differensiallanuvchi, $\lim\limits_{t \to b-0}(uv)$ mavjud bo'lib $\int_a^b udv$, $\int_a^b vdu$ integrallarning birortasi mavjud bo'lsa,

$$\int_a^b udv = (u \cdot v)\Big|_a^b - \int_a^b vdu \qquad (11.2)$$

tenglik o'rinli, bu yerda

$$(uv)\Big|_a^b = \lim_{t\to b-0} u(t)v(t) - u(a)v(a).$$

11.2- eslatma. Agar $\int_a^b uv'dx$ yoki $\int_a^b vu'dx$ integral yaqinlashuvchi bo'lib, $\lim_{t\to b-0} u(x)v(x)$ mavjud va chekli bo'lsa, (11.2) formula o'rinli bo'ladi.

11.5- misol. $\int_0^1 \frac{\left(\sqrt[8]{x}+2\right)^2}{\sqrt{x}}dx$ integralni hisoblang.

Yechilishi. Xosmas integralning 1-chiziqlilik xossasidan foydalanib, quyidagiga ega bo'lamiz:

$$\int_0^1 \frac{(\sqrt[8]{x}+2)^2}{\sqrt{x}}dx = \int_0^1 \frac{dx}{\sqrt[4]{x}} + 4\int_0^1 \frac{dx}{\sqrt[8]{x^3}} + 4\int_0^1 \frac{dx}{\sqrt{x}}.$$

(0;1] da berilgan $\frac{1}{\sqrt[4]{x^3}}$, $\frac{1}{\sqrt[8]{x^3}}$, $\frac{1}{\sqrt{x}}$ funksiyalar uchun, mos ravishda,

$$\frac{4}{3}\sqrt[4]{x^3},\quad \frac{8}{5}\sqrt[8]{x^5},\quad 2\sqrt{x}$$

funksiyalar boshlang'ich funksiyalar bo'ladi. Nyuton — Leybnis formulasi bo'yicha:

$$\int_0^1 \frac{dx}{\sqrt[4]{x}} = \frac{4}{3}\sqrt[4]{x^3}\Big|_{+0}^1 = \frac{4}{3},\quad \int_0^1 \frac{dx}{\sqrt[8]{x}} = \frac{8}{5}\sqrt[8]{x^5}\Big|_{+0}^1 = \frac{8}{5},\quad \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_{+0}^1 = 2.$$

Demak,

$$\int_0^1 \frac{\left(\sqrt[8]{x}+2\right)^2}{\sqrt{x}}dx = \frac{4}{3} + 4\frac{8}{5} + 8 = 15\frac{11}{15}.$$

11.6- misol. $\int_0^{\frac{\pi}{2}} \ln\cos x dx$ integralni hisoblang.

Yechilishi. $x=\frac{\pi}{2}$ nuqta atrofida integral ostidagi funksiya chegaralanmagan. Berilgan integralda $x=\frac{\pi}{2}-t$ almashtirish bajaramiz. Natijada

$$\int_0^{\frac{\pi}{2}} \ln\cos x dx = \int_0^{\frac{\pi}{2}} \ln\sin t dt \qquad (*)$$

hosil bo'ladi. Tenglikning o'ng tomonidagi integral ostidagi funksiya uchun $t=0$ nuqta maxsus nuqta bo'ladi. $\int_0^{\frac{\pi}{2}} \ln\sin t dt$ integralning mavjudligini ko'rsatish uchun bo'laklab integrallash formulasidan foydalanamiz: $u=\ln\sin t$, $dv=dt$, $du=\operatorname{ctg} t dt$, $v=t$, u holda

$$\int_0^{\frac{\pi}{2}} \ln\sin t dt = t\cdot\ln\sin t\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} t\cdot\operatorname{ctg} t dt = \lim_{t\to 0}(t\cdot\ln\sin t) -$$

$$-\int_0^{\frac{\pi}{2}} t\cdot\operatorname{ctg} t dt = -\int_0^{\frac{\pi}{2}} t\cdot\operatorname{ctg} t dt.$$

$t\cdot\operatorname{ctg} t$ funksiya $\left(0;\frac{\pi}{2}\right]$ da chegaralanganligi uchun oxirgi integral mavjud.

Demak, $\int_0^{\frac{\pi}{2}} \ln\sin t dt$ integral ham mavjud bo'ladi, (*) ga asosan berilgan integral ham mavjud. $J=\int_0^{\frac{\pi}{2}} \ln\sin t dt$ integralda $t=2u$ almashtirishni bajaramiz:

$$J=2\int_0^{\pi/4}\ln\sin 2u\,du=2\int_0^{\pi/4}(\ln 2+\ln\sin u+\ln\cos u)du=$$

$$=\frac{\pi}{2}\ln 2+\int_0^{\pi/4}\ln\sin u\,du+2\int_0^{\pi/4}\ln\cos u\,du.$$

Keyingi integralda $u=\frac{\pi}{2}-z$ almashtirishni bajarib, uni $\int_0^{\frac{\pi}{2}}\ln(\sin z)dz$ ko'rinishga keltiramiz. Natijada

$$J=\frac{\pi}{2}\ln 2+2\int_0^{\frac{\pi}{2}}\ln\sin u\,du+2\int_0^{\frac{\pi}{2}}\ln\sin z\,dz=$$

$$=\frac{\pi}{2}\ln 2+2\int_0^{\frac{\pi}{2}}\ln\sin z\,dz=\frac{\pi}{2}\ln 2+2J.$$

Bu tenglamadan $J=-\frac{\pi}{2}\ln 2$ bo'ladi. Shunday qilib,

$$\int_0^{\frac{\pi}{2}}\ln\cos x\,dx=-\frac{\pi}{2}\ln 2\,.$$

11.7- misol. $\int_0^2\frac{dx}{(4-x)\sqrt{2-x}}$ integralni hisoblang.

Yechilishi. Berilgan xosmas integralni hisoblash uchun o'zgaruvchilarni almashtirish formulasidan foydalanamiz. $2-x=t^2$ deb belgilaymiz, $t>0$, bu yerdan $x=2-t^2$, $dx=2tdt$, $\alpha=\sqrt{2}$, $\beta=0$ bo'ladi. Demak, quyidagiga ega bo'lamiz:

$$\int_0^2\frac{dx}{(4-x)\sqrt{2-x}}=-2\int_{\sqrt{2}}^0\frac{tdt}{t(t^2+2)}=2\int_0^{\sqrt{2}}\frac{dt}{t^2+2}=\frac{\sqrt{2}}{4}\pi.$$

Bu yerda o'zgaruvchilarni almashtirgandan so'ng, xosmas integral xos integralga aylantirildi.

11.8- misol. $\lim\limits_{x\to 0} x\int\limits_x^1 \frac{\cos^2 t}{t^2}dt$ integralni hisoblang.

Yechilishi. Berilgan integralda $f(t)=\cos^2 t,\ g(t)=\frac{1}{t^2}$ deb, o'rta qiymat haqidagi teoremaga asosan,

$$\int\limits_x^1 \frac{\cos^2 t}{t^2}dt=\left(\frac{1}{x}-1\right)\cdot\cos^2\xi,\quad x<\xi<1$$

ni hosil qilamiz. $\forall\varepsilon>0$ son berilgan bo'lsin, u holda $\forall x\in\left(0;\frac{\varepsilon}{1+\varepsilon}\right)$ lar uchun

$$\cos^2\xi\left(\frac{1}{x}-1\right)>\frac{\cos^2\xi}{\varepsilon}$$

bo'lgani sababli $x\to+0$ da $\int\limits_x^1\frac{\cos^2 t}{t^2}dt\to+\infty$. Berilgan limitni hisoblash uchun Lopital qoidasini qo'llaymiz:

$$\lim_{x\to 0} x\cdot\int\limits_x^1\frac{\cos^2 t}{t^2}dt=\lim_{x\to 0}\frac{\frac{\cos^2 x}{x^2}}{\frac{1}{x^2}}=\lim_{x\to 0}\cos^2 x=1.$$

11.3. Chegaralanmagan funksiya xosmas integrallarining ba'zi bir tatbiqlari.

11.3.1. Yuzni xosmas integral yordamida hisoblash. $f(x)$ funksiya $[a;b)$ da aniqlangan, uzluksiz va $\forall x\in[a;b)$ uchun $f(x)\ge 0$ bo'lsin. Unda $D=\{(x;y): a\le x<b,\ 0\le y\le f(x)\}$ sohaning yuzi ushbu $S=\int\limits_a^b f(x)dx$ xosmas integral orqali ifoda qilinadi.

11.9- misol. $y=x^{-\frac{2}{3}},\quad y=0,\quad x=-1,\quad x=1$ chiziqlar bilan chegaralangan shaklning yuzini hisoblang.

Yechilishi. Talab qilingan yuzni quyidagi xosmas integral orqali hisoblaymiz:

$$S=\int_{-1}^{1}x^{-\frac{2}{3}}dx=\lim_{\substack{\varepsilon\to 0+0\\ \eta\to 0-0}}\left(\int_{-1}^{\varepsilon}x^{-\frac{2}{3}}dx+\int_{\eta}^{1}x^{-\frac{2}{3}}dx\right)=$$

$$=\lim_{\substack{\varepsilon\to 0+0\\ \eta\to 0-0}}\left(3\left(\varepsilon^{\frac{1}{3}}+1\right)+3\left(1-\eta^{1/3}\right)\right)=6\ \text{(kv.birl.)}.$$

11.3.2. Aylanma jismning hajmini xosmas integral yordamida hisoblash. Ushbu $D=\{(x;y):a\le x<b,\quad 0\le y\le f(x)\}$ egri chiziqli trapetsiyani Ox va Oy o‘qlar atrofida aylantirish natijasida hosil bo‘lgan aylanma jismlarning hajmi, mos ravishda,

$$V_x=\pi\int_a^b f^2(x)dx,\quad V_y=2\pi\int_a^b|xy|\,dx \qquad (11.3)$$

xosmas integrallar orqali hisoblanadi.

11.10- misol. $y=\dfrac{1}{\sqrt{x-1}},\quad x\in(1;2]$ chiziq bilan chegaralangan shaklni Ox o‘q atrofida aylantirish natijasida hosil bo‘lgan aylanma jism hajmini toping.

Yechilishi. Talab qilingan aylanma jismning hajmini (11.3) formula orqali topamiz:

$$V_x=\pi\int_1^2\frac{dx}{\sqrt{x-1}}=\lim_{\varepsilon\to 0}\pi\int_{1+\varepsilon}^{2}\frac{dx}{\sqrt{x-1}}=\pi\lim_{\varepsilon\to 0}2\sqrt{x-1}\Big|_{1+\varepsilon}^{2}=2\pi\ \text{(kv.birl.)}.$$

11.3.3. Aylanma sirtning yuzini xosmas integrallar yordamida hisoblash. $f(x)$ funksiya $[a;b)$ da aniqlangan, uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo‘lib, u $f(x)\ge 0$ bo‘lsin. $f(x)$ funksiya grafigini Ox o‘q atrofida aylantirish natijasida hosil bo‘lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

$$S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx\,. \qquad (11.4)$$

Shunga o‘xshash, Oy o‘q atrofida aylantirish natijasida hosil bo‘lgan aylanma sirtning yuzi ushbu formula orqali hisoblanadi:

$$S = 2\pi \int_c^d x(y)\sqrt{1+(x'(y))^2}\,dy \,. \qquad (11.5)$$

Mustaqil yechish uchun misollar

Xosmas integrallarning yaqinlashuvchiligini ko'rsating va qiymatini toping:

11.1. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$. **11.2.** $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. **11.3.** $\int_1^e \frac{dx}{x\sqrt{\ln x}}$. **11.4.** $\int_0^4 \frac{dx}{x+\sqrt{x}}$.

11.5. $\int_1^2 \frac{xdx}{\sqrt{x-1}}$. **11.6.** $\int_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}}dx$. **11.7.** $\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}dx$.

11.8. $\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$. **11.9.** $\int_0^3 \frac{x^2dx}{\sqrt{9-x^2}}$. **11.10.** $\int_{-1}^1 \frac{\arccos x}{\sqrt{1-x^2}}dx$.

11.11. $\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx$. **11.12.** $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx$.

Xosmas integrallarning uzoqlashuvchi ekanligini isbotlang:

11.13. $\int_{-1}^3 \frac{dx}{x}$. **11.14.** $\int_0^e \frac{dx}{e^x-1}$. **11.15.** $\int_{-3}^3 \frac{xdx}{x^2-1}$. **11.16.** $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin^3 x}}dx$.

11.17. $\int_0^{\frac{1}{3}} \frac{dx}{x\ln x}$. **11.18.** $\int_{-1}^1 \frac{e^{\frac{1}{x}}}{x^3}dx$. **11.19.** $\int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx$. **11.20.** $\int_0^{\frac{\pi}{2}} \mathrm{tg}x\,dx$.

Xosmas integrallarni hisoblang:

11.21. $\int_0^1 \frac{2-\sqrt[3]{x}-x^3}{\sqrt[5]{x^3}}dx$. **11.22.** $\int_{\sqrt{2}}^2 \frac{dx}{(x-1)\sqrt{x^2-2}}$.

11.23 $\int_0^1 \frac{dx}{\sqrt{x\sqrt{x}}}$. **11.24.** $\int_0^{\frac{\pi}{2}} \sqrt{\mathrm{tg}x}\,dx$. **11.25.** $\int_0^{\frac{\pi}{2}} \ln\cos x dx$.

11.26. $\int_0^{\pi} x\ln\sin x dx$. **11.27.** $\int_0^{\frac{\pi}{4}} \sqrt{\mathrm{ctg}x}\,dx$.

11.28. $\int\limits_{-1}^{1} \frac{dx}{(16-x^2)\sqrt{1-x^2}}$. **11.29.** $\int\limits_{a}^{b} \frac{xdx}{\sqrt{(x-a)(b-x)}}$.

11.30. $\int\limits_{0}^{1} \frac{x^3 \arcsin x}{\sqrt{1-x^2}} dx$.

Limitlarni hisoblang:

11.31. $\lim\limits_{x\to+\infty} \frac{\int\limits_0^x \sqrt{1+t^{10}}\,dt}{x^6}$. **11.32.** $\lim\limits_{x\to+\infty} \frac{\int\limits_0^x t^{-1}e^{-t}\,dt}{\ln\frac{1}{x}}$.

11.33. $\lim\limits_{x\to0} x^2 \int\limits_x^1 \frac{\sin 2t}{t^3} dt$. **11.34.** $\lim\limits_{x\to0} \sqrt{x} \int\limits_x^1 \frac{e^{-t}}{t^{3/2}} dt$.

Berilgan funksiyaning grafigi va abssissalar o‘qi bilan chegaralangan shaklning yuzini toping:

11.35. $y = \frac{-x}{\sqrt{x+1}}, \quad x \in (-1; 0]$.

11.36. $y = \frac{1}{\sqrt{2-5x}}, \quad x \in [0; 0,4)$.

11.37. $y = \frac{x}{\sqrt{(x-2)(5-x)}}, \quad x \in (2; 5)$.

11.38. $y = \frac{1}{\sqrt{1-x}}, \quad x \in [0; 1)$.

11.39. $y = \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}, \quad x \in [0; 1)$.

11.40. $y = \sqrt{\frac{x^3}{2a-x}}, \quad x \in [0; 2a)$.

11.41. $y = x\ln\frac{1+x}{1-x}, \quad x \in [0; 1)$.

Berilgan chiziq va uning asimptotalari bilan chegaralangan shaklning yuzini toping.

11.42. $y^2 = \frac{8-4x}{x}$. **11.43.** $y^2(x+1) = x^2, \quad x < 0$.

11.44. $(1-x^2)y^2 = x^2, \quad x > 0$.

11.45. $x = \cos t, y = \cos 2t \operatorname{tg} t, \quad t \in \left[\frac{\pi}{4}; \ \frac{3\pi}{4}\right]$.

Misollarning javoblari

11.1. $\frac{3}{2}$. **11.2.** $\frac{\pi}{2}$. **11.3.** 2. **11.4.** 2ln3. **11.5.** $\frac{8}{3}$. **11.6.** $\frac{10}{7}$.

11.7. $\frac{\pi^2}{8}$. **11.8.** $\frac{1}{\ln 2}$. **11.9.** $\frac{9\pi}{4}$. **11.10.** $\frac{\pi^2}{2}$. **11.11.** $-2e^{-1}$. **11.12.** 2.

11.21. $\frac{625}{187}$. **11.22.** $\frac{\pi}{2}$. **11.23.** $\frac{4}{3}$. **11.24.** $\frac{\pi}{\sqrt{2}}$. **11.25.** $-\frac{\pi \ln 2}{2}$.

11.26. $-\frac{\pi^2 \ln 2}{2}$. **11.27.** $\frac{1}{2\sqrt{2}}\left(\pi + \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$. **11.28.** $\frac{\pi}{4\sqrt{15}}$.

11.29. $\frac{\pi(a+b)}{2}$. **11.30.** $\frac{7}{9}$. **11.31.** $\frac{1}{6}$. **11.32.** 1. **11.33.** 0. **11.34.** 2.

11.35. $\frac{4}{3}$. **11.36.** $\frac{2\sqrt{2}}{5}$. **11.37.** $\frac{7\pi}{2}$. **11.38.** 2. **11.39.** 2.

11.40. $\frac{3\pi a^2}{2}$. **11.41.** 1. **11.42.** 4π. **11.43.** $\frac{8}{3}$. **11.44.** 2. **11.45.** $2+\frac{\pi}{2}$.

12 - §. Xosmas integrallarning yaqinlashuvchiligi haqidagi teoremalar

12.1. Xosmas integrallarning yaqinlashuvchiligi. $f(x)$ funksiya $[a;b)$ da berilgan bo'lib, b nuqta shu funksiyaning maxsus nuqtasi bo'lsin.

12.1- teorema. $[a;b)$ da manfiy bo'lmagan $f(x)$ funksiyadan olingan

$$\int_a^b f(x)dx$$

xosmas integralning yaqinlashuvchi bo'lishi uchun $\forall t \in [a;b)$ da

$$\{F(t)\} = \left\{\int_a^t f(x)dx\right\} \leq C \quad (C = \text{const})$$

bo'lishi zarur va yetarli.

12.2- teorema. $f(x)$ va $g(x)$ funksiyalar $[a;b)$ da berilgan bo'lib, b nuqta shu funksiyalarning maxsus nuqtasi bo'lsin. Agar $\forall x \in [a;b)$ da $0 \leq f(x) \leq g(x)$ tengsizlik bajarilsa, $\int_a^b g(x)dx$ integralning yaqinlashuvchiligidan $\int_a^b f(x)dx$ integralning yaqinlashuvchiligi; $\int_a^b f(x)dx$ integralning uzoqlashuvchiligidan $\int_a^b g(x)dx$ integralning uzoqlashuvchiligi kelib chiqadi.

12.3- teorema. $f(x)$ va $g(x)$ funksiyalar $[a;b)$ da aniqlangan, $f(x) \geq 0,\ g(x) > 0$ bo'lib,

$$\lim_{x \to b-0} \frac{f(x)}{g(x)} = k \quad (0 \leq k \leq +\infty)$$

mavjud bo'lsin. Agar $k < +\infty$ bo'lib, $\int_a^b g(x)dx$ yaqinlashuvchi bo'lsa, $\int_a^b f(x)dx$ ham yaqinlashuvchi bo'ladi. Agar $k > 0$ bo'lib, $\int_a^b g(x)dx$ uzoqlashuvchi bo'lsa, $\int_a^b f(x)dx$ ham uzoqlashuvchi bo'ladi.

12.1- misol. $\int_0^1 \frac{e^{-x}\sin^2 x}{(e^{1-x}-1)^{1/2}}dx$ integralning yaqinlashuvchiligini ko‘rsating.

Yechilishi. Integral ostidagi funksiya [0;1) da musbat. $\forall t\in[0;1)$ da $F(t)=\int_0^t \frac{e^{-x}\sin^2 x}{(e^{1-x}-1)^{1/2}}dx$ funksiya o‘suvchi. So‘ngra

$$F(t)=\int_0^t \frac{e^{-x}\sin^2 x}{(e^{1-x}-1)^{1/2}}dx\le\int_0^t \sin^2 1\int_0^t \frac{e^{-x}}{(e^{1-x}-1)^{1/2}}dx=$$

$$=-\frac{2\sin^2 1}{e}(e^{1-x}-1)^{1/2}\Big|_0^t=\frac{2\sin^2 1}{e}\left[(e-1)^{1/2}-(e^{1-t}-1)^{1/2}\right]=$$

$$=\frac{2\sin^2 1}{e}\frac{e-e^{1-t}}{(e-1)^{1/2}+(e^{1-t}-1)^{1/2}}\le\frac{2\sin^2 1}{e}\frac{e-1}{(e-1)^{1/2}}=\frac{2\sin^2 1}{e}\sqrt{e-1}=C.$$

Demak, $F(t)$ funksiya yuqoridan chegaralangan, ya’ni $F(t)\le C$.

Shunday qilib, 12.1- teoremaga asosan, berilgan xosmas integral yaqinlashuvchi bo‘ladi.

12.2- misol. $\int_0^1 \frac{\operatorname{arctg}^2 x}{\sqrt{1-x^4}}dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $x=1$ nuqta maxsus nuqta bo‘ladi. Ravshanki, $\forall x\in[0;1)$ da

$$0\le\frac{\operatorname{arctg}^2 x}{\sqrt{1-x^4}}=\frac{\operatorname{arctg}^2 x}{\sqrt{1+x^2}\sqrt{1-x^2}}\le\frac{\frac{\pi^2}{16}}{\sqrt{1-x^2}}$$

tengsizlik o‘rinli. Ushbu xosmas integral yaqinlashuvchi:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\arcsin x\Big|_0^{1-0}=\frac{\pi}{2}$$

Demak, 12.2- teoremaga binoan, $\int_0^1 \frac{\operatorname{arctg}^2 x}{\sqrt{1-x^4}}dx$ integral ham yaqinlashuvchi bo‘ladi.

12.3- misol. $\int\limits_0^1 \frac{\sqrt{1+x^2}}{1-x}dx$ integralning uzoqlashuvchiligini ko'rsating.

Yechilishi. Ma'lumki, $\forall x \in [0;1)$ da $\sqrt{1+x^2} \geq 1$ bo'ladi. Demak, [0; 1) dagi barcha x lar uchun

$$\frac{\sqrt{1+x^2}}{1-x} > \frac{1}{1-x}$$

o'rinli. Ma'lumki, $\int\limits_0^1 \frac{dx}{1-x}$ xosmas integral uzoqlashuvchi. Demak,

12.2- teoremaga binoan, $\int\limits_0^1 \frac{\sqrt{1+x^2}}{1-x}dx$ integral ham uzoqlashuvchi bo'ladi.

12.4- misol. $\int\limits_0^1 \frac{dx}{\sqrt[5]{\arcsin x}}$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun x=0 nuqta maxsus nuqta bo'ladi. Bu integral bilan birga ushbu $\int\limits_0^1 \frac{dx}{\sqrt[5]{x}}$ yaqinlashuvchi integralni qaraymiz. Ravshanki,

$$\lim_{x \to +0} \frac{\frac{1}{\sqrt[5]{\arcsin x}}}{\frac{1}{\sqrt[5]{x}}} = \lim_{x \to +0} \sqrt[5]{\frac{x}{\arcsin x}} = 1.$$

Demak, k=1 bo'lgani uchun, 12.3- teoremaga asosan, berilgan integral ham yaqinlashuvchi bo'ladi.

12.5- misol. $\int\limits_0^{+\infty} \frac{\ln x}{1+x^2}dx$ xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun $+\infty$ va 0 nuqtalar maxsus nuqtalardan iborat. Shuning uchun

$$\int_0^{+\infty} \frac{\ln x}{1+x^2}dx = \int_0^1 \frac{\ln x}{1+x^2}dx + \int_1^{+\infty} \frac{\ln x}{1+x^2}dx .$$

$0<\lambda<1$ bo'lganda $\lim\limits_{x\to 0} \frac{\ln x}{1+x^2} : \frac{1}{x^\lambda} = \lim\limits_{x\to 0} \frac{x^\lambda \ln x}{1+x^2} = 0$ bo'lgani uchun, 12.3-teoremaga asosan, $\int_0^1 \frac{\ln x}{1+x^2}dx$ yaqinlashuvchi.

$1<\lambda<2$ bo'lganda, $\lim\limits_{x\to \infty} \frac{\ln x}{1+x^2} : \frac{1}{x^\lambda} = \lim\limits_{x\to \infty} \frac{x^2}{1+x^2} : \frac{\ln x}{x^{2-\lambda}} = 0$.

Demak, $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx$ ham, 10.3- teoremaga asosan yaqinlashuvchi bo'ladi. Shunday qilib, berilgan xosmas integral yaqinlashuvchi.

12.4- teorema. $f(x)$ funksiyani x ning b ga yetarli yaqin qiymatlarida

$$f(x) = \frac{\varphi(x)}{(b-x)^\alpha} \quad (\alpha > 0)$$

ko'rinishida tasvirlash mumkin bo'lsin. U holda $\varphi(x) \le C < +\infty$ va $\alpha < 1$ bo'lganda, $\int_a^b f(x)dx$ yaqinlashuvchi; $\varphi(x) \ge C > 0$ va $\alpha \ge 1$ bo'lganda esa $\int_a^b f(x)dx$ uzoqlashuvchi bo'ladi.

12.5- teorema. $f(x)$ funksiya $x \to b-0$ da $\frac{1}{b-x}$ ga nisbatan α $(\alpha > 0)$ tartibli cheksiz katta miqdor bo'lsin. U holda $\int_a^b f(x)dx$ integral $\alpha < 1$ bo'lganda yaqinlashuvchi, $\alpha \ge 1$ bo'lganda esa uzoqlashuvchi bo'ladi.

Natija. $x \to b-0$ da $f(x) \sim g(x)$ bo‘lsin. U holda, $\int\limits_a^b f(x)\,dx$ va $\int\limits_a^b g(x)\,dx$ integrallar bir vaqtda yaqinlashadi yoki uzoqlashadi.

12.6- misol. $\int\limits_0^1 \frac{\sin x}{x^2}\,dx$ integralning uzoqlashuvchiligini ko‘rsating.

Yechilishi. Integral ostidagi funksiya uchun x=0 nuqta maxsus nuqta bo‘ladi. Bu integral bilan birga $\int\limits_0^1 \frac{dx}{x}$ uzoqlashuvchi integralni qaraymiz. Ravshanki,

$$\lim_{x\to+0} \frac{\frac{\sin x}{x^2}}{\frac{1}{x}} = \lim_{x\to+0} \frac{\sin x}{x} = 1\,.$$

12.3- teoremaga asosan, $\int\limits_0^1 \frac{\sin x}{x^2}\,dx$ integral ham, uzoqlashuvchi bo‘ladi.

12.7- misol. $\int\limits_{-2}^2 \frac{dx}{(4-x)\sqrt{4-x^2}}$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiya uchun x=±2 nuqtalar maxsus nuqtalar bo‘ladi. So‘ngra

$$\int\limits_{-2}^2 \frac{dx}{(4-x)\sqrt{4-x^2}} = \int\limits_{-2}^0 \frac{dx}{(4-x)\sqrt{4-x^2}} + \int\limits_0^2 \frac{dx}{(4-x)\sqrt{4-x^2}}\,. \qquad (*)$$

(*) ning o‘ng tomonidagi integrallar ostidagi funksiyalarni, mos ravishda, quyidagi ko‘rinishda tasvirlaymiz:

$$\frac{1}{(4-x)\sqrt{4-x^2}} = \frac{\frac{1}{(4-x)\sqrt{2-x}}}{\sqrt{2+x}} = \frac{\varphi(x)}{\sqrt{2+x}}\,;$$

$$\frac{1}{(4-x)\sqrt{4-x^2}}=\frac{\dfrac{1}{(4-x)\sqrt{2+x}}}{\sqrt{2-x}}=\frac{\varphi_1(x)}{\sqrt{2-x}},$$

bunda $\varphi(x)$ va $\varphi_1(x)$ funksiyalarning yuqoridan chegaralanganligini va $\alpha=\frac{1}{2}<1$ ekanligini e'tiborga olsak, u vaqtda 12.4- teoremaga asosan, berilgan xosmas integral yaqinlashuvchi bo'ladi.

12.8- misol. $\int_0^1 \frac{\sin x}{x^\alpha}dx$ integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Integral ostidagi funksiyani

$$f(x)=\frac{\sin x}{x^\alpha}=\frac{1}{x^{\alpha-1}}\frac{\sin x}{x}=\frac{1}{x^{\alpha-1}}\varphi(x)$$

ko'rinishda tasvirlaymiz, bunda $\varphi(x)=\frac{\sin x}{x}$ absolut qiymati bo'yicha chegaralangan funksiya. Bu yerda $\alpha-1<1$ bo'lganda integral yaqinlashuvchi, $\alpha-1\geq 1$ bo'lganda esa uzoqlashuvchi bo'ladi.

Demak, 12.4- teoremaga asosan, berilgan integral $\alpha<2$ bo'lganda yaqinlashuvchi, $\alpha\geq 2$ bo'lganda esa uzoqlashuvchi bo'ladi.

12.9- misol. Ushbu integralni yaqinlashuvchilikka tekshiring:

$$\int_0^2 \frac{e^x-1}{\sqrt[5]{x^4}\cdot \operatorname{tg}\sqrt[3]{x}}dx.$$

Yechilishi. Integral ostidagi funksiya x=0 nuqtaning o'ng atrofida chegaralanmagan va $x\to 0+0$ da

$$\frac{e^x-1}{\sqrt[3]{x^4}\operatorname{tg}\sqrt[3]{x}}\sim\frac{x}{\sqrt[3]{x^4}\sqrt[3]{x}}=\frac{1}{\sqrt[3]{x^2}}.$$

bo'ladi; $\int_0^2 \frac{dx}{\sqrt[3]{x^2}}$ integral esa yaqinlashuvchi. U holda, yuqoridagi natijaga ko'ra, berilgan integral yaqinlashuvchi bo'ladi.

12.6- teorema (Koshi teoremasi). $\int_a^b f(x)dx$ xosmas integral (b – maxsus nuqta) yaqinlashuvchi bo'lishi uchun, $\forall\varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ topilib, $b-\delta < t_1 < b$, $b-\delta < t_2 < b$ tengsizliklarni qanoatlantiruvchi ixtiyoriy t_1 va t_2 lar uchun

$$|F(t_2)-F(t_1)| = \left|\int_a^{t_2} f(x)dx - \int_a^{t_1} f(x)dx\right| = \left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Koshi teoremasidan ko'p hollarda xosmas integrallarning uzoqlashuvchiligini isbotlashda foydalaniladi: agar $\exists\varepsilon_0 > 0$, $\forall\eta \in [a;b)$ uchun va $\exists\eta_1 \in [\eta;b)$ va $\eta_2 \in [\eta;b)$ uchun

$$\left|\int_{\eta_1}^{\eta_2} f(x)dx\right| \geq \varepsilon_0$$

tengsizlik bajarilsa, $\int_a^b f(x)dx$ xosmas integral uzoqlashuvchi bo'ladi.

12.10- misol. Ushbu integralning yaqinlashuvchiligini Koshi teoremasidan foydalanib ko'rsating:

$$\int_0^1 \frac{(1-x)\sin\frac{1}{1-x}}{x^2-2x+2}dx.$$

Yechilishi. $\forall\varepsilon>0$ berilgan songa ko'ra,

$$\exists t_0(\varepsilon) = \frac{2-e^{2\varepsilon}}{2\pi(e^{2\varepsilon}-1)} \quad (t_0(\varepsilon) < 1)$$

topilib, $t' = 1-\sqrt{\frac{1}{2\pi n}} > t_0$, $t'' = 1-\sqrt{\frac{1}{n\pi}} > t_0$ $(n \in N)$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ lar uchun

$$|F(t'')-F(t')|=\left|\int_{1-\sqrt{\frac{1}{n\pi}}}^{1-\sqrt{\frac{1}{2n\pi}}}\frac{(1-x)\sin\frac{1}{1-x}}{x^2-2x+2}dx\right|\le\int_{1-\sqrt{\frac{1}{n\pi}}}^{1-\sqrt{\frac{1}{2n\pi}}}\frac{(1-x)}{x^2-2x+2}dx=$$

$$=-\frac{1}{2}\ln((1-x)^2+1)\Big|_{1-\sqrt{\frac{1}{n\pi}}}^{1-\sqrt{\frac{1}{2n\pi}}}=\frac{1}{2}\left[\ln(\frac{1}{n\pi}+1)-\ln\left(\frac{1}{2n\pi}+1\right)\right]=$$

$$=\frac{1}{2}\ln\frac{\frac{1}{\pi n}+1}{\frac{1}{2\pi n}+1}<\varepsilon$$

tengsizlik bajariladi, ya'ni $n\to\infty da$ $|F(t'')-F(t')|\to 0$, shunday qilib, berilgan integral yaqinlashuvchi.

12.11- misol. Ushbu $\int\limits_0^1\sin^2\left(\frac{1}{1-x}\right)\frac{dx}{1-x}$ integralning uzoqlashuvchiligini isbotlang.

Yechilishi. $\forall\theta\in[0;1)$ sonni va $n>\frac{1}{\pi(1-\theta)}$ tengsizlikni qanoatlantiruvchi n natural sonni olamiz. $\left[1-\frac{1}{\pi n};1-\frac{1}{2n\pi}\right]$ oraliqda

$$\int_{1-\frac{1}{n\pi}}^{1-\frac{1}{2n\pi}}\sin^2\left(\frac{1}{1-x}\right)\frac{dx}{1-x}$$

integralni quyidan baholaymiz:

$$\left|\int_{1-\frac{1}{n\pi}}^{1-\frac{1}{2n\pi}}\sin^2\left(\frac{1}{1-x}\right)\frac{dx}{1-x}\right|=\int_{n\pi}^{2n\pi}\frac{\sin^2 t}{t}dt\ge\frac{1}{2n\pi}\int_{n\pi}^{2n\pi}\sin^2 t\,dt=$$

$$=\frac{1}{2n\pi}\int_{n\pi}^{2n\pi}\frac{1-\cos 2t}{2}dt=\frac{1}{4}.$$

Bu tengsizlikdan $\exists \varepsilon = \frac{1}{4}$ son mavjud bo'lib, $\forall \theta \in [0;1)$ uchun shunday $\theta_1 = 1 - \frac{1}{n\pi}$ va $\theta_2 = 1 - \frac{1}{2n\pi}$ sonlar mavjud bo'ladiki,

$$\left| \int_{\theta_1}^{\theta_2} \sin^2 \left(\frac{1}{1-x} \right) \frac{dx}{1-x} \right| \geq \varepsilon = \frac{1}{4}$$

tengsizlik o'rinli bo'ladi. Bu esa, Koshi teoremasiga asosan, berilgan integralning uzoqlashuvchi ekanligini ko'rsatadi.

Koshi teoremasi muhim ahamiyatga ega bo'lgan teorema, lekin, uning yordamida, amaliyotda, xosmas integrallarning yaqinlashuvchiligini tekshirish har doim yengil bo'lavermaydi. Shuning uchun xosmas integral yaqinlashuvchiligining yetarli shartlaridan foydalanish qulay bo'ladi.

12.2. Xosmas integrallarning absolut va shartli yaqinlashuvchiligi. $f(x)$ funksiya $[a;b-\eta]$ $(\eta > 0)$ da xos ma'noda integrallanuvchi bo'lsin.

12.7- teorema. Agar $\int_a^b |f(x)| dx$ integral yaqinlashuvchi bo'lsa, $\int_a^b f(x)\, dx$ integral ham yaqinlashuvchi bo'ladi.

12.1- eslatma. $\int_a^b |f(x)|\, dx$ integralning uzoqlashuvchi bo'lishidan $\int_a^b f(x)\, dx$ integralning uzoqlashuvchi bo'lishi har doim ham kelib chiqavermaydi.

12.1- ta'rif. Agar $\int_a^b |f(x)|\, dx$ integral yaqinlashuvchi bo'lsa, $\int_a^b f(x)\, dx$ integral *absolut yaqinlashuvchi* deyiladi, $f(x)$ funksiya esa $[a;\, b)$ da *absolut integrallanuvchi funksiya* deb ataladi.

12.2- ta'rif. Agar $\int_a^b f(x)\,dx$ integral yaqinlashuvchi bo'lib, $\int_a^b |f(x)|\,dx$ integral uzoqlashuvchi bo'lsa, $\int_a^b f(x)\,dx$ *shartli yaqinlashuvchi integral* deb ataladi.

12.12- misol. $\int_{0,5}^{1} \frac{\cos 2\pi x}{\sqrt{1-x^2}}\,dx$ xosmas integralning absolut yaqinlashuvchiligini ko'rsating.

Yechilishi. $\forall x \in [0,5;1)$ uchun $|f(x)| = \left|\frac{\cos 2\pi x}{\sqrt{1-x^2}}\right| \le \frac{1}{\sqrt{1-x^2}}$ tengsizlik o'rinli. $\int_{0,5}^{1} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big|_{0,5}^{1} = \frac{\pi}{3}$. Demak, 12.2- teoremaga asosan, $\int_{0,5}^{1} \left|\frac{\cos 2\pi x}{\sqrt{1-x^2}}\right| dx$ integral yaqinlashuvchi. Shunday qilib, 12.1- ta'rifga asosan, berilgan integral absolut yaqinlashuvchi bo'ladi.

12.13- misol. $J = \int_0^1 \frac{\sin \frac{1}{x}}{x^{3/2}}\,dx$ xosmas integralning shartli yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. (11.2) formulaga asosan, berilgan integralni bo'laklab integrallaymiz:

$$J = \sqrt{x} \cos \frac{1}{x}\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{\cos \frac{1}{x}}{\sqrt{x}}\,dx,$$

bunda $\lim_{x\to 0} \sqrt{x} \cos \frac{1}{x} = 0$. $\int_0^1 \frac{\cos \frac{1}{x}}{\sqrt{x}}\,dx$ integral absolut yaqinlashuvchi bo'lganligi uchun, 12.7- teoremaga asosan, $\int_0^1 \frac{\cos \frac{1}{x}}{\sqrt{x}}\,dx$

yaqinlashuvchi, u holda J integral ham yaqinlashuvchi bo'ladi. Endi $\int_0^1 \frac{\left|\sin\frac{1}{x}\right|}{x^{3/2}}dx$ integralni absolut yaqinlashishga tekshiramiz: $|\sin x| \geq \sin^2 x$ tengsizlikdan foydalanib,

$$\int_0^1 \frac{\left|\sin\frac{1}{x}\right|}{x^{3/2}}dx \geq \frac{1}{2}\int_0^1 \frac{\cos\frac{2}{x}}{x^{3/2}}dx = \frac{1}{2}\int_0^1 \frac{dx}{x^{3/2}} - \frac{1}{2}\int_0^1 \frac{\cos\frac{2}{x}}{x^{3/2}}dx$$

munosabatga ega bo'lamiz. Bunda $\int_0^1 \frac{dx}{x^{3/2}}$ integral uzoqlashuvchi.

Demak, 12.2- teoremaga ko'ra, $\int_0^1 \frac{\left|\sin\frac{1}{x}\right|}{x^{3/2}}dx$ integral uzoqlashuvchi. Shunday qilib, 12.2- ta'rifga ko'ra, J integral shartli yaqinlashuvchi bo'ladi.

12.3. Xosmas integrallar yaqinlashuvchiligining yetarli shartlari. $f(x)$ va $g(x)$ funksiyalar $[a;b)$ da berilgan bo'lib, b shu funksiyalarning maxsus nuqtasi bo'lsin.

12.8- teorema (Dirixle teoremasi). $f(x)$ va $g(x)$ funksiyalar $[a;b)$ da berilgan bo'lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a;b)$ da uzluksiz va uning shu oraliqdagi boshlang'ich $F(x)$ $(F'(x) = f(x))$ funksiyasi chegaralangan, ya'ni

$$\exists M > 0 : \forall x \in [a,b) \to |F(x)| \leq M;$$

2) $g(x)$ funksiya $[a;b)$ da $g'(x)$ uzluksiz hosilaga ega;

3) $g(x)$ funksiya $[a;b)$ da monoton, ya'ni $\forall x \in [a,b)$ lar uchun $g'(x) \geq 0$ yoki $g'(x) \leq 0$;

4) $\lim\limits_{x\to b-0} g(x) = 0$.

U holda

$$\int_a^b f(x)g(x)dx \qquad (*)$$

integral yaqinlashuvchi bo‘ladi.

12.9- teorema (Abel teoremasi). $f(x)$ va $g(x)$ funksiyalar $[a;b)$ da berilgan bo‘lib, ular quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a;b)$ da uzluksiz va $\int_a^b f(x)dx$ integral yaqinlashuvchi;

2) $g(x)$ funksiya $[a;b)$ da chegaralangan;

3) $g(x)$ funksiya uzluksiz, differensiallanuvchi va $[a;b)$ da monoton, ya’ni $\forall x \in [a,b)$ uchun $g'(x) \geq 0$ yoki $g'(x) \leq 0$ bo‘lsin.

U holda (*) integral yaqinlashuvchi bo‘ladi.

12.14- misol. $\int_{-1}^{0} \sin\left(\frac{1}{\sin x}\right)\frac{dx}{\sin x}$ xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Berilgan integralni Dirixle teoremasidan foydalanib, yaqinlashuvchilikka tekshiramiz:

$$f(x) = \frac{\cos x}{\sin^2 x}\sin\left(\frac{1}{\sin x}\right), \quad g(x) = \mathrm{tg}x$$

deb belgilaymiz. $f(x)$funksiya $[-1;0)$ da uzluksiz va uning boshlang‘ich funksiyasi $\cos\frac{1}{\sin x}$ ga teng bo‘lib, u chegaralangan. $g(x)$ funksiya esa $[-1;0)$ da uzluksiz differensiallanuvchi va o‘suvchi bo‘lib, $\lim\limits_{x\to 0-0} \mathrm{tg}x = 0$ bo‘ladi.

Demak, integral ostidagi funksiya Dirixle teoremasining hamma shartlarini qanoatlantiradi. Shuning uchun berilgan xosmas integral yaqinlashuvchi bo‘ladi.

12.15- misol. $\int_0^1 \frac{1}{e^x - 1}\sin\frac{1}{x}dx$ xosmas integralni yaqinlashuvchilikka tekshiring.

Yechilishi. Berilgan integral ostidagi funksiyani $f(x)=\frac{1}{x}\sin\frac{1}{x}\,dx,\ g(x)=\frac{x}{e^x-1}$ kabi belgilab, integralni yaqinlashishga tekshirishda Abel teoremasidan foydalanamiz:

1) $f(x)$ funksiya (0;1] da uzluksiz va $\int_0^1 \frac{1}{x}\sin\frac{1}{x}\,dx=\int_1^{+\infty}\frac{\sin t}{t}\,dt$ yaqinlashuvchi;

2) x=0 nuqtaning atrofida $g(x)=\frac{x}{e^x-1}$ chegaralangan funksiya, chunki $\lim\limits_{x\to 0}\frac{x}{e^x-1}=1$;

3) $g(x)$ funksiya (0;1] da uzluksiz va differensiallanuvchi bo'lib, uning hosilasi $\forall x\in(0;1]$ da $g'(x)=\frac{e^x(1-x)-1}{(e^x-1)^2}<0$, demak, $g(x)$ funksiya kamayuvchi.

Shunday qilib, berilgan integral ostidagi funksiya Abel teoremasining hamma shartlarini qanoatlantiradi. Shuning uchun berilgan integral yaqinlashuvchi.

12.4. Xosmas integrallarning bosh qiymati.

12.4.1.Chegarasi cheksiz integralning bosh qiymati. $f(x)$ funksiya $(-\infty;+\infty)$ oraliqda berilgan bo'lib, bu oraliqning istalgan chekli qismida xos ma'noda (Riman ma'nosida) integrallanuvchi bo'lsin.

Ma'lumki, $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integral ushbu [(10.2) ga q.] tenglik orqali aniqlanar edi:

$$\int_{-\infty}^{+\infty} f(x)dx=\lim_{\substack{\tau\to-\infty\\ t\to+\infty}}\int_{\tau}^{t} f(x)dx\,. \qquad (1)$$

Bunda t bilan τ larning bir-biriga bog'liq bo'lmasdan o'z limitlariga intilishi talab qilinadi. t va τ larning bir-biriga bog'liq bo'lmagan (1) limiti mavjud bo'lmagan, ya'ni $\int_{-\infty}^{+\infty} f(x)dx$ integral uzoqlashuvchi bo‘lgan holda t va τ lar $t=-\tau$ shartni qanoatlantirib, o‘z limitlariga intilganda (1) limit mavjud bo‘lishi ham mumkin. Shuning uchun bu holni qarash muhim ahamiyatga ega. Masalan,

$$\int_{-\infty}^{+\infty} xdx = \lim_{\substack{\tau\to-\infty \\ t\to+\infty}} \int_{\tau}^{t} xdx = \lim_{\substack{\tau\to-\infty \\ t\to+\infty}} \frac{x^2}{2}\Big|_{\tau}^{t} = \lim_{\substack{\tau\to-\infty \\ t\to+\infty}} \frac{1}{2}(t^2-\tau^2).$$

Bu limit mavjud emas. Agar t va τ lar $t=-\tau$ shartni qanoatlantirsa, u holda $\lim\limits_{\substack{\tau\to-\infty \\ t\to+\infty}} \frac{1}{2}(t^2-\tau^2)=0$ bo‘ladi.

12.3- ta’rif. Agar $t=-\tau$ bo‘lib, $t\to+\infty$ da $\int_{\tau}^{t} f(x)dx$ ifodaning limiti mavjud va chekli bo‘lsa, $\int_{-\infty}^{+\infty} f(x)dx$ uzoqlashuvchi xosmas integral bosh qiymat ma’nosida Koshi ma’nosida yaqinlashuvchi deyilib, $\lim\limits_{t\to+\infty}\int_{-t}^{-t} f(x)dx$ limit esa $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integralning *bosh qiymati* deyiladi va $V.P.\int_{-\infty}^{+\infty} f(x)dx$ kabi belgilanadi.

Demak,

$$V.P.\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_{-t}^{t} f(x)dx.$$

Bunda V.P. belgisi fransuzcha «valcur principiale» — «bosh qiymat» so‘zlarining birinchi harflarini ifodalaydi.

12.2- eslatma. $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u bosh qiymat ma’nosida ham yaqinlashuvchi bo‘ladi va ular bir-biriga teng bo‘ladi. Lekin $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integralning bosh qiymat ma’nosida yaqinlashuvchi bo‘lishidan uning xosmas ma’noda yaqinlashuvchi bo‘lishi har doim ham kelib chiqavermaydi.

12.3- eslatma. $f(x)$ toq funksiya bo'lsa, har doim

$$V.P.\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_{-t}^{t} f(x)dx = 0$$

bo'ladi.

Agar $f(x)$ juft funksiya bo'lsa,

$$V.P.\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_{-t}^{t} f(x)dx = 2\lim_{t\to+\infty}\int_{0}^{t} f(x)dx = 2\lim_{t\to+\infty}\int_{-t}^{0} f(x)dx$$

bo'ladi.

Shuning uchun $\int_{-\infty}^{0} f(x)dx$ va $\int_{0}^{+\infty} f(x)dx$ integrallarning birortasi uzoqlashuvchi bo'lsa, $V.P.\int_{-\infty}^{+\infty} f(x)dx$ ham mavjud bo'lmaydi.

Ma'lumki, $(-\infty; +\infty)$ ning istalgan chekli qismida xos ma'noda integrallanuvchi ixtiyoriy $f(x)$ funksiyani (shu funksiya kabi xossalarga ega bo'lgan) juft va toq funksiyalar yig'indisi shaklida tasvirlash mumkin:

$$f(x) = \varphi(x) + \psi(x) \ ,$$

bunda $\varphi(x) = \dfrac{f(x)+f(-x)}{2}$ — juft funksiya, $\psi(x) = \dfrac{f(x)-f(-x)}{2}$ — toq funksiya. 12.3- eslatmaga asosan, agar $\int_{-\infty}^{+\infty} \varphi(x)dx$ integral yaqinlashuvchi bo'lsa,

$$V.P.\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{+\infty} \varphi(x)dx \qquad (2)$$

bo'ladi.

12.16- misol. $V.P.\int_{-\infty}^{\infty} \dfrac{x+5}{x^2+16}dx$ integralni toping.

Yechilishi. Integral ostidagi $f(x)$ funksiyani ushbu

$$f(x) = \frac{x+5}{x^2+16} = \frac{5}{x^2+16} + \frac{x}{x^2+16}$$

ko‘rinishda tasvirlaymiz, bunda $\varphi(x)=\frac{5}{x^2+16}$ — juft funksiya, $\psi(x)=\frac{x}{x^2+16}$ — toq funksiya. (2) formulaga asosan,

$$V.P.\int_{-\infty}^{+\infty}\frac{x+5}{x^2+16}dx=\int_{-\infty}^{+\infty}\frac{5}{x^2+16}dx=2\int_{0}^{+\infty}\frac{5}{x^2+16}dx=\frac{10}{4}\operatorname{arctg}\frac{x}{4}\Big|_0^{\infty}=\frac{5\pi}{4}.$$

12.17- misol. $V.P.\int_{-\infty}^{+\infty}\frac{dx}{x^2+4}$ integralni toping.

Yechilishi. Integral ostidagi $f(x)=\frac{1}{x^2+4}$ juft funksiya bo‘lgani uchun, 12.2- eslatmaga asosan,

$$V.P.\int_{-\infty}^{+\infty}\frac{dx}{x^2+4}=2\int_{0}^{+\infty}\frac{dx}{x^2+4}=2\cdot\frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_0^{\infty}=\frac{\pi}{2}.$$

Demak, $\int_{-\infty}^{+\infty}\frac{dx}{x^2+4}$ — xosmas integral yaqinlashuvchi bo‘lgani uchun bu integral bosh qiymat ma’nosida ham mavjud va ularning qiymati bir-biriga teng.

4.2. Chegaralanmagan funksiya xosmas integralining bosh qiymati. $f(x)$ funksiya $[a;b]$ kesmaning c $(a<c<b)$ nuqtasidan tashqari hamma nuqtalarida aniqlangan bo‘lib, $(a;c)$ va $(c;b)$ ning qismidan iborat bo‘lgan istalgan kesmada integrallanuvchi bo‘lsin. U holda agar

$$\lim_{\varepsilon\to 0}\left[\int_{a}^{c-\varepsilon}f(x)dx+\int_{c+\varepsilon}^{b}f(x)dx\right]$$

limit mavjud va chekli bo‘lsa, $f(x)$ funksiya $[a;b]$ kesmada *Koshi ma’nosida integrallanuvchi* deyiladi va limitning bu qiymati integralning *Koshi ma’nosidagi bosh qiymati* deb ataladi va u

$$V.P.\int_{a}^{b}f(x)dx$$

kabi belgilanadi.

Demak,

$$V.P.\int_a^b f(x)dx=\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon} f(x)dx+\int_{x+\varepsilon}^b f(x)dx\right].$$

12.4- eslatma. $\int_a^b f(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u holda u bosh qiymat ma’nosida ham yaqinlashuvchi bo‘ladi va ular bir-biriga teng bo‘ladi, $\int_a^b f(x)dx$ lekin xosmas integralning bosh qiymat ma’nosida yaqinlashuvchi bo‘lishidan uning yaqinlashuvchi bo‘lishi har doim ham kelib chiqavermaydi.

12.18- misol. $\int_a^b \frac{dx}{x-c}$ $(a<c<b)$ integralning xosmas integral ma’nosida mavjud emasligini, bosh qiymat ma’nosida esa mavjudligini ko‘rsating.

Yechilishi. Xosmas integralning ta’rifiga ko‘ra,

$$\int_a^b \frac{dx}{x-c}=\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\left[\int_a^{c-\varepsilon_1}\frac{dx}{x-c}+\int_{c+\varepsilon_2}^b\frac{dx}{x-c}\right]=\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\left[-\ln(c-x)\Big|_a^{c-\varepsilon_1}+\ln(x-c)\Big|_{c+\varepsilon_2}^b\right]=$$

$$\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\left[\ln\varepsilon_1-\ln(c-a)+\ln(b-c)-\ln\varepsilon_2\right]=\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\left[\ln\frac{\varepsilon_1}{\varepsilon_2}+\ln\frac{b-c}{c-a}\right].$$

$\lim\limits_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\ln\frac{\varepsilon_1}{\varepsilon_2}$ — mavjud emas. Shunga ko‘ra, berilgan xosmas integral mavjud emas. Lekin $\varepsilon_1=\varepsilon_2=\varepsilon$ shartga ko‘ra $\lim\limits_{\varepsilon_1\to 0}\ln\frac{\varepsilon_1}{\varepsilon_2}=0$.

Demak, berilgan integral xosmas integral ma’nosida mavjud emas, lekin bosh qiymat ma’nosida mavjud va u

$$V.P.\int_a^b\frac{dx}{x-c}=\ln\frac{b-c}{c-a}.$$

Mustaqil yechish uchun misollar

Integrallarning yaqinlashuvchiligini isbotlang:

12.1. $\int_0^{+\infty} \frac{dx}{x^2+\sqrt[3]{x}}$. **12.2.** $\int_0^{2\pi} \frac{dx}{\sqrt[3]{x}}$. **12.3.** $\int_0^{8} \frac{\sqrt{x}\,dx}{e^{\sin x}-1}$.

12.4. $\int_0^1 \frac{dx}{\sqrt{x}+\mathrm{arctg}x}$. **12.5.** $\int_0^4 \sqrt{\frac{16+x^2}{16-x^2}}\,dx$. **12.6.** $\int_0^1 \frac{dx}{\sqrt[5]{1-x^{10}}}$.

12.7. $\int_0^{\pi} \sin\left(\frac{1}{\cos x}\right)\frac{dx}{\sqrt{x}}$. **12.8.** $\int_0^1 \frac{dx}{\arccos x}$. **12.9.** $\int_0^{\pi} \frac{\ln(\sin x)}{\sqrt[3]{x}}\,dx$.

12.10. $\int_0^1 \frac{dx}{e^{\sqrt{x}}-1}$.

Integrallarning uzoqlashuvchiligini isbotlang:

12.11. $\int_2^4 \frac{dx}{\ln(x-1)}$. **12.12.** $\int_{-1}^1 \frac{dx}{\ln(1+x)}$. **12.13.** $\int_0^1 \frac{dx}{e^x-\cos x}$.

12.14. $\int_1^2 \frac{x-2}{x^3-3x^2+4}\,dx$. **12.15.** $\int_0^{\pi} \frac{\ln(\sin x)}{x\sqrt{\sin x}}\,dx$.

12.16. $\int_0^1 \frac{x^2dx}{\sqrt[3]{(1-x^2)^5}}$.

Xosmas integrallar α ning qanday qiymatlarida absolut va shartli yaqinlashuvchi bo‘ladi:

12.17. $\int_0^1 \frac{x^{\alpha}}{x^2+1}\sin\frac{1}{x}\,dx$. **12.18.** $\int_0^{0,5} \left(\frac{x}{1-x}\right)^{\alpha} \cos\frac{1}{x^2}\,dx$.

12.19. $\int_0^1 x^{\alpha}\mathrm{arctg}x\cos\frac{1}{x}\,dx$. **12.20.** $\int_0^{\frac{\pi}{4}} \mathrm{tg}^{\alpha}x\cos(\mathrm{ctg}x)dx$.

12.21. $\int_0^1 \frac{(1-x)^{\alpha}}{x}\sin\frac{1}{x}\,dx$. **12.22.** $\int_0^1 \frac{x^{\alpha}}{e^x-1}\sin\frac{1}{x}\,dx$.

12.23. $\int_0^1 \frac{\sin x^{\alpha}}{x^2}\,dx$.

Xosmas integrallarning Koshi ma'nosidagi bosh qiymatni toping:

12.24. $V.P.\int_{-\infty}^{+\infty}\cos x dx$. **12.25.** $V.P.\int_{-\infty}^{+\infty}\operatorname{arctg} x\, dx$.

12.26. $V.P.\int_{-\infty}^{+\infty}\frac{x+5}{x^2+25}dx$.

12.27. $V.P.\int_{-\infty}^{+\infty}\frac{\operatorname{arctg}\frac{x}{2}}{x^2+4}dx$. **12.28.** $V.P.\int_{-1}^{1}\frac{dx}{\sqrt[5]{x}}$. **12.29.** $V.P.\int_{-1}^{1}\frac{dx}{x^4}$.

12.30. $V.P.\int_{-4}^{4}\frac{dx}{x^2-1}$. **12.31.** $V.P.\int_{0}^{\pi}x\operatorname{tg}x$. **2.32.** $V.P.\int_{0,5}^{4}\frac{dx}{x\ln x}$.

12.33. $V.P.\int_{1}^{5}\frac{dx}{(x-3)^5}$. **12.34.** $V.P.\int_{4}^{10}\frac{dx}{(x-6)^4}$.

12.35. α ning qanday qiymatida ushbu integral $V.P.\int_{0}^{2}\frac{x^\alpha}{1-x}dx$ mavjud?

Misollarning javoblari

12.17. $\alpha>-1$ da absolut yaqinlashuvchi, $-2<\alpha\leq-1$ da shartli yaqinlashuvchi. **12.18.** $\alpha>-1$ da absolut yaqinlashuvchi, $-3<\alpha\leq-1$ da shartli yaqinlashuvchi. **12.19.** $\alpha>-2$ da absolut yaqinlashuvchi, $-3<\alpha\leq-2$ da shartli yaqinlashuvchi. **12.20.** $\alpha>-1$ da absolut yaqinlashuvchi, $-2<\alpha\leq-1$ da shartli yaqinlashuvchi. **12.21.** $\alpha>-1$ da shartli yaqinlashuvchi, $\alpha\leq-1$ da uzoqlashuvchi. **12.22.** $\alpha>0$ da absolut yaqinlashuvchi, $-1<\alpha\leq 0$ da shartli yaqinlashuvchi. **12.23.** $\alpha>1$ da absolut yaqinlashuvchi, $\alpha<-1$ da shartli yaqinlashuvchi. **12.24.** Mavjud emas. **12.25.** 0. **12.26.** π. **12.27.** 0. **12.28.** 0. **12.29.** $-\frac{2}{3}$. **12.30.** $\frac{1}{2}\ln\frac{3}{5}$. **12.31.** $-\pi\ln 2$. **12.32.** ln2. **12.33.** 0. **12.34.** Mavjud emas. **12.35.** $\alpha>-1$.

IV BOB

PARAMETRGA BOG'LIQ BO'LGAN INTEGRALLAR

Matematika va matematik fizikaning ko'p sohalarida parametrga bog'liq bo'lgan integrallar asosiy apparat sifatida ishlatiladi. Shuning uchun biz bu bobda parametrga bog'liq bo'lgan integrallarning funksional xossalarini o'rganish bilan shug'ullanamiz.

13- §. Parametrga bog'liq bo'lgan xos integral tushunchasi

$f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;b], y\in E\subset R\}$ to'plamda berilgan bo'lib, y ning E to'plamdan olingan har bir tayinlangan qiymatida, $f(x,y)$ funksiya x ning funksiyasi sifatida $[a;b]$ da (xos ma'noda) integrallanuvchi, ya'ni

$$\int_a^b f(x,y)dx$$

integral mavjud bo'lsin. Bu integral y o'zgaruvchining E dan olingan qiymatiga bog'liq bo'ladi va uni

$$I(y)=\int_a^b f(x,y)dx \tag{13.1}$$

deb belgilaymiz. Odatda (13.1) *integral parametrga bog'liq bo'lgan integral*, y o'zgaruvchi esa *parametr* deb ataladi.

13.1- misol. $f(x,y)=\cos xy$ funksiyaning x o'zgaruvchi bo'yicha $[a;b]$ dagi integrali y ning funksiyasi ekanligini ko'rsating, bunda $y\neq 0$.

Yechilishi. $f(x,y)=\cos xy$ funksiya y ning $E=R\setminus\{0\}$ to'plamdan olingan har bir o'zgarmas qiymatida, x o'zgaruvchining funksiyasi sifatida $[a;b]$ da xos ma'noda integrallanuvchi, ya'ni

$$\int_a^b \cos xydx=\frac{1}{y}\int_a^b \cos xyd(xy)=\frac{\sin by-\sin ay}{y}.$$

Demak, integral $E=R\setminus\{0\}$ to'plamda berilgan

$$I(y)=\frac{\sin by-\sin ay}{y}$$

funksiyadan iborat bo'lar ekan.

I(y) funksiyaning funksional xossalarini (limiti, uzluksizligi, differensiallanuvchiligi, integrallanuvchiligi va h.k) o'rganishda, (13.1) integral ostidagi *f(x,y)* funksiyaning *y* bo'yicha limiti va unga intilish xarakteri muhim rol o'ynaydi.

13.1. Limit funksiya. Tekis yaqinlashish. Limit funksiyaning uzluksizligi. *f(x,y)* funksiya $M=\{(x,y)\in R^2:\ a\le x\le b,\ y\in E\subset R\}$ to'plamda berilgan bo'lib, y_0 esa E to'plamning limit nuqtasi bo'lsin. Agar $y\to y_0$ da *f(x,y)* funksiyaning limiti mavjud bo'lsa, bu limit *x* o'zgaruvchining $[a;b]$ dan olingan qiymatiga bog'liq bo'ladi, ya'ni $\lim\limits_{y\to y_0} f(x,y)=f(x,y_0)=\varphi(x)$.

13.1- ta'rif. Agar $\forall\varepsilon>0$ olinganda ham, $\forall x\in[a;b]$ uchun shunday $\delta=\delta(\varepsilon,x)>0$ topilib, $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon$$

tengsizlik bajarilsa, $\varphi(x)$ funksiya *f(x,y)* funksiyaning $y\to y_0$ dagi *limit funksiyasi* deyiladi.

f(x,y) funksiya *M* to'plamda aniqlangan bo'lib, ∞ esa *E* to'plamning limit nuqtasi bo'lsin.

13.2- ta'rif. Agar $\forall\varepsilon>0$ olinganda ham, $\forall x\in[a,b]$ uchun shunday $\Delta=\Delta(\varepsilon,x)>0$ topilib, $|y|>\Delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon$$

tengsizlik bajarilsa, $\varphi(x)$ funksiya *f(x,y)* funksiyaning $y\to+\infty$ dagi *limit funksiyasi* deyiladi.

13.2- misol. $f(x,y)=x^2\text{arctg}\,y$ funksiya

$$M=\{(x,y)\in R^2:0\le x\le 1,\ 0\le y\le\frac{\pi}{4}\}$$

to'plamda berilgan bo'lsa, $y \to \frac{\pi}{4}$ da $f(x,y) = x^2 \mathrm{arctg} y$ funksiyaning limit funksiyasini toping.

Yechilishi. Ravshanki, $y \to \frac{\pi}{4}$ da $f(x,y) = x^2 \mathrm{arctg} y$ funksiyaning limiti $\varphi(x) = x^2$ bo'ladi. $\forall \varepsilon > 0$ sonni olaylik. Agar $\delta = \varepsilon$ desak, u holda $\left| y - \frac{\pi}{4} \right| < \delta$ tengsizlikni qanoatlantiruvchi

$\forall y \in [0; \frac{\pi}{4}]$ va $\forall x \in [0,1]$ lar uchun

$$|f(x,y) - \varphi(x)| = |x^2 \mathrm{arctg} y - x^2| = |x^2| \, |\mathrm{arctg} y - 1| =$$

$$= |x^2| \left| \mathrm{arctg} y - \mathrm{arctg} \frac{\pi}{4} \right| \leq \left| y - \frac{\pi}{4} \right| < \delta = \varepsilon$$

munosabat o'rinli bo'ladi. Demak, 13.1- ta'rifga ko'ra, $y \to \frac{\pi}{4}$ da $f(x,y) = x^2 \mathrm{arctg} y$ funksiyaning limit funksiyasi

$$\varphi(x) = \lim_{y \to \frac{\pi}{4}} x^2 \mathrm{arctg} y = x^2$$

bo'ladi.

13.3- misol. $f(x,y) = (1-x)\mathrm{arctg} x^y$ funksiya $M = \{(x,y) \in R^2 : : 0 < x \leq 1,\ 0 \leq y \leq 1\}$ to'plamda berilgan bo'lsa, $y \to 0$ da bu funksiyaning limit funksiyasini toping.

Yechilishi. Agar x o'zgaruvchi tayinlangan bo'lsa,

$$\lim_{y \to 0} x^y = 1, \quad \lim_{y \to 0} (1-x)\mathrm{arctg} x^y = \frac{\pi}{4}(1-x) = \varphi(x).$$

Haqiqatan ham, $\forall \varepsilon > 0$ songa ko'ra, $\delta = \log_x (1-\varepsilon)$, $(x \neq 1)$ deb olinsa, unda $|y - y_0| = |y| < \delta$ tengsizlikni qanoatlantiradigan $\forall y \in [0;1]$ uchun

$$\left|f(x,y)-\varphi(x)\right|=\left|(1-x)\operatorname{arctg}x^{y}-\frac{\pi}{4}(1-x)\right|=\left|1-x\right|\left|\operatorname{arctg}x^{y}-\operatorname{arctg}1\right|\le$$

$$\le\left|1-x^{y}\right|<1-x^{\log_{x}(1-\varepsilon)}=1-(1-\varepsilon)=\varepsilon$$

tengsizlik bajariladi. Demak, 13.1- ta'rifga asosan $y\to 0$ da berilgan $f(x,y)$ funksiyaning limit funksiyasi $\varphi(x)=\frac{\pi}{4}(1-x)$ bo'ladi.

13.4- misol. $f(x,y)=\frac{1}{x^{3}}\cos\frac{x}{y}$ funksiya $M=\{(x,y)\in R^{2}:$ $:0<x<1,\ 0<y<\infty\}$ to'plamda berilgan bo'lsa, $y\to+\infty$ da berilgan funksiyaning limit funksiyasini toping.

Yechilishi. Agar x o'zgaruvchi tayinlangan bo'lsa,

$$\lim_{y\to+\infty}\frac{1}{x^{3}}\cos\frac{x}{y}=\frac{1}{x^{3}}=\varphi(x)$$

bo'ladi. Haqiqatan ham, $\forall\varepsilon>0$ songa asosan, $\Delta=\frac{1}{\sqrt{2x\varepsilon}}$ deb olinsa, unda $|y|>\Delta$ tengsizlikni qanoatlantiruvchi $\forall y\in(0;+\infty)$ uchun

$$\left|\frac{1}{x^{3}}\cos\frac{x}{y}-\frac{1}{x^{3}}\right|=2\left|\frac{1}{x^{3}}\right|\left|\sin\frac{x}{2y}\right|\cdot\left|\sin\frac{x}{2y}\right|\le\frac{1}{2xy^{2}}<\varepsilon$$

tengsizlik o'rinli bo'ladi. Demak, ta'rifga asosan, $y\to+\infty$ da $f(x,y)=\frac{1}{x^{3}}\cos\frac{x}{y}$ funksiyaning limit funksiyasi $\varphi(x)=\frac{1}{x^{3}}$ bo'ladi.

13.1- eslatma. Yuqorida keltirilgan 13.2- misolda limit funksiya ta'rifida $\delta=\varepsilon$ bo'lib, y faqat ε gagina bog'liq. 13.3, 13.4- misollarda esa $\delta=\log_{x}(1-x)$, $\Delta=\frac{1}{\sqrt{2x\varepsilon}}$ bo'lib, y berilgan $\varepsilon>0$ bilan birga qaralayotgan x nuqtaga ham bog'liq ekanligini ko'ramiz.

Limit funksiya ta'rifidagi $\delta>0$ ning qaralayotgan x nuqtalarga bog'liq bo'lmay, faqat $\varepsilon>0$ gagina bog'liq ravishda tanlab olinishi mumkin bo'lgan hol, muhimdir.

13.3- ta'rif. $f(x,y)$ funksiya M to'plamda berilgan bo'lib, uning $y \to y_0$ dagi limit funksiyasi $\varphi(x)$ bo'lsin. Agar $\forall \varepsilon > 0$ olinganda ham $\exists \delta = \delta(\varepsilon) > 0, |y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $\forall y \in E$ va $\forall x \in [a;b]$ uchun

$$|f(x,y) - \varphi(x)| < \varepsilon$$

tengsizlik bajarilsa, $f(x,y)$ funksiya $[a;b]$ da o'z limit funksiyasi $\varphi(x)$ ga *tekis yaqinlashadi* deyiladi.

13.4- ta'rif. $f(x,y)$ funksiya M to'plamda $y \to y_0$ da $\varphi(x)$ limit funksiyaga ega bo'lsin. $\forall \delta$ olinganda ham, shunday $\varepsilon_0 > 0$, $x_0 \in [a,b]$ va $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $y_1 \in E$ topilib,

$$|f(x_0, y_1) - \varphi(x_0)| \geq \varepsilon_0$$

tengsizlik o'rinli bo'lsa, $f(x,y)$ funksiya $\varphi(x)$ limit funksiyaga *notekis yaqinlashadi* deyiladi.

Yuqoridagi keltirilgan 13.2- misolda berilgan funksiya o'zining limit funksiyasiga tekis yaqinlashuvchi, 13.3- va 13.4- misollarda esa notekis yaqinlashuvchi bo'ladi.

$f(x,y)$ funksiya M to'plamda berilgan bo'lib, y_0 esa E to'plamning limit nuqtasi bo'lsin.

13.1- teorema. $f(x,y)$ funksiya $y \to y_0$ da $\varphi(x)$ limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi uchun, $\forall \varepsilon > 0$ olinganda ham x ga ($x \in [a;b]$) bog'liq bo'lmagan $\exists \delta > 0$ topilib, $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $\forall y', y \in E$ va $\forall x \in [a,b]$ uchun $|f(x,y) - f(x,y')| < \varepsilon$ tengsizlikning bajarilishi zarur va yetarli.

13.2- teorema. Agar $f(x,y)$ funksiya y ning E to'plamdan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a,b]$ da uzluksiz bo'lsa va $y \to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ limit funksiyaga tekis yaqinlashsa, u holda $\varphi(x)$ funksiya ham $[a,b]$ da uzluksiz bo'ladi.

13.5- misol. Ushbu $f(x,y)=\dfrac{2y^2x}{1+y^\alpha x^2}$, $\alpha>4$ funksiya

$$M=\{(x,y)\in R^2 : x\in R, y\in(0;+\infty)\}$$

to'plamda berilgan bo'lsin. $y_0\to+\infty$ da $f(x,y)$ funksiyaning limit funksiyasini toping va uni tekis yaqinlashishga tekshiring.

Yechilishi. Agar $x=0$ bo'lsa, $\forall y\in(0;+\infty)$ uchun $f(0,y)=0$ bo'ladi. Agar $x\neq 0$ bo'lsa, $\forall y\in(0;+\infty)$ uchun

$$|f(x,y)|=\left|\frac{2xy^2}{1+x^2y^\alpha}\right|\le\frac{2y^2|x|}{|x|^2y^\alpha}=\frac{2y^2}{|x|y^\alpha}=\frac{2}{|x|y^{\alpha-2}}$$

bo'ladi. Bundan $\lim\limits_{y\to+\infty}f(x,y)=0$.

Shunday qilib, M to'plamda berilgan $f(x,y)=\dfrac{2y^2x}{1+y^\alpha x^2}$ funksiyaning limit funksiyasi $\varphi(x)=0$ bo'lar ekan. $x\neq 0$ uchun $1+y^\alpha x^2\ge 2y^{\frac{\alpha}{2}}|x|$ tengsizlik o'rinli.

$\forall\varepsilon>0$ berilganda $\Delta=\dfrac{1}{\sqrt[\alpha-4]{\varepsilon^2}}$ deb olsak, u holda $y>\Delta$ ni qanoatlantiruvchi $\forall y\in(0;+\infty)$ uchun

$$|f(x,y)-\varphi(x)|=\left|\frac{2y^2x}{1+y^\alpha x^2}\right|\le\frac{2y^2|x|}{2y^{\frac{\alpha}{2}}|x|}=\frac{1}{y^{\frac{\alpha}{2}-2}}<\varepsilon$$

tengsizlik o'rinli bo'ladi. Demak, $\forall x\in R$ uchun $y\to+\infty$ da berilgan $f(x,y)$ funksiya o'zining $\varphi(x)=0$ limit funksiyasiga tekis yaqinlashadi.

13.6- misol. Ushbu

$$f(x,y)=\frac{y+1}{y+x^2},\ M=\{(x,y)\in R^2 : x\in[-1;1], y\in(0;+\infty)\},$$

funksiyaning limit funksiyasini toping va unga intilish xarakterini aniqlang:

Yechilishi. $\lim\limits_{y\to+\infty} f(x,y)=\lim\limits_{y\to+\infty}\dfrac{1+\frac{1}{y}}{1+\frac{x^2}{y}}=1$; $\varphi(x)=1$. $\forall\varepsilon>0$ songa

ko'ra, $\Delta=\dfrac{1}{\varepsilon}$ deb olsak, u holda $y>\Delta$ tengsizlikni qanoatlantiruvchi $\forall y\in(0;+\infty)$ uchun

$$|f(x,y)-\varphi(x)|=\left|\frac{y+1}{y+x^2}-1\right|=\frac{1-x^2}{y+x^2}\leq\frac{1}{y}<\varepsilon$$

tengizlik o'rinli bo'ladi. Demak, $f(x,y)$ funksiya $\varphi(x)=1$ limit funksiyaga tekis yaqinlashadi.

13.7- misol. Ushbu funksiyaning limit funksiyasini toping va unga intilish xarakterini aniqlang:

$$f(x,y)=y\cdot\sin\frac{1}{xy},\ M=\{(x,y)\in R^2:0<x<+\infty, y\in(0;+\infty)\}.$$

Yechilishi. Ma'lumki, $t\to0$ da $\sin t\sim t$. Buni e'tiborga olgan holda, $y\to+\infty$ da $y\sin\dfrac{1}{yx}\sim y\dfrac{1}{yx}$. Berilgan funksiyaning $y\to+\infty$ dagi limit funksiyasi $(0,+\infty)$ da $\varphi(x)=\dfrac{1}{x}$ funksiyadan iborat.

$0<x\leq1$ da $y\sin\dfrac{1}{yx}$ funksiya $y\to+\infty$ da $\dfrac{1}{x}$ limit funksiyaga notekis yaqinlashuvchi bo'ladi. Haqiqatan ham, $x_1=\dfrac{1}{y}$ deb olsak, u holda

$$|f(x_1,y)-\varphi(x_1)|=\left|y\sin\frac{1}{yx_1}-\frac{1}{x_1}\right|=|y\sin1-y|=$$
$$=y(1-\sin1)\geq1-\sin1=\varepsilon_0,\ y\geq1.$$

Shunday qilib, $y \to +\infty$ da $y\sin\frac{1}{yx}$ funksiya $(0;1]$ da $\varphi(x)=\frac{1}{x}$ limit funksiyaga notekis intiladi, $1 \le x < +\infty$ da esa tekis intiladi.

Haqiqatan ham, $\forall\varepsilon > 0$ songa ko'ra, $\Delta = \frac{1}{\varepsilon}$ deb olsak, u holda $y > \Delta$ tengsizlikni qanoatlantiruvchi $\forall y \in (0;+\infty)$ uchun

$$\left|y\sin\frac{1}{yx}-\frac{1}{x}\right| = y\left|\sin\frac{1}{yx}-\frac{1}{yx}\right| \le \frac{y}{2y^2x^2} \le \frac{1}{y} < \varepsilon,$$

chunki $\forall t \in R$ uchun $|\sin t - t| < \frac{t^2}{2}$ tengsizlik o'rinli.

13.2. Parametrga bog'liq bo'lgan integrallarning xossalari. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a,b], y\in E\subset R\}$ to'plamda berilgan bo'lib, y_0 shu E to'plamning limit nuqtasi bo'lsin.

13.3- teorema (integral belgisi ostida limitga o'tish). Agar $f(x,y)$ funksiya:

1) y ning E to'plamdagi har bir tayin qiymatida x ning funksiyasi sifatida $[a;b]$ da uzluksiz bo'lsa;

2) $y \to y_0$ da $\varphi(x)$ limit funksiyaga ega va unga x ga nisbatan tekis yaqinlashsa,

$$\lim_{y\to y_0} I(y) = \lim_{y\to y_0}\int_a^b f(x,y)dx = \int_a^b \varphi(x)dx$$

tenglik o'rinli, ya'ni parametr bo'yicha integral belgisi ostida limitga o'tish mumkin.

13.8- misol. $\lim\limits_{y\to 0}\int_{-\pi}^{\pi}(x^2+\cos^2 xy)dx$ ni toping.

Yechilishi. 1) Integral ostidagi $f(x,y)=(x^2+\cos^2 xy)$ funksiyaning $M=\{(x,y)\in R^2 : x\in[-\pi;\pi], y\in[-1;1]\}$ da uzluksizligi ravshan.

2) Limit funksiyani topamiz: $\lim\limits_{y\to 0}(x^2+\cos^2 xy) = x^2+1$.

Demak, limit funksiya $\varphi(x)=x^2+1$ ekan. $\forall\varepsilon>0$ olinganda, $\delta=\frac{\sqrt{\varepsilon}}{\pi}$ deb olinsa, $\forall x\in[-\pi;\pi]$ va $\forall y\in[-1;1]$ uchun

$$|f(x,y)-\varphi(x)| = |x^2+\cos^2 xy - x^2 - 1| = |\cos^2 xy - 1| =$$
$$= |\sin^2 xy| \le |xy|^2 < \varepsilon$$

bo'ladi. 13.3- teoremaga asosan,

$$\lim_{y\to 0}\int_{-\pi}^{\pi}(x^2+\cos^2 xy)dx = \int_{-\pi}^{\pi}\lim_{y\to 0}(x^2+\cos^2 xy)dx = \int_{-\pi}^{\pi}(x^2+1)dx =$$
$$=2\int_0^{\pi}(x^2+1)dx = \frac{2\pi}{3}(\pi^2+3).$$

13.9- misol. Ushbu integralda limit belgisini integral ostiga kiritish mumkinmi:

$$\lim_{y\to 0}\int_0^1 \frac{2xy^2}{(y^2+x^2)^2}dx\,?$$

Yechilishi. Faraz qilaylik, limit belgisini integral ostiga kiritish mumkin bo'lsin. U holda, tayinlangan x lar uchun $\lim\limits_{y\to 0}\frac{2xy^2}{(y^2+x^2)^2}=0$ ekanini ko'rish qiyin emas. Demak,

$$\int_0^1 \lim_{y\to 0}\frac{2xy^2}{(y^2+x^2)^2}dx = 0$$

bo'ladi. Endi integralning qiymatini hisoblab, so'ngra limitga o'tamiz:

$$\int_0^1 \frac{2xy^2}{(y^2+x^2)^2}dx = y^2\int_0^1\frac{d(y^2+x^2)}{(y^2+x^2)^2} = -\frac{y^2}{y^2+x^2}\Big|_0^1 = \frac{1}{y^2+1},$$

$$\lim_{y\to 0}\frac{1}{y^2+1}=1\,.$$

Shunday qilib, limit belgisini integral ostiga kiritish mumkin emas ekan, chunki $f(x,y)=\frac{2xy^2}{(y^2+x^2)^2}$ funksiya (0;0) nuqtada uzilishga ega va $M=\{(x,y)\in R^2 : x\in[0;1], y\in[-1;0)\cup(0;1]\}$ to'plamda esa o'zining limit funksiyasiga tekis intilmaydi.

13.10- misol. $\lim_{y\to+\infty}\int_1^2 \frac{\ln(x+|y|)}{\ln(x^2+y^2)}dx$ integralni hisoblang.

Yechilishi. $f(x,y)=\frac{\ln(x+|y|)}{\ln(x^2+y^2)}$ funksiya tayinlangan $y\ (|y|>1)$ da x bo‘yicha [1; 2] da uzluksiz va $y\to+\infty$ da $f(x,y)\to\frac{1}{2}$.

Haqiqatan ham, $\forall x\in[1;\ 2]$ va $|y|>\sqrt{e^{\frac{1}{\varepsilon}}-1}$ bo‘lganda,

$$\left|\frac{\ln(x+|y|)}{\ln(x^2+y^2)}-\frac{1}{2}\right|=\left|\frac{\ln(1+\frac{2x|y|}{x^2+y^2})}{2\ln(x^2+y^2)}\right|\le\frac{x|y|}{(x^2+y^2)\ln(x^2+y^2)}\le$$

$$\le\frac{2|y|}{(1+y^2)\ln(1+y^2)}\le\frac{1}{\ln(1+y^2)}<\varepsilon$$

bo‘ladi. Shunday qilib, $\forall\varepsilon>0$ songa ko‘ra, $\Delta=\sqrt{e^{\frac{1}{\varepsilon}}-1}$ deb olinsa, $|y|>\Delta$ va $\forall x\in[1;\ 2]$ lar uchun

$$\left|\frac{\ln(x+|y|)}{\ln(x^2+y^2)}-\frac{1}{2}\right|\le\frac{1}{\ln(1+y^2)}<\varepsilon$$

tengsizlik bajariladi.

Demak, 13.3- teoremaga asosan, parametr bo‘yicha integral ostida limitga o‘tish mumkin, ya’ni

$$\lim_{y\to+\infty}\int_1^2\frac{\ln(x+|y|)}{\ln(x^2+y^2)}dx=\int_1^2\lim_{y\to+\infty}\frac{\ln(x+|y|)}{\ln(x^2+y^2)}dx=\frac{1}{2}.$$

13.11- misol. Ushbu limitni hisoblang:

$$\lim_{y\to0}\int_0^1\frac{\ln(1+x+y\sin x)}{1+x}dx.$$

Yechilishi. $f(x,y)=\dfrac{\ln(1+x+y\sin x)}{1+x}$ funksiya tayinlangan $y\in[0;1]$ larda x bo'yicha $[0;1]$ da uzluksiz va $y\to 0$ da $\varphi(x)=\dfrac{\ln(1+x)}{1+x}$ ga tekis yaqinlashuvchi bo'ladi.

Haqiqatan ham, $\forall\varepsilon>0$ songa ko'ra, $\delta=\varepsilon$ deb olsak,

$$|f(x,y)-\varphi(x)|=\left|\frac{\ln(1+x+y\sin x)}{1+x}-\frac{\ln(1+x)}{1+x}\right|=$$

$$=\frac{1}{1+x}\ln\left(1+\frac{y\sin x}{1+x}\right)\le\frac{y|\sin x|}{(1+x)^2}<y<\varepsilon.$$

Demak, 13.3- teoremaga asosan,

$$\lim_{y\to 0}\int_0^1\frac{\ln(1+x+y\sin x)}{1+x}dx=\int_0^1\lim_{y\to 0}\frac{\ln(1+x+y\sin x)}{1+x}dx=\frac{\ln^2 2}{2}.$$

13.4- teorema (Integralning parametr bo'yicha uzluksizligi). Agar $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;b], y\in[c;d]\}$ to'plamda berilgan va uzluksiz bo'lsa, u holda

$$I(y)=\int_a^b f(x,y)dx$$

funksiya y bo'yicha $[c,d]$ da uzluksiz bo'ladi.

13.12- misol. $I(y)=\int_0^1\dfrac{x^2}{x^3+y^2+1}dx,\ y\in[0;1]$ integralni $M=\{(x,y)\in R^2 : x\in[0,1],\ y\in[0,1]\}$ to'plamda uzluksizlikka tekshiring.

Yechilishi. $f(x,y)=\dfrac{x^2}{x^3+y^2+1}$ funksiya M to'plamda uzluksiz.

13.4-teoremaga asosan, $I(y)$ funksiya $[0;1]$ da uzluksiz bo'ladi.

Haqiqatan ham,

$$I(y)=\int_0^1\frac{x^2}{x^3+y^2+1}dx=\frac{1}{3}\ln\left|1+x^3+y^2\right|\Big|_0^1=\frac{1}{3}\ln\frac{2+y^2}{1+y^2}$$

funksiya $[0;1]$ da bo'yicha uzluksiz.

13.2- eslatma. Agar $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;b], y\in[c;d]\}$ to'plamda uzluksiz va $y_0\in[c;d]$ bo'lsa, u holda

$$\lim_{y\to y_0}\int_a^b f(x,y)dx=\int_a^b \lim_{y\to y_0} f(x,y)dx$$

formula o'rinli.

13.13- misol. $\lim_{y\to 0}\int_{-1}^{1}\sqrt{x^2+y^2}\,dx,\ y\in[-1;1]$ ni toping.

Yechilishi. Integral ostidagi $f(x,y)=\sqrt{x^2+y^2}$ funksiya $M=\{(x,y)\in R^2 : x\in[-1;1], y\in[-1;1]\}$ to'plamda uzluksiz va $y_0=0\in[-1;1]$ bo'ladi, u holda, 13.2- eslatmaga ko'ra,

$$\lim_{y\to 0}\int_{-1}^{1}\sqrt{x^2+y^2}dx=\lim_{y\to 0}\int_{-1}^{1}\sqrt{x^2+y^2}dx=\int_{-1}^{1}|x|\,dx=1.$$

13.5- teorema (integralni parametr bo'yicha differensiallash). $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;b], y\in[c;d]\}$ to'plamda berilgan va y o'zgaruvchining $[c;d]$ dan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a;b]$ da uzluksiz bo'lsin. Agar $f(x,y)$ funksiya M to'plamda $f'_y(x,y)$ xususiy hosilaga ega bo'lib, u M da uzluksiz bo'lsa, u holda

$$I(y)=\int_a^b f(x,y)dx$$

funksiya $[c;d]$ da $I'(y)$ hosilaga ega va ushbu

$$I'(y)=\int_a^b f'(x,y)dx$$

tenglik o'rinli. Bunga *Leybnis qoidasi* ham deyiladi.

13.14- misol. $I(y)=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\ln(y^2\sin^2 x)dx$ funksiyaning $I'(y)$ hosilasini toping.

Yechilishi. Integral ostidagi $f(x,y)=\ln(y^2\sin^2 x)$ funksiya $M=\left\{(x,y)\in R^2: x\in\left[\frac{\pi}{4};\frac{3\pi}{4}\right],\ 0<y_0\le y\le y_1<+\infty\right\}$ to'plamda uzluksiz hamda $f_y'(x,y)=\frac{2}{y}$ hosilaga ega va u ham M da uzluksiz. U holda 13.5- teorema bo'yicha (ya'ni Leybnis qoidasini qo'llash mumkin):

$$I'(y)=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\ln(y^2\sin^2 x))_y'\,dx=\frac{2}{y}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}dx=\frac{\pi}{y}.$$

13.15- misol. $I(y)=\int_0^{\frac{\pi}{2}}\ln(\sin^2 x+y^2\cos^2 x)dx,\quad y\neq 0$ integralni hisoblang.

Yechilishi. $y>0$ va $y\neq 0$ bo'lmasin. $f(x,y)=\ln(\sin^2 x+$ $+y^2\cos^2 x)$ funksiya $M=\left\{(x,y)\in R^2: x\in\left[0;\frac{\pi}{2}\right],\ y\in[y_1,y_2]\right\}$ to'plamda uzluksiz, $\frac{\partial f}{\partial y}=\frac{2y\cos^2 x}{\sin^2 x+y^2\cos^2 x}$ hosilaga ega va u ham M da uzluksiz. U holda, 13.5- teoremaga asosan,

$$I'(y)=\int_0^{\frac{\pi}{2}}\frac{2y\cos^2 x}{\sin^2 x+y^2\cos^2 x}dx$$

bo'ladi. Bu integralni hisoblash uchun $t=\mathrm{tg}x$ almashtirishni olamiz. Unda

$$I'(y)=2y\int_0^{+\infty}\frac{dt}{(t^2+1)(t^2+y^2)}=\frac{2y}{y^2-1}\int_0^{+\infty}\left(\frac{1}{t^2+1}-\frac{1}{t^2+y^2}\right)dt=$$

$$=\frac{2y}{y^2-1}\left(\mathrm{arctg}t-\frac{1}{y}\mathrm{arctg}\frac{t}{y}\right)\Bigg|_0^{+\infty}=\frac{\pi}{y+1}$$

bo'ladi va bu yerdan $I(y)$ topiladi:

$$I(y)=\pi\ln|y+1|+C.$$

y>0 bo‘lganda, $I(y)$ uzluksiz va $I(1)$=0 bo‘lgani uchun $C=-\pi\ln 2$. Demak, y>0 bo‘lganda, $I(y)=\pi\ln\frac{y+1}{2}$. $I(y)$ juft funksiya ekanligini hisobga olgan holda, $y\neq 0$ uchun

$$I(y)=\pi\ln\frac{|y|+1}{2}$$

ifodani hosil qilamiz.

13.16- misol. $I(y)=\int\limits_0^1 \operatorname{arctg}\frac{x}{y}dx$ funksiyaning y=0 nuqtadagi hosilasining mavjud yoki mavjud emasligini tekshirishda Leybnis qoidasidan foydalanish mumkinmi?

Yechilishi. y>0 bo‘lganda, berilgan integralga Leybnis qoidasini qo‘llash mumkin bo‘lsin deb, faraz qilamiz:

$$I'(y)=-\int\limits_0^1\frac{xdx}{x^2+y^2}=\frac{1}{2}\ln\frac{y^2}{1+y^2}.$$

Endi berilgan integralni bevosita hisoblaymiz:

$$I(y)=\int\limits_0^1\operatorname{arctg}\frac{x}{y}dx=x\operatorname{arctg}\frac{x}{y}\bigg|_0^1-\int\limits_0^1\frac{xy}{y^2+x^2}dx=\operatorname{arctg}\frac{1}{y}-$$

$$-\frac{y}{2}\ln(y^2+x^2)\bigg|_0^1=\operatorname{arctg}\frac{1}{y}+\frac{1}{2}y\ln\frac{y^2}{1+y^2}.$$

Agar integral ostidagi $f(x,y)=\operatorname{arctg}\frac{x}{y}$ funksiyaning y=0 va y>0 dagi qiymatini $\frac{\pi}{2}$ ga teng desak, u holda $I(0)=\frac{\pi}{2}$. Demak, $I(y)$ funksiya y=0 nuqtada uzluksiz bo‘ladi.

$I(y)$ funksiyaning y=0 nuqtadagi hosilasini ta’rif bo‘yicha topamiz:

$$I'(0)=\lim_{y\to 0}\frac{J(y)-J(0)}{y}=\lim_{y\to 0}\frac{\operatorname{arctg}\frac{t}{y}+\frac{y}{2}\ln\frac{y^2}{1+y^2}-\frac{\pi}{2}}{y}.$$

Bu limitni hisoblashda arctg$x$ ning $|x|>1$ bo‘lgandagi yoyilmasidan foydalanamiz:

$$I'(0)=\lim_{y\to 0}\frac{\frac{\pi}{2}-y^2+\frac{1}{3}y^6-...+\frac{y}{2}\ln\frac{y^2}{1+y^2}-\frac{\pi}{2}}{y}=-\infty.$$

Shunday qilib, berilgan $I(y)$ funksiya y=0 nuqtada chekli hosilaga ega emas. Bu holda Leybnis qoidasini qo'llash mumkin emas, chunki $f'_y(x,y)=-\frac{x}{x^2+y^2}$ funksiya (0;0) nuqtada uzulishga ega.

13.6- teorema (integralni parametr bo'yicha integrallash). Agar $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a;b],\ y\in[c;d]\}$ to'plamda uzluksiz bo'lsa, u holda

$$\int_c^d I(y)dy=\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_a^b dx\int_c^d f(x,y)dy \qquad (*)$$

formula o'rinli, ya'ni integral ostida parametr bo'yicha integrallash mumkin.

13.17- misol. Ushbu integralni hisoblang:

$$J=\int_0^1\cos(\ln\frac{1}{x})\frac{x^{y_2}-x^{y_1}}{\ln x}dx \quad (y_1>0,\ y_2>0).$$

Yechilishi. Ravshanki, x>0 da

$$\frac{x^{y_2}-x^{y_1}}{\ln x}=\int_{y_1}^{y_2}x^y dy.$$

Bu tenglikni e'tiborga olib, berilgan integralni quyidagi ko'rinishga keltiramiz:

$$J=\int_0^1 dx\int_{y_1}^{y_2}x^y\cos\left(\ln\frac{1}{x}\right)dx.$$

Bunda $f(x,y)=x^y\cos(\ln\frac{1}{x})$ funksiya $M=\{(x,y)\in R^2: 0\le x\le 1,$ $y_1\le y\le y_2\}$ to'plamda uzluksiz (f(0;y)=0 deb qaraymiz). Shuning uchun, 13.6- teoremaga ko'ra

$$J=\int_0^1 dx\int_{y_1}^{y_2}x^y\cos(\ln\frac{1}{x})dx=\int_{y_1}^{y_2}dy\int_0^1 x^y\cos(\ln\frac{1}{x})dx$$

deb yozish mumkin. Keyingi ichki integralda $x=e^{-t}$ almashtirish bajarib, quyidagiga ega bo'lamiz:

$$J=\int_{y_1}^{y_2}dy\int_0^{+\infty}e^{-t(y+1)}\cos tdt.$$

Endi $J_1=\int_0^{+\infty}e^{-t(y+1)}\cos tdt$ integralni ikki marta bo'laklab integrallash natijasida, J_1 ga nisbatan chiziqli tenglamani hosil qilib, undan $J_1=\frac{y+1}{(y+1)^2+1}$ ekanligini topamiz. Natijada

$$J=\int_{y_1}^{y_2}\frac{y+1}{(y+1)^2+1}dy$$

integralga ega bo'lamiz, bundan

$$J=\frac{1}{2}\ln\left|y^2+2y+2\right|\Big|_{y_1}^{y_2}=\frac{1}{2}\ln\frac{y_2{}^2+2y_2+2}{y^2{}_1+2y_1+2}$$

kelib chiqadi.

13.18- misol. Quyidagi integrallar o'zaro teng bo'ladimi:

$$\int_0^1 dx\int_0^1\frac{y-x}{(x+y)^3}dy \text{ va } \int_0^1 dy\int_0^1\frac{y-x}{(x+y)^3}dx\,?$$

Yechilishi. Integral ostidagi $f(x,y)=\frac{y-x}{(x+y)^3}$ funksiya (0;0) nuqtada uzilishga ega. Shuning uchun 13.6- teorema o'rinli bo'lmaydi.

Haqiqatan ham,

$$\int_0^1\frac{y-x}{(x+y)^3}dx=\frac{x}{(x+y)^2}\Bigg|_0^1=\frac{1}{(1+y)^2};$$

$$\int_0^1\frac{y-x}{(1+y)^2}dy=-\frac{1}{1+y}\Bigg|_0^1=\frac{1}{2};$$

$$\int_0^1\frac{y-x}{(x+y)^3}dy=\left(-\frac{1}{y+x}+\frac{x}{(x+y)^2}\right)\Bigg|_0^1=-\frac{1}{(1+x)^2};$$

$$-\int_0^1\frac{1}{(1+x)^2}dx=\frac{1}{x+1}\Bigg|_0^1=-\frac{1}{2}.$$

Shunday qilib, bu integrallar bir-biriga teng emas.

Yuqorida biz ko‘rib o‘tgan integrallarning chegaralari o‘zgarmas edi. Endi integralning chegaralari ham parametrning funksiyalari bo‘lgan holni qaraymiz. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a,b]\ y\in[c,d]\}$ to‘plamda berilgan. y o‘zgaruvchining $[c,d]$ dagi har bir tayin qiymatida $f(x,y)$ funksiya x o‘zgaruvchining funksiyasi sifatida $[a,b]$ da integrallanuvchi bo‘lsin.

$x=\alpha(y)$ va $x=\beta(y)$ funksiyalarning har biri $[c,d]$ da berilgan va $\forall y\in[c,d]$ uchun

$$a\le\alpha(y)\le\beta(y)\le b \tag{13. 2}$$

shartni qanoatlantirsin. Bu shartlarda

$$\int_{\alpha(y)}^{\beta(y)} f(x,y)dx \tag{13. 3}$$

integral mavjud va u y o‘zgaruvchi (parametr) ning funksiyasi bo‘ladi:

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx\,.$$

Agar (13.3) da $\alpha(y)=a,\quad \beta(y)=b\quad (y\in[c,d])$ deb olsak, u holda (13.3) integral (13.1) integralga aylanadi.

13.7- teorema. $f(x,y)$ funksiya M to‘plamda uzluksiz, $\alpha(y)$, $\beta(y)$ funksiyalarning har biri $[c,d]$ da uzluksiz va ular (13.2) shartni qanoatlantirsin. U holda

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya ham $[c,d]$ da uzluksiz bo‘ladi.

13.8- teorema. $f(x,y)$ funksiya M to‘plamda uzluksiz, $f'_y(x,y)$ xususiy hosilaga ega va u ham M da uzluksiz, $\alpha(y)$, $\beta(y)$ funksiyalar esa $\alpha'(y)$, $\beta'(y)$ hosilalarga ega hamda ular (13.2) shartni qanoatlantirsin. U holda

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c;d]$ da $F'(y)$ hosilaga ega va u

$$F'(y)=\int_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx+\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y) \quad (13.4)$$

formula orqali topiladi.

13.19- misol. $F(y)=\int\limits_{\cos y}^{\sin y} \text{ch}yx^2dx$ funksiyaning $F'(y)$ hosilasini toping.

Yechilishi. $f(x,y)=\text{ch}yx^2$ funksiya $M=\{(x,y)\in R^2 :$ $: x\in[a,b],\ y\in[\alpha_1,\alpha_2]\}$ to'plamda uzluksiz, $f'_y(x,y)=x^2\text{sh}yx^2$ hosilaga ega va u ham M da uzluksiz, $x=\cos y$, $x=\sin y$ funksiyalar $[\alpha_1;\alpha_2]$ da mos ravishda, $x'=-\sin y$, $x'=\cos y$ hosilalarga ega hamda ular (13.2) shartni qanoatlantiradi. U holda (13.4) formulaga asosan,

$$F'(y)=\cos y\ \ \text{ch}(y\sin^2 y)+\sin y\ \ \text{ch}(y\cos^2 y)+\int_{\cos y}^{\sin y} x^2\text{sh}(yx^2)dx$$

bo'ladi.

13.20- misol. $F(y)=\int\limits_{2y}^{4y}\frac{\text{arctg}xy}{x}dx$ funksiyaning $F'(y)$ hosilasini toping.

Yechilishi. Integral ostidagi $f(x,y)=\frac{\text{arctg}xy}{x}$ funksiya $M=\{(x,y)\in R^2 : x\neq 0, x\in R,\ y\in[\alpha_1,\alpha_2]\}$ to'plamda uzluksiz, $f'_y(x,y)=\frac{1}{1+x^2y^2}$ hosilaga ega. $x=2y$, $x=4y$ funksiyalar ham $[\alpha_1,\alpha_2]$ da mos ravishda, $x'=2$, $x'=4$ hosilalarga ega va (13.2) shartni qanoatlantiradi. Shuning uchun (13.4) formulaga asosan

$$F'(y)=\int_{2y}^{4y}\frac{dx}{1+x^2y^2}+\frac{\text{arctg}4y^2}{y}-\frac{\text{arctg}2y^2}{y}=2\frac{\text{arctg}4y^2-\text{arctg}2y^2}{y}.$$

13.21- misol. $F(\alpha)=\int\limits_0^{\alpha}dx\int\limits_{x-\alpha}^{x+\alpha}\cos(x^2+y^2-\alpha^2)dy$ funksiyaning $F'(\alpha)$ hosilasini toping.

Yechilishi. Agar

$$f(x,\alpha)=\int\limits_{x-\alpha}^{x+\alpha}\cos(x^2+y^2-\alpha^2)dy$$

deb belgilasak, u holda $F(\alpha)=\int\limits_0^{\alpha}f(x,\alpha)dx$ ni hosil qilamiz. Ravshanki, $f(x,y)$ funksiya uchun 13.8- teoremaning shartlari bajariladi. Bunga (13.4) formulani qo'llasak,

$$F'(\alpha)=\int\limits_0^{\alpha}f'(x,\alpha)dx+f(\alpha,\alpha)$$

bo'ladi, bunda

$$f'_\alpha(x,\alpha)=\cos(x^2+(x+\alpha)^2-\alpha^2)+\cos(x^2+(x-\alpha)^2-\alpha^2)+$$

$$+2\alpha\int\limits_{x-\alpha}^{x+\alpha}\sin(x^2+y^2-\alpha^2)dy.$$

Shunday qilib,

$$F'(\alpha)=\int\limits_0^{2\alpha}\cos y^2dy+2\int\limits_0^{\alpha}\cos 2x^2\cos 2\alpha x dx+$$

$$+2\alpha\int\limits_0^{\alpha}dx\int\limits_{x-\alpha}^{x+\alpha}\sin(x^2+y^2-\alpha^2)dx.$$

13.22- misol. $F(\alpha)=\int\limits_0^{\alpha}f(x+\alpha,x-\alpha)dx$ funksiyaning $F'(a)$ hosilasini toping.

Yechilishi. Integral ostidagi funksiyani $f(u,v)$ deb belgilaymiz, bunda $u=x+\alpha$, $v=x-\alpha$. Bu $f(u,v)$ funksiya u va v o'zgaruvchilari bo'yicha uzluksiz xususiy hosilalarga ega, ya'ni 13.8- teorema shartlarini qanoatlantiradi, deb faraz qilamiz. U holda (13.4) formulaga asosan,

$$F''(\alpha) = f(2\alpha,0) + \int_0^{\alpha} \left(\frac{\partial f(u,v)}{\partial u} - \frac{\partial f(u,v)}{\partial v}\right)dx,$$

bunda $\frac{df}{dx} = f'_u + f'_v$ ekanligini e'tiborga olsak,

$$\int_0^{\alpha} (f'_u - f'_v)dx = 2\int_0^{\alpha} f'_v dx - f(2\alpha,0) + f(\alpha,-\alpha)$$

bo'ladi. Shunday qilib,

$$F'(\alpha) = f(\alpha,-\alpha) + 2\int_0^{\alpha} f'_v dx .$$

Mustaqil yechish uchun misollar

Berilgan to'plamda funksiyaning limit funksiyalarini toping:

13.1. $f(x,y) = \frac{1}{y}\sin xy;\ \ M = \{(x,y) \in R^2 : x \in (0,+\infty),$
$y \in (0,+\infty)\},\ \ y_0 = +\infty.$

13.2. $f(x,n) = \frac{nx}{1+n+x},\ M = \{(x,n) \in R^2 : x \in [0,1],\ n \in N\},$
$n_0 = +\infty.$

13.3. $f(x,n) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right),$

$M = \{(x,n) \in R^2 : x \in (0,+\infty),\ n \in N\}, n_0 = +\infty.$

13.4. $f(x,n) = n\left(x^{\frac{1}{n}} - 1\right),$

$M = \{(x,n) \in R^2 : x \in [1,\alpha],\ n \in N\},\ \ n_0 = +\infty.$

13.5. $f(x,y) = \sqrt{x^2 + \frac{1}{y}},$

$M = \{(x,y) \in R^2 : x \in R,\ y \in (0,+\infty)\},\ \ y_0 = +\infty.$

13.6. $f(x,n)=\operatorname{arctg} nx;$

$M=\{(x,n)\in R^2 : x\in(0,+\infty),\ n\in N\},\quad n_0=+\infty.$

13.7. $f(x,\alpha)=x^2\cos\dfrac{1}{\alpha x},$

$M=\{(x,\alpha)\in R^2 : x\in[0,2],\alpha\in[0,2]\},\quad \alpha_0=+\infty.$

13.8. $f(x,y)=\sqrt{x}\sin y,$

$M=\{(x,y)\in R^2 : x\in[0,+\infty),\ y\in(0,\pi)\},\quad y_0=\dfrac{\pi}{3}.$

13.9. $f(x,n)=nx^2\sin\dfrac{x}{n},$

$M=\{(x,n)\in R^2 : x\in R,\ n\in N\},\quad n_0=+\infty.$

13.10. $f(x,n)=n^2x(1-x^2)^n,\quad M=\{(x,n)\in R^2 : x\in[0,1],\ n\in N\},\quad n_0=+\infty.$

Berilgan to‘plamda funksiyaning limit funksiyalarini toping va yaqinlashish xarakterini tekshiring:

13.11. $f(x,n)=x^n,$

$M=\{(x,n)\in R^2 : x\in[0,\dfrac{1}{2}],\ n\in N\},\quad n_0=+\infty.$

13.12. $f(x,n)=\dfrac{nx^2}{n+x},$

$M=\{(x,n)\in R^2 : x\in[1,+\infty),\ n\in N\},\quad n_0=+\infty.$

13.13. $f(x,n)=x^n-x^{n+1},$

$M=\{(x,n)\in R^2 : x\in[0,1],\ n\in N\},\quad n_0=+\infty.$

13.14. $f(x,n)=x^n-x^{2n},$

$M=\{(x,n)\in R^2 : x\in[0,1],\ n\in N\},\quad n_0=+\infty.$

13.15. *a)* $f(x,n)=\dfrac{\sin nx}{n}$; *b)* $f(x,n)=\sin\dfrac{x}{n}$;

$M=\{(x,n)\in R^2 : x\in R,\ n\in N\},\quad n_0=+\infty.$

13.16. *a*) $f(x,n) = \text{arctg}nx$; *b*) $f(x,n) = x\text{arctg}nx$;

$$M = \{(x,n) \in R^2 : x \in (0,+\infty), n \in N\}, \quad n_0 = +\infty.$$

13.17. *a*) $f(x,n) = \left(1 + \frac{x}{n}\right)^n$, $M = \{(x,n) \in R^2 : x \in R, n \in N\}$;

b) $f(x,n) = \left(1 + \frac{x}{n}\right)^n$,

$$M = \{(x,n) \in R^2 : x \in (a,b) - \text{chekli oraliq}, n \in N\}, \quad n_0 = +\infty.$$

13.18. $f(x,n) = n\left(x^{\frac{1}{n}} - 1\right)$,

$$M = \{(x,n) \in R^2 : x \in [1,\alpha], n \in N\}, \quad n_0 = +\infty.$$

13.19. $f(x,n) = nxe^{-nx^2}$,

$$M = \{(x,n) \in R^2 : x \in [0,1], n \in N\}, \quad n_0 = +\infty.$$

13.20. $f(x,n) = nx(1-x)^n$,

$$M = \{(x,n) \in R^2 : x \in [0,1], n \in N\}, \quad n_0 = +\infty.$$

13.21. Ushbu funksiyalarning R da uzluksiz ekanligini isbotlang:

1) $I(\alpha) = \int_0^1 \sin^2 \alpha x^2 dx$; 2) $I(\alpha) = \int_{-1}^{2} \frac{x^2}{1 + x^2 + \alpha^2 x^4} dx$.

13.22. $I(\alpha) = \int_0^1 \text{sign}(x - \alpha) dx$ funksiyaning R da uzluksizligini isbotlang.

13.23. Limitlarni hisoblang:

1) $\lim_{\alpha \to 0} \int_{-1}^{1} \sqrt[4]{x^4 + \alpha^4} dx$; 2) $\lim_{\alpha \to 0} \int_{\alpha}^{1+\alpha} \frac{dx}{1 + x^2 + \alpha^2}$; 3) $\lim_{\alpha \to 0} \int_0^1 \sqrt{1 + \alpha^2 x^4} dx$;

4) $\lim_{\alpha \to 1} \int_2^4 \frac{xdx}{1 + x^2 + \alpha^6}$; 5) $\lim_{\alpha \to 1} \int_0^1 x^2 e^{\alpha x^3} dx$; 6) $\lim_{R \to +\infty} \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta$.

13.24. Ushbu limitlarni integral ostiga kiritish mumkinmi:

1) $\lim\limits_{y\to 0}\int\limits_0^1 \frac{x}{y^2}e^{-\frac{x^2}{y^2}}dx$; 2) $\lim\limits_{y\to 0}\int\limits_{-1}^3 \operatorname{arctg}\frac{xy}{1+y}dx$?

13.25. $f(x)$ funksiya $[a,b]$ da uzluksiz, $\alpha<\alpha_0<x<b$.

$$\lim_{y\to 0}\frac{1}{y}\int\limits_{\alpha_0}^{x}[f(t+y)-f(t)]dt = f(x)-f(\alpha_0)$$

ekanligini isbotlang.

Funksiyalarning hosilalarini toping:

13.26. $I(y)=\int\limits_0^1 \sin yx dx$. **13.27.** $I(y)=\int\limits_1^3 \frac{\cos(yx^3)}{x}dx$.

13.28. $I(y)=\int\limits_1^2 \frac{e^{yx^2}}{x}dx$. **13.29.** $I(y)=\int\limits_0^y \frac{\ln(1+yx)}{x}dx$.

13.30. $I(y)=\int\limits_y^{2y} \frac{\sin yx}{x}dx$. **13.31.** $I(y)=\int\limits_{\sin y}^{\cos y} e^{y\sqrt{1-x^2}}dx$.

13.32. $I(y)=\int\limits_{ye^{-y}}^{ye^{y}} \ln(1+(xy)^2)dx$.

13.33. $I(y)=\int\limits_0^{\alpha^2} dx\int\limits_{x-\alpha}^{x+\alpha} \sin(x^2+y^2-\alpha^2)dy$.

13. 34. $\alpha=0$ bo‘lganda, $I(\alpha)=\int\limits_0^1 \ln(x^2+\alpha^2)dx$ funksiyaning hosilasini Leybnis qoidasi bilan topish mumkinmi?

13.35. $\int\limits_0^b \frac{dx}{x^2+\alpha^2}$ integralni $\alpha(\alpha>0)$ — parametr bo‘yicha differensiallab, $\int\limits_\alpha^b \frac{dx}{(x^2+\alpha^2)^2}$ integralni hisoblang.

13.36. $F(\alpha)=\int_0^{\alpha}(x+\alpha)f(x)dx$ funksiya berilgan, bunda $f(x)$ — differensiallanuvchi funksiya bo'lsa, $F(\alpha)$ ni toping.

13.37. $\int_0^1\left(\int_0^1 f(x,\alpha)d\alpha\right)dx$ va $\int_0^1\left(\int_0^1 f(x,\alpha)dx\right)d\alpha$ integrallarning teng yoki teng emasligini aniqlang.

Agar: 1) $f(x,\alpha)=\dfrac{\alpha^2-x^2}{(\alpha^2+x^2)^2}$, 2) $f(x,\alpha)=\left(\dfrac{x^5}{\alpha^4}-\dfrac{2x^3}{\alpha^3}\right)e^{-\frac{x^2}{\alpha}}$ bo'lsa.

Misollarning javoblari

13.1. $\varphi(x)=0$. **13.2.** $\varphi(x)=x$. **13.3.** $\varphi(x)=\dfrac{1}{2\sqrt{x}}$.

13.4. $\varphi(x)=\ln x$. **13.5.** $\varphi(x)=|x|$. **13.6.** $\varphi(x)=\dfrac{\pi}{2}$.

13.7. $\varphi(x)=x^2$. **13.8.** $\varphi(x)=\dfrac{\sqrt{3}}{2}\sqrt{x}$. **13.9.** $\varphi(x)=x^3$.

13.10. $\varphi(x)=0$. **13.11.** $\varphi(x)=0$ ga tekis yaqinlashadi.

13.12. $\varphi(x)=x^2$ ga tekis yaqinlashadi. **13.13.** $\varphi(x)=0$ ga tekis yaqinlashadi. **13.14.** $\varphi(x)=0$ ga notekis yaqinlashadi.

13.15. a) $\varphi(x)=0$ ga tekis yaqinlashadi; b) $\varphi(x)=0$ ga notekis yaqinlashadi. **13.16.** a) $\varphi(x)=\dfrac{\pi}{2}$ ga notekis yaqinlashadi; b) $\varphi(x)=\dfrac{x\pi}{2}$ ga tekis yaqinlashadi. **13.17.** a) $\varphi(x)=e^x$ ga tekis yaqinlashadi; b) $\varphi(x)=e^x$ ga notekis yaqinlashadi. **13.18.** $\varphi(x)=\ln x$ ga tekis yaqinlashadi. **13.19.** $\varphi(x)=0$ ga notekis yaqinlashadi.

13.20. $\varphi(x)=0$ ga notekis yaqinlashadi. **13.23.** 1) 1; 2) $\frac{\pi}{4}$; 3) 1; 4) $\frac{\ln 3}{2}$;

5) $\frac{e-1}{3}$; 6) 0. **13.24.** 1) yo'q; 2) ha. **13.26.** $I'(y)=\frac{y\sin y+\cos y-1}{y^2}$.

13.27. $I'(y)=\frac{\cos 27y-\cos y}{3y}$. **13.28.** $I'(y)=\frac{e^{4y}-e^{y}}{2y}$.

13.29. $I'(y)=\frac{2\ln(1+y^2)}{y}$. **13.30.** $I'(y)=2\frac{\sin 2y^2-\sin y^2}{y}$.

13.31. $\int\limits_{\sin y}^{\cos y}\sqrt{1-x^2}e^{y\sqrt{1-x^2}}dx-\sin y e^{y|\sin y|}-\cos y\cdot e^{y|\cos y|}$.

13.32. $I'(y)=4\mathrm{sh}y+\frac{2}{y^2}(\mathrm{arctg}(y^2e^{-y})-\mathrm{arctg}(y^2e^{y}))+$
$+(y+1)e^{y}\ln(1+y^4e^{2y})-(y-1)e^{-y}\ln(1+y^4e^{-2y})$.

13.33. $I'(\alpha)=2\alpha\int\limits_{\alpha^2-\alpha}^{\alpha^2+\alpha}\sin(y^2+\alpha^4-\alpha^2)dy+2\int\limits_{0}^{\alpha^2}\sin 2x^2\cos 2\alpha x dx-$

$-2\alpha\int\limits_{0}^{\alpha^2}dx\int\limits_{x-\alpha}^{x+\alpha}\cos(x^2+y^2-\alpha^2)dy$. **13.34.** Yo'q.

13.35. $\frac{1}{2\alpha^3}\mathrm{arctg}\frac{b}{\alpha}+\frac{b}{2\alpha^2(\alpha^2+b^2)}$.

13.36. $F''(\alpha)=3f(\alpha)+2\alpha f'(\alpha)$. **13.37.** 1) Teng emas; 2) Teng emas.

14- §. Parametrga bog'liq bo'lgan xosmas integrallar

14.1. Parametrga bog'liq bo'lgan xosmas integral tushunchasi.

1. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$ da berilgan bo'lib, y o'zgaruvchining E dagi har bir o'zgarmas qiymatida x bo'yicha $[a,+\infty)$ da integrallanuvchi, ya'ni

$$\int_a^{+\infty} f(x,y)dx \quad (y\in E\subset R)$$

xosmas integral mavjud va chekli bo'lsin. Bu integral y ning qiymatiga bog'liq bo'ladi:

$$I(y)=\int_a^{+\infty} f(x,y)dx. \tag{14.1}$$

(14.1) integral *parametrga bog'liq* (chegarasi cheksiz) bo'lgan *xosmas integral* deyiladi.

$f(x,y)$ funksiya $M'=\{(x,y)\in R: x\in(-\infty,a],\ \ y\in E\subset R\}$ $(M''=\{(x,y)\in R: x\in(-\infty;+\infty),\ \ y\in E\subset R\})$ to'plamda berilgan va o'zgaruvchining E dan olingan har bir tayin qiymatida x ning funksiyasi sifatida $(-\infty,a]$ ($(-\infty;+\infty)$) da integrallanuvchi bo'lsin , ya'ni

$$\int_{-\infty}^{a} f(x,y)dx \quad \left(\int_{-\infty}^{+\infty} f(x,y)dx\right) \tag{14.2}$$

integral mavjud bo'lsin. (14.2) integral ham, *parametrga bog'liq bo'lgan xosmas* integral deb ataladi.

2. $f(x,y)$ funksiya $M_1=\{(x,y)\in R^2: x\in[a,b), y\in E\subset R\}$ to'plamda berilgan bo'lib, y o'zgaruvchining E dan olingan har bir o'zgarmas qiymatida x ning funksiyasi sifatida qaraganda $x=b$ nuqta maxsus nuqta bo'lsin va bu funksiya $[a,b)$ da integrallanuvchi, ya'ni

$$\int_a^{b)} f(x,y)dx \quad (y\in E\subset R)$$

mavjud bo'lsin. Ravshanki, bu integral ham y ning qiymatlariga bog'liq bo'ladi:

$$I_1(y)=\int_a^{b)} f(x,y)dx. \tag{14.3}$$

Bu integralga *chegaralanmagan funksiyaning parametrga bog'liq bo'lgan xosmas integrali* deyiladi.

Xuddi shunday, ushbu

$$\int_{a)}^{b} f(x,y)dx, \quad \int_{a)}^{b)} f(x,y)dx$$

xosmas parametrga bog‘liq bo‘lgan integrallarning ham ta’riflari yuqoridagi kabi beriladi.

3. $f(x,y)$ funksiya $M_2 = \{(x,y) \in R^2 : x \in (c,+\infty),\ y \in E \subset R\}$ to‘plamda berilgan bo‘lib, y o‘zgaruvchining E to‘plamdan olingan har bir tayin qiymatida x o‘zgaruvchining funksiyasi sifatida qaralganda, uning uchun $x=c$ maxsus nuqta bo‘lsin va bu funksiya $(c;+\infty)$ da integrallanuvchi, ya’ni

$$\int_{c)}^{+\infty} f(x,y)dx$$

chegaralanmagan funksiyaning chegarasi cheksiz xosmas integrali mavjud bo‘lsin. Bu integral y ning qiymatiga bog‘liq bo‘ladi:

$$I_2(y) = \int_{c)}^{+\infty} f(x,y)dx. \tag{14.4}$$

(14.4) integral chegaralanmagan funksiyaning *parametrga bog‘liq bo‘lgan chegarasi cheksiz xosmas integrali* deb ataladi.

Masalan, ushbu

$$I_1(\alpha) = \int_{2}^{+\infty} \frac{dx}{x^\alpha} \quad (\alpha > 0), I_2(\alpha) = \int_{a)}^{b} \frac{dx}{(x-a)^\alpha} \quad (\alpha > 0),$$

$$I_3(\alpha) = \int_{0)}^{+\infty} x^{\alpha-1}e^{-x}dx \qquad (\alpha > 0)$$

integrallar parametrga bog‘liq bo‘lgan xosmas integrallardir. Parametrga bog‘liq bo‘lgan xosmas integrallarni o‘rganishda integralning tekis yaqinlashish tushunchasi muhim ahamiyatga ega.

14.2. Parametrga bog‘liq bo‘lgan xosmas integrallarning tekis yaqinlashishi

1. $f(x,y)$ funksiya $M = \{(x,y) \in R^2 : x \in [a,+\infty),\ y \in E \subset R\}$ da aniqlangan bo‘lib, y ning E dan olingan har bir tayin qiymatida

$$I(y)=\int_a^{+\infty} f(x,y)dx$$

integral mavjud bo'lsin. U holda, cheksiz oraliq bo'yicha olingan xosmas integrallarning ta'rifiga ko'ra,

$$I(y)=\int_a^{+\infty} f(x,y)dx=\lim_{t\to+\infty}\int_a^t f(x,y)dx=\lim_{t\to+\infty}F(t,y)$$

ko'rinishda yozamiz, bunda $F(t,y)=\int_a^t f(x,y)dx$.

Shunday qilib, $I(y)$ funksiya $F(t,y)$ funksiyaning $(y=\text{const})$ $t\to+\infty$ dagi limit funksiyasi bo'ladi.

14.1- ta'rif. Agar $t\to+\infty$ da $F(t,y)$ funksiya o'z limit funksiyasi $I(y)$ ga E da tekis yaqinlashsa, ya'ni $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topilib, $\forall t>\delta$ va $\forall y\in E$ uchun

$$\left|\int_t^{+\infty} f(x,y)dx\right|<\varepsilon \qquad (14.5)$$

tengsizlik bajarilsa, u holda $\int_a^{+\infty} f(x,y)dx$ integral E to'plamda *tekis yaqinlashuvchi* deyiladi. (14.5) tengsizlik

$$\sup_{y\in E}\left|\int_t^{+\infty} f(x,y)dx\right|<\varepsilon$$

tengsizlikka teng kuchli.

14.2- ta'rif. Agar $t\to+\infty$ da $F(t,y)$ funksiya o'z limit funksiyasi $I(y)$ ga E da notekis yaqinlashsa, ya'ni $\forall\delta>0$ olinganda ham, shunday $\varepsilon_0>0$, $y_0\in E$ va $t_1>\delta$ tengsizlikni qanoatlantiruvchi $t_1\in[a,+\infty)$ topilib,

$$\left|\int_{t_1}^{+\infty} f(x,y_0)dx\right|\geq\varepsilon_0 \qquad (14.6)$$

tengsizlik bajarilsa, u holda $\int\limits_a^{+\infty} f(x,y)dx$ integral E to'plamda *notekis yaqinlashuvchi* deyiladi.

2. $f(x,y)$ funksiya $M_1 = \{(x,y) \in R^2 : x \in [a,b), y \in E \subset R\}$ to'plamda berilgan bo'lsin. y o'zgaruvchining E dan olingan har bir tayin qiymatida ushbu

$$I(y) = \int\limits_a^{b)} f(x,y)dx$$

integral mavjud bo'lsin. Xosmas integralning ta'rifiga ko'ra, ixtiyoriy $[a,t]$ $(a < t < b)$ da $F_1(t,y) = \int\limits_a^t f(x,y)dx$ integral mavjud va

$$I_1(y) = \int\limits_a^{b)} f(x,y)dx = \lim_{t \to b-0} F_1(t,y)$$

bo'ladi. Demak, $I_1(y)$ funksiya $F_1(t,y)$ funksiyaning $t \to b-0$ dagi limit funksiyasi bo'ladi.

14.3- ta'rif. Agar $t \to b-0$ da $F_1(t,y)$ funksiya o'z limit funksiyasi $I_1(y)$ ga E to'plamda tekis (notekis) yaqinlashsa, ya'ni $\forall \varepsilon > 0$ son olinganda ham, shunday $b' = b'(\varepsilon) > 0$ son topilib, $(b' \in [a,b))$ $\forall \xi \in [b',b)$ va $\forall y \in E$ uchun

$$\left| \int\limits_\xi^{b)} f(x,y)dx \right| < \varepsilon$$

$(\exists \varepsilon_0 > 0, \exists \xi \in [b',b), \exists y_0 \in E$ va $\forall b' \in [a,b)$ uchun $\left| \int\limits_\xi^b f(x,y_0)dx \right| \geq \varepsilon_0)$ tengsizlik bajarilsa, u holda $\int\limits_a^{b)} f(x,y)dx$ integral E to'plamda *tekis (notekis) yaqinlashuvchi* deyiladi.

14.1- misol. $\int\limits_0^{+\infty} e^{-x} \operatorname{arctg} xy dx$ integralning $(-\infty;+\infty) = R$ da y parametr bo'yicha tekis yaqinlashish xarakterini tekshiring.

Yechilishi. $\forall\varepsilon>0$ son uchun $\exists\delta=\ln\frac{\pi}{\varepsilon}>0$ topiladiki, $\forall t>\delta$ va $\forall y\in R$ uchun

$$\left|\int_t^{+\infty} e^{-x}\operatorname{arctg}xy dx\right|\le\frac{\pi}{2}\int_t^{+\infty} e^{-x}dx=\frac{\pi}{2}e^{-t}\le\frac{\pi}{2}e^{-\delta}=\frac{\varepsilon}{2}<\varepsilon.$$

Demak, 14.1- ta'rifga asosan, berilgan integral $(-\infty;+\infty)$ da y parametr bo'yicha tekis yaqinlashuvchi bo'ladi.

14.2- misol. $I(y)=\int_0^{+\infty}\frac{dx}{(x-y)^2+4}$ integralning *a*) $E=[0;+\infty)$; *b*) $E_1=(-\infty;0]$ to'plamlarda y parametr bo'yicha tekis yaqinlashish xarakterini tekshiring.

Yechilishi. *a*) Dastlab, berilgan integralni yaqinlashishga tekshiramiz. Chegarasi cheksiz xosmas integralning ta'rifiga ko'ra

$$F(t,y)=\int_0^t\frac{dx}{(x-y)^2+4}=\frac{1}{2}\operatorname{arctg}\frac{x-y}{2}\Big|_0^t=\frac{1}{2}\left(\operatorname{arctg}\frac{t-y}{2}+\operatorname{arctg}\frac{y}{2}\right).$$

Bundan, har bir tayinlangan y uchun $t\to+\infty$ da $F(t,y)$ funksiyaning limitini topamiz:

$$I(y)=\lim_{t\to+\infty}F(t,y)=\lim_{t\to+\infty}\frac{1}{2}\left(\operatorname{arctg}\frac{t-y}{2}+\operatorname{arctg}\frac{y}{2}\right)=\frac{\pi}{4}+\frac{1}{2}\operatorname{arctg}y.$$

Demak, berilgan integral yaqinlashuvchi. Endi qaralayotgan integralni tekis yaqinlashishga tekshiramiz. $\varepsilon_0=\frac{\pi}{6}$ desak, u holda $\forall\delta\in(0,+\infty)$ olinganda ham, $t_1>\delta$ ni qanoatlantiruvchi barcha t_1 va $\exists y_0=t_1$ uchun

$$F(t_1,y_0)=\int_{t_1}^{+\infty}\frac{dx}{(x-y_0)^2+4}=\frac{\pi}{4}-\operatorname{arctg}\frac{t_1-y_0}{2}=\frac{\pi}{4}>\frac{\pi}{6}=\varepsilon_0.$$

Shunday qilib, $\forall\varepsilon_0\in(0,\frac{\pi}{4})$ uchun (14.6) tengsizlik o'rinli. Demak, 14.2- ta'rifga ko'ra berilgan integral $E=[0;+\infty)$ to'plamda y parametr bo'yicha notekis yaqinlashadi.

b) $\forall\varepsilon>0$ son olinganda ham, $\delta=\delta(\varepsilon)=2\text{ctg}2\varepsilon$ deyilsa, u vaqtda $\forall t>\delta$ va $\forall y\in E_1$ uchun

$$\sup_{y\in E_1}\left|F(t,y)-I(y)\right|=\sup_{y\in E_1}\int_t^{+\infty}\frac{dx}{(x-y)^2+4}=$$

$$=\sup_{y\in E_1}\left\{\frac{1}{2}\left(\frac{\pi}{2}-\text{arctg}\frac{t-y}{2}\right)\right\}=\frac{1}{2}\text{arcctg}\frac{t}{2}<\varepsilon$$

tengsizlik bajariladi. Demak, berilgan integral, 14.1-ta'rifga ko'ra, E_1 da y parametr bo'yicha tekis yaqinlashuvchi bo'ladi.

14.3- misol. $I_1(y)=\int_0^1\frac{dx}{x^y}$ integralni *a*) $E=(0;1)$; *b*) $E_1=(0;\alpha)(\alpha<1)$ to'plamlarda tekis yaqinlashishga tekshiring.

Yechilishi. Xosmas integralning ta'rifiga ko'ra

$$I_1(y)=\int_0^1\frac{dx}{x^y}=\lim_{\lambda\to 0}\int_\lambda^1\frac{dx}{x^y}=\lim_{\lambda\to 0}\left(\frac{x^{1-y}}{1-y}\right)\Bigg|_\lambda^1=\frac{1}{1-y}.$$

Demak, berilgan integral (0;1) da yaqinlashuvchi. Endi, bu yaqinlashishni tekis yaqinlashishga tekshiramiz:

a) $y\in(0;1)$ bo'lsin. $x\to 0+0$ da integral ostidagi funksiya chegaralanmagan bo'ladi. $\int_0^\lambda\frac{dx}{x^y}=\frac{\lambda^{1-y}}{1-y}$ integralni baholaymiz.

$$\sup_{y\in E}\left|\int_0^\lambda\frac{dx}{x^y}\right|=\sup_{y\in E}\frac{\lambda^{1-y}}{1-y}=\lim_{y\to 1-0}\frac{\lambda^{1-y}}{1-y}=+\infty$$

bo'lgani uchun

$$\forall\delta=\delta(\varepsilon),\ \exists\varepsilon_0>0,\ \exists\lambda>0(0<\lambda<\delta(\varepsilon)),\exists y_0\in(0;1)\quad\left|\int_0^\lambda\frac{dx}{x^{y_0}}\right|\geq\varepsilon_0$$

tengsizlik bajariladi.

Shunday qilib, berilgan integral y parametr bo'yicha $E_1=(0;\alpha)$ to'plamda notekis yaqinlashadi.

b) $y\in(0,\alpha)$ $(\alpha<1)$ bo'lsin. $\forall\varepsilon>0$ son olinganda ham, $\delta=\left(\varepsilon(1-\alpha)\right)^{\frac{1}{1-\alpha}}$ deb olinganda, $0<\lambda<\delta(\varepsilon)$ va $\forall y\in(0,\alpha)$ uchun $\int_0^\lambda\frac{dx}{x^y}=\frac{\lambda^{1-y}}{1-y}<\varepsilon$ tengsizlik bajariladi.

Demak, 14.3 -ta'rifga asosan, berilgan integral $E_1 = (0;\alpha)$ to'plamda y parametr bo'yicha tekis yaqinlashadi.

14.4- misol. $\int_0^{+\infty} e^{-xy}dx$ integralning: $a)$ $E = [b;+\infty)$ $(b > 0)$ to'plamda tekis yaqinlashuvchi; $b)$ $E_1 = (0;+\infty)$ to'plamda esa notekis yaqinlashuvchiligini isbotlang.

Isbot. $a)$ $t > 0,$ $y \geq b > 0$ bo'lsin.

$$\int_t^{+\infty} e^{-xy}dx = \frac{e^{-yt}}{y},$$

$y \geq b$ bo'lganligi uchun, agar $t > \frac{1}{b}\ln\frac{1}{\varepsilon b}$ bo'lsa, $\forall\varepsilon > 0$ va $\forall y \in E$ uchun

$$0 < \int_t^{+\infty} e^{-xy}dx = \frac{e^{-tb}}{b} < \varepsilon$$

tengsizlik bajariladi. $\delta(\varepsilon) = \max(\sigma(\varepsilon);0)$, bunda $\sigma(\varepsilon) = \frac{2}{b}\ln\frac{1}{\varepsilon b}$ deb belgilasak, $\forall t \in [\sigma(\varepsilon);+\infty)$ va $\forall y \in E$ uchun

$$\left|\int_t^{+\infty} e^{-xy}dx\right| < \varepsilon$$

tengsizlik bajariladi. Demak, 14.1- ta'rifga asosan, berilgan integral E da tekis yaqinlashuvchi bo'ladi.

$b)$ $\forall\delta > 0$ uchun $t_1 = 1 + \delta$, $y_0 = \frac{1}{1+\delta}$ deb olsak,

$$\int_{t_1}^{+\infty} e^{-xy_0}dx = \frac{e^{-y_0 t_1}}{y_0} = (1+\delta)e^{-1} \geq e^{-1},$$

ya'ni $\varepsilon_0 = e^{-1}$ da (14.5) tengsizlik bajariladi. Shunday qilib, 14.2- ta'rifga asosan, berilgan integral $(0;+\infty)$ da notekis yaqinlashadi.

14.3. Parametrga bog'liq bo'lgan xosmas integrallarning yaqinlashish alomatlari.

1. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;+\infty), y\in E\subset R\}$ da berilgan bo'lib, y ning E dagi har bir tayin qiymatida x ning funksiyasi sifatida $[a;+\infty)$ da integrallanuvchi, ya'ni

$$I(y)=\int_a^{+\infty} f(x,y)dx \tag{14.6}$$

xosmas integral mavjud bo'lsin.

14.3- ta'rif. Agar $\forall\varepsilon>0$ $\exists\delta=\delta(\varepsilon)>0,\ t'>\delta, t''>\delta$ ni qanoatlantiruvchi $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

bajarilsa, (14.6) xosmas integral E da *fundamental integral* deyiladi.

Koshi teoremasi. $I(y)=\int_a^{+\infty} f(x,y)dx$ integralning E to'plamda tekis yaqinlashuvchi bo'lishi uchun uning E da fundamental bo'lishi zarur va yetarlidir.

14.1- natija. Agar $\forall\delta\in[a;+\infty)$ uchun $\exists\varepsilon_0, \exists t', t''\in[\delta;+\infty), \exists y_0\in E$ topilib,

$$\left|\int_{t'}^{t''} f(x,y_0)dx\right|\geq\varepsilon_0 \tag{14.7}$$

tengsizlik bajarilsa, u holda (14.6) integral y bo'yicha E to'plamda notekis yaqinlashuvchi bo'ladi.

Koshi teoremasidan quyidagi natija kelib chiqadi.

14.2- natija. Agar ixtiyoriy $x\in[a;+\infty), y\in E$ uchun $0\leq f(x,y)\leq\varphi(x,y)$ tengsizlik bajarilib, (14.6) integral yaqinlashuvchi va $\int_a^{+\infty}\varphi(x,y)dx$ integral y bo'yicha tekis yaqinlashuvchi bo'lsa, u holda (14.6) integral E to'plamda y bo'yicha tekis yaqinlashuvchi bo'ladi.

2. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;b), y\in E\subset R\}$ to'plamda berilgan bo'lib, y ning E dan olingan har bir tayin qiymatida x ning funksiyasi sifatida $[a,b)$ (b — maxsus nuqta) da integrallanuvchi, ya'ni

$$I_1(y)=\int_a^{b)} f(x,y)dx \qquad (14.6')$$

xosmas integral mavjud bo'lsin.

14.4- ta'rif. Agar $\forall\varepsilon>0$ son olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topilsaki, $b-\delta<t'<b$, $b-\delta<t''<b$ bo'lgan $\forall t',t''$ lar va $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

tengsizlik bajarilsa, u holda (14.6') integral E to'plamda *fundamental integral* deb ataladi.

Koshi teoremasi. $I(y)=\int_a^{b)} f(x,y)dx$ integralning E to'plamda tekis yaqinlashuvchi bo'lishi uchun uning E da fundamental bo'lishi zarur va yetarlidir.

Natija. Agar $\forall\delta\in[a;b)$ uchun $\exists\varepsilon_0>0, \exists\, t',t''\in[b-\delta;b)$ va $\exists\, y_0\in E$ topilib,

$$\left|\int_{t'}^{t''} f(x,y_0)dx\right|\geq\varepsilon_0 \qquad (14.7')$$

tengsizlik bajarilsa, u holda (14.6') integral E to'plamda notekis yaqinlashuvchi bo'ladi.

14.5- misol. $I(y)=\int_0^{+\infty}\frac{\sin yx}{x}dx$ integralning $a)$ $E=[a;1] (a>0)$; $b)$ $E_1=[0;1]$ to'plamlarda y parametr bo'yicha tekis yaqinlashish xarakterini Koshi teoremasidan foydalanib tekshiring.

a) $\forall\varepsilon>0$ son olinganda ham, $\delta=\delta(\varepsilon)=\dfrac{3}{a\pi\varepsilon}$ deyilsa, u holda $\forall t'=n\pi>\delta, t''=2n\pi>\delta\,(n\in N)$ va $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''}\frac{\sin yx}{x}dx\right|=\left|-\frac{1}{x}\frac{\cos xy}{y}\Big|_{t'}^{t''}-\frac{1}{y}\int_{t'}^{t''}\frac{\cos xy}{x^2}dx\right|=$$

$$=\left|\frac{1}{t'}\frac{\cos yt'}{y}-\frac{1}{t''}\frac{\cos yt''}{y}-\frac{1}{y}\int_{t'}^{t''}\frac{\cos xy}{x^2}dx\right|\leq$$

$$\leq\left|\frac{1}{t'y}\right|+\left|\frac{1}{t''y}\right|+\frac{1}{y}\left|\int_{t'}^{t''}\frac{|\cos xy|}{x^2}dx\right|\leq\frac{1}{at'}+\frac{1}{at''}+\frac{1}{a}\left|\int_{t'}^{t''}\frac{dx}{x^2}\right|<\frac{2}{at'}+\frac{2}{at''}<\varepsilon.$$

Demak, berilgan parametrga bog‘liq bo‘lgan xosmas integral Koshi teoremasiga asosan E to‘plamda tekis yaqinlashuvchi bo‘ladi.

b) $\forall\delta>0$ son uchun $y_\delta=\delta$, $t'_\delta=\dfrac{\pi}{3\delta}$, $t''_\delta=\dfrac{\pi}{2\delta}$ deb tanlasak, u holda

$$\left|\int_{t'_\delta}^{t''_\delta}\frac{\sin y_\delta x}{x}dx\right|=\left|\int_{\frac{\pi}{3\delta}}^{\frac{\pi}{2\delta}}\frac{\sin\delta x}{x}dx\right|=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sin t}{t}dt=\varepsilon_0$$

bo‘ladi, bunda $\varepsilon_0>0$. Demak, berilgan integral, yuqoridagi natijaga asosan, $y\in E_1$ da notekis yaqinlashadi.

14.6- misol. $\displaystyle\int_0^{+\infty}\frac{dx}{(x-y)^4+16}$ integralning $E\in(-\infty;0]$ to‘plamda y parametr bo‘yicha tekis yaqinlashish xarakterini tekshiring.

Yechilishi. $f(x,y)=\dfrac{1}{(x-y)^4+16}\leq\dfrac{1}{(x-y)^2+4}=\varphi(x,y)$ tengsizlik $\forall x\in[0;+\infty]$, $y\in(-\infty;0]$ uchun bajariladi. $\displaystyle\int_0^{+\infty}\frac{dx}{(x-y)^4+16}$ integralning har bir tayinlangan y uchun yaqinlashuvchi ekanligini ko‘rsatamiz. Ma’lumki,

$$\int_0^{+\infty}\frac{dx}{(x-y)^4+16}=\frac{1}{32\sqrt{2}}\ln\left|\frac{(x-2)^2+2\sqrt{2}(x-y)+4}{(x-y)^2-2\sqrt{2}(x-y)+4}\right|\Bigg|_0^{+\infty}+$$

$$+\frac{1}{16\sqrt{2}}\operatorname{arctg}\frac{2\sqrt{2}(x-y)}{4-(x-y)^2}\Bigg|_0^{+\infty}=\frac{1}{32\sqrt{2}}\ln\left|\frac{y^2-2\sqrt{2}y+4}{y^2+2\sqrt{2}y+4}\right|+\frac{1}{16\sqrt{2}}\operatorname{arctg}\frac{2\sqrt{2}y}{4-y^2}.$$

Demak, bu integral yaqinlashuvchi ekan. Ravshanki, $\int_0^{+\infty}\frac{dx}{(x-y)^2+4}$ integral E to'plamda tekis yaqinlashuvchi (14.2-misolning b) bandiga qarang). Shunday qilib, berilgan integral, 14.2-natijaga asosan, E to'plamda yaqinlashuvchi bo'ladi.

14.7-misol. $\int_0^1\frac{\operatorname{arctg}yx}{(1-x^2)^y}dx$ integralni Koshi teoremasidan foydalanib, $E=[0;\frac{1}{2}]$ to'plamda tekis yaqinlashuvchi ekanligini ko'rsating.

Yechilishi. $\forall\varepsilon>0$ son olinganda ham, $\delta=\frac{1}{16\varepsilon^2}$ deyilsa, u holda $\forall t'=1-\frac{1}{2n}>\delta,\quad t''=1-\frac{1}{4n}>\delta\ (n\in N)$ va $\forall y\in E$ uchun

$$\left|\int_{1-\frac{1}{2n}}^{1-\frac{1}{4n}}\frac{\operatorname{arctg}xy}{(1-x^2)^y}dx\right|\le\frac{1}{2}\int_{1-\frac{1}{2n}}^{1-\frac{1}{4n}}\frac{dx}{(1-x)^y}=\frac{1}{2}\left[-\frac{(1-x)^{1-y}}{1-y}\right]\Bigg|_{1-\frac{1}{2n}}^{1-\frac{1}{4n}}= \tag{*}$$

$$=\frac{1}{2(1-y)}\left[\left(\frac{1}{2n}\right)^{1-y}-\left(\frac{1}{4n}\right)^{1-y}\right].$$

Ma'lumki, barcha $x\ge y>0,\ \mu\le1$ lar uchun

$$\left|x^\mu-y^\mu\right|\le|\mu|(x-y)\cdot y^{\mu-1}$$

tengsizlik o'rinli. Bu tengsizlikni e'tiborga olgan holda (*) tengsizlikdan

$$\left|\int_{1-\frac{1}{2n}}^{1-\frac{1}{4n}} \frac{\operatorname{arctg} xy}{(1-x^2)^y}dx\right| \le \frac{1}{2}\left(\frac{1}{4n}\right)^{1-y} \le \frac{1}{2}\frac{1}{(4n)^{1/2}} < \varepsilon$$

bajariladi. Shunday qilib, berilgan integral Koshi teoremasiga asosan, $E=[0;\frac{1}{2}]$ to‘plamda tekis yaqinlashuvchi bo‘lar ekan.

Veyershtrass alomati

3. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;+\infty),\ y\in E\subset R\ \}$ da berilgan bo‘lib, y ning E dagi har bir tayin qiymatida $f(x,y)$ funksiya x ning funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi bo‘lsin. Agar shunday $\varphi(x)$ funksiya $(x\in[a;+\infty))$ topilsaki:

1) $\forall x\in[a;+\infty)$ va $\forall y\in E$ uchun $|f(x,y)|\le\varphi(x)$ bo‘lsa;

2) $\int_a^{+\infty}\varphi(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u holda

$I(y)=\int_a^{+\infty} f(x,y)dx$ integral E da tekis yaqinlashuvchi bo‘ladi.

4. $f(x,y)$ funksiya $M_1=\{(x,y)\in R^2 : x\in[a,b),y\in E\subset R\}$ to‘plamda berilgan bo‘lib, uning E dan olingan har bir tayin qiymatida $f(x,y)$ funksiya x ning funksiyasi sifatida $[a,b)$ da integrallanuvchi bo‘lsin, ya’ni $I_1(y)=\int_a^{b)} f(x,y)dx$ integral mavjud bo‘lsin. Agar $[a,b)$ da shunday $\varphi(x)$ funksiya topilsaki:

1) $\forall x\in[a,b)$ va $\forall y\in E$ uchun $|f(x,y)|\le\varphi(x)$ bo‘lsa;

2) $\int_a^{b)}\varphi(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u holda

$I_1(y)=\int_a^{b)} f(x,y)dx$ integral E to'plamda tekis yaqinlashuvchi bo'ladi.

14.8- misol. Ushbu integralning $E=[-a,a]$ $(a>0)$ to'plamda tekis yaqinlashuvchiligining Veyershtrass alomatidan foydalanib ko'rsating:

$$I(y)=\int_0^{+\infty}\frac{\ln(1+x)\text{arctg}yx}{x^2}dx.$$

Yechilishi. Berilgan integralni ikkita integralga ajratib, ularning har birini tekis yaqinlashishga tekshiramiz: $I=I_1+I_2$;

$$I_1(y)=\int_0^1\frac{\ln(1+x)\text{arctg}yx}{x^2}dx,\quad I_1(y)=\int_1^{+\infty}\frac{\ln(1+x)\text{arctg}yx}{x^2}dx.$$

I_1 integralda $\forall x\in[0;1]$ va $y\in E$ uchun

$$|f(x,y)|=\left|\frac{\ln(1+x)\text{arctg}yx}{x^2}\right|\le\frac{\ln(1+x)|yx|}{x^2}\le a\frac{\ln(1+x)}{x}=O(1),\ x\to0 \text{ da.}$$

Demak, taqqoslash alomatiga ko'ra $I_1(y)$ integral E to'plamda tekis yaqinlashuvchi, $I_2(y)$ integralda $\forall x\in[1;+\infty)$ va $\forall y\in[-a;a]$ lar uchun $|f(x,y)|\le\frac{\pi}{2}\frac{\ln(1+x)}{x^2}=\varphi(x)$ tengsizlik o'rinli.

$$\int_1^{\infty}\frac{\ln(1+x)}{x^2}dx=-\frac{\ln(1+x)}{x}\Bigg|_1^{+\infty}+\int_1^{\infty}\left(\frac{1}{x}-\frac{1}{1+x}\right)dx=\ln2+\ln\frac{x}{1+x}\Bigg|_1^{+\infty}=\ln4.$$

Veyershtrass alomatiga ko'ra $I_2(y)$ integral $E=[-a,a]$ $(a>0)$ to'plamda tekis yaqinlashuvchi. Shunday qilib, berilgan integral ham E to'plamda tekis yaqinlashuvchi.

14.9- misol. $\int_1^{+\infty}\frac{dx}{x^y}$ integralni *a*) $E_1=[y_0;+\infty)$ $(y_0>0)$, *b*) $E_2=(1;+\infty)$ to'plamlarda tekis yaqinlashishga tekshiring.

Yechilishi. *a*) Agar $\forall x\in[1;+\infty)$ va $\forall y\in E_1$ uchun $\left|\frac{1}{x^y}\right|\le\frac{1}{x^{y_0}}$ ekanligini e'tiborga olsak va $\varphi(x)=\frac{1}{x^{y_0}}$ deb belgilasak, u holda

$$\int_1^{+\infty}\varphi(x)dx=\int_1^{+\infty}\frac{dx}{x^{y_0}}=\frac{1}{y_0-1}$$

integralning yaqinlashuvchiligidan, Veyershtrass alomatiga asosan, berilgan integral E_1 da tekis yaqinlashuvchi bo'ladi.

b) $y\in E_2$ bo'lsin.

$$\int_A^{+\infty}\frac{dx}{x^y}=\frac{A^{1-y}}{y-1}\ \text{va}\ \lim_{y\to 1+0}\frac{A^{1-y}}{y-1}=+\infty$$

bo'lganligidan, $\forall A>0$ va $\forall\varepsilon>0$ uchun $\exists y>1$ topiladiki,

$\int_A^{+\infty}\frac{dx}{x^y}>\varepsilon$ bo'ladi. Demak, bu holda berilgan integral E_2 to'plamda notekis yaqinlashadi.

14.10- misol. Ushbu integralning R to'plamda tekis yaqinlashishga tekshiring:

$$\int_1^{+\infty}\operatorname{arctg}\frac{2y}{y^2+x^3}dx,\ y\in R.$$

Yechilishi. $\forall y\in R,\forall x\in(1;+\infty)$ uchun $\left|\operatorname{arctg}\frac{2y}{y^2+x^3}\right|\le\frac{2|y|}{y^2+x^3}$ tengsizlik bajariladi. Tayinlangan $x\in(1;+\infty)$ larda $\varphi(y)=\frac{2|y|}{y^2+x^3}$ funksiya $y=\pm\sqrt{x^3}$ nuqtada ekstremumga erishadi.

$\forall y\in R,\ \forall x\in(1;+\infty)$ uchun $\left|\operatorname{arctg}\frac{2y}{y^2+x^3}\right|\le\frac{1}{x^{3/2}}$ tengsizlik o'rinli bo'ladi. $\int_1^{+\infty}\frac{dx}{x^{3/2}}$ xosmas integral yaqinlashuvchi bo'lganligi uchun Veyershtrass alomatiga ko'ra, berilgan integral R to'plamda y parametr bo'yicha tekis yaqinlashuvchi bo'ladi.

Abel alomati. $f(x,y)$ va $g(x,y)$ funksiyalar $M=\{(x,y)\in R^2:$ $:x\in[a;+\infty),y\in E\subset R'\}$ da berilgan. y ning E dagi har bir tayin qiymatida $g(x,y)$ funksiya x ning funkiyasi sifatida $[a;+\infty)$ da

monoton funksiya bo'lsin. Agar $\int_a^{+\infty} f(x,y)\,dx$ integral E to'plamda

tekis yaqinlashuvchi va $\forall(x,y)\in M$ uchun $|g(x,y)|\le C(C=\text{const})$ bo'lsa, u holda

$$\int_a^{+\infty} f(x,y)g(x,y)dx$$

xosmas integral E to'plamda tekis yaqinlashuvchi bo'ladi.

14.11- misol. Integralni tekis yaqinlashishga tekshiring:

$$\int_1^{+\infty} \frac{\text{arctg}yx}{x^{\frac{3}{2}}} e^{-xy} dx \quad (y\in E=[a;+\infty)) \ (a>0) \ .$$

Yechilishi. $f(x,y)=\dfrac{\text{arctg}yx}{x^{\frac{3}{2}}}$, $g(x,y)=e^{-xy}$ deb olib, Abel alomati shartining bajarilishini tekshiramiz. Ushbu $\int_1^{+\infty} \dfrac{\text{arctg}yx}{x^{\frac{3}{2}}} dx$ integral Veyershtrass alomatiga ko'ra tekis yaqinlashuvchi, $g(x,y)=e^{-xy}$ esa y ning $E=[a;+\infty)$ dan olingan har bir tayin qiymatida x ning kamayuvchi funksiyasi va $\forall x\in[1;+\infty)$, $\forall y\in E=[a;+\infty)$ uchun $|g(x,y)|=e^{-xy}\le 1$ bo'ladi. Demak, berilgan integral Abel alomatiga asosan, $[a;+\infty)$ da tekis yaqinlashuvchi.

Dirixle alomati. $f(x,y)$ va $g(x,y)$ funksiyalar M da berilgan bo'lsin. Agar $\forall t\ge a$ hamda, $\forall y\in E$ uchun

$$\left|\int_a^t f(x,y)dx\right|\le C \quad (C=\text{const})$$

bo'lsa va y ning E to'plamdagi har bir o'zgarmas tayin qiymatida $x\to+\infty$ da $g(x,y)$ funksiya $\varphi(y)=0$ limit funksiyaga tekis yaqinlashsa, u holda, $\int_a^{+\infty} f(x,y)g(x,y)dx$ integral E to'plamda tekis yaqinlashuvchi bo'ladi.

14.12- misol. Ushbu integralni tekis yaqinlashishga tekshiring:

$$\int_2^{+\infty} \frac{\cos xy \cdot \ln x}{\sqrt{x}} dx \quad \left(y \in E = [a;+\infty), a > 0\right).$$

Yechilishi. Agar $f(x,y) = \cos xy,\quad g(x,y) = \frac{\ln x}{\sqrt{x}}$ deyilsa, u holda $\forall t > 0, \forall y \in E$ uchun

$$\left|\int_2^t f(x,y)dx\right| = \left|\int_2^t \cos xydx\right| = \left|\frac{\sin ty - \sin 2y}{y}\right| \leq 2$$

bo'ladi. $x \to +\infty$ da $g(x,y) = \frac{\ln x}{\sqrt{x}}$ funksiya E da nolga tekis yaqinlashadi, ya'ni $g(x,y) = \frac{\ln x}{\sqrt{x}} \overset{E}{\rightrightarrows} 0$.

Demak, berilgan integral Dirixle alomatiga asosan, $E = [a;+\infty)$ da tekis yaqinlashuvchi.

14.13- misol. Ushbu

$$\int_1^{+\infty} e^{-y \cdot x} \frac{\cos x}{x^p} dx \quad (p > 0)$$

integralni $E = [0;+\infty)$ to'plamda tekis yaqinlashishga tekshiring.

Yechilishi. Integral ostidagi funksiyani $f(x,y) = \frac{\cos x}{x^p}$, $g(x,y) = e^{-y \cdot x}$ deb belgilab, Abel alomati shartlarining bajarilishini tekshiramiz: $\int_1^{+\infty} \frac{\cos x}{x^p} dx$ integral $p > 0$ da Dirixle alomatiga ko'ra tekis yaqinlashuvchi bo'ladi, $g(x;y) = e^{-y \cdot x}$ funksiya y ning $E = [0;+\infty)$ dan olingan tayin qiymatida x ning kamayuvchi funksiyasi va $\forall x \in [1;+\infty)$, $y \in [0;+\infty)$ lar uchun $g(x;y) = e^{-y \cdot x} \leq 1$. Demak, berilgan integral Abel alomatiga ko'ra, $[0;+\infty)$ to'plamda tekis yaqinlashuvchi.

Mustaqil yechish uchun misollar

I(y) integrallarning *E* to‘plamda tekis yaqinlashuvchanligini isbotlang:

14.1. $I(y)=\int\limits_{1}^{+\infty} x^{y}e^{-3x}dx, y\in E=[1;4].$

14.2. $I(y)=\int\limits_{2}^{+\infty} \frac{dx}{x(\text{in}\,x)^{y}}, y\in E=[a;+\infty), a>1.$

14.3. $I(y)=\int\limits_{1}^{+\infty} \frac{\ln^{3}x}{x^{2}+y^{4}}dx,\ y\in E=(-\infty;+\infty).$

14.4. $I(y)=\int\limits_{-\infty}^{+\infty} \frac{\cos yx}{9+x^{2}}dx,\ y\in E=(-\infty;+\infty).$

14.5. $I(y)=\int\limits_{0}^{1} x^{y-1}\ln^{3}x dx, y\in E=[a;4], a>0.$

14.6. $I(y)=\int\limits_{0}^{1} \frac{x^{y}\text{arctg}xy}{\sqrt{1-x^{2}}}, y\in E=[0;2].$

14.7. $I(y)=\int\limits_{2}^{+\infty} \frac{\sin xy\ln x}{\sqrt[4]{x}}dx, y\in E=[1;+\infty).$

14.8. $I(y)=\int\limits_{1}^{+\infty} \frac{\sin x}{\sqrt[5]{x}}e^{-xy}dx, y\in E=[0;+\infty).$

14.9. $I(y)=\int\limits_{0}^{+\infty} \cos(x^{2}y)dx, y\in E=[1;+\infty).$

14.10. $I(y)=\int\limits_{0}^{+\infty} \frac{\sin(y^{4}x)}{x\ +y^{4}}dx, y\in E=(1;+\infty).$

14.11. $I(y)=\int\limits_{2}^{+\infty} x^{y}e^{-2x'}dx, y\in E=[1;2].$

14.12. $I(y)=\int\limits_{0}^{\infty} \frac{\cos(yx^{3})}{x}dx,\ y\in E=[a;+\infty), a>0.$

$I(y)$ integrallarning E to'plamda notekis yaqinlashuvchiligini ko'rsating:

14.13. $I(y)=\int\limits_{2}^{3}\frac{dx}{(x-2)^{y}},y\in E=[-1;1).$

14.14. $I(y)=\int\limits_{0}^{+\infty}\frac{dx}{9+(x-y)^{4}},y\in E=[0;+\infty).$

14.15. $I(y)=\int\limits_{0}^{+\infty}x^{2}e^{-yx^{4}}dx,y\in E=(0;+\infty).$

14.16. $I(y)=\int\limits_{0}^{+\infty}e^{-(x-y)^{2}}dx,y\in E=[0;+\infty).$

14.17. $I(y)=\int\limits_{0}^{+\infty}\frac{dx}{(x+1)^{y}},y\in E=(1;+\infty).$

$I(y)$ integrallarni E to'plamda tekis yaqinlashishga tekshiring:

14.18. $I(y)=\int\limits_{0}^{+\infty}\frac{dx}{1+x^{y}},\ y\in E=(1;+\infty).$

14.19. $I(y)=\int\limits_{1}^{+\infty}\frac{\cos x^{2}}{1+x^{y}}dx,y\in E=[0;+\infty).$

14.20. $I(y)=\int\limits_{0}^{+\infty}\frac{\cos yx}{e^{y^{2}(1+x^{2})}}dx,y\in E=(-\infty;+\infty).$

14.21. $I(y)=\int\limits_{0}^{1}\sin\frac{1}{x}\cdot\frac{dx}{x^{y}},y\in E=(0;3).$

14.22. $I(y)=\int\limits_{0}^{1}\frac{\cos yxdx}{\sqrt{|x-y|}},y\in E=[0;1].$

14.23. $I(y)=\int\limits_{0}^{2\pi}\frac{e^{-x}dx}{|\sin x|^{y}},y\in E=[0;1).$

14.24. $I(y)=\int\limits_0^{+\infty}\frac{\cos e^x dx}{1+x^y}, y\in E=(0;+\infty)$.

14.25. $I(y)=\int\limits_0^{+\infty}\frac{\operatorname{arctg}(xy)\operatorname{arctg}(xy^2)}{1+x^2}e^{-xy}dx, y\in E=[0;+\infty)$.

14.26. $I(y)=\int\limits_0^{+\infty}\sin x^2\operatorname{arctg}(xy)dx, y\in E=(-\infty;+\infty)$.

14.27. $I(y)=\int\limits_0^{1}\frac{1}{x}\cdot\cos\frac{1}{x}2^{xy}dx,\ y\in E=(-\infty;1]$.

14.28. $I(y)=\int\limits_1^{+\infty}\frac{\sin 2x}{\sqrt{x}}\frac{dx}{9+x^2y^2}, y\in E=(-\infty;+\infty)$.

14.29. $I(y)=\int\limits_0^{+\infty}\frac{\ln(e^x-x)}{x^y}dx, y\in E=(2;3)$.

14.30. $I(y)=\int\limits_1^{+\infty}\frac{\sin(y^2x)\operatorname{arctg}(xy)}{\sqrt[3]{x^2}}dx, y\in E=(-\infty;-1)\cup[1;+\infty)$.

14.31. $I(y)=\int\limits_0^{+\infty}\frac{\sin x}{x}e^{-xy}dx, y\in E=[0;+\infty)$.

14.32. $I(y)=\int\limits_0^{+\infty}\frac{\sin xy}{x}dx, y\in E=[1;2]$.

Misollarning javoblari

14.18. Notekis yaqinlashadi. **14.19**. Tekis yaqinlashadi. **14.20**. Notekis yaqinlashadi. **14.21**. Notekis yaqinlashadi. **14.22**. Tekis yaqinlashadi. **14.23**. Notekis yaqinlashadi. **14.24**. Tekis yaqinlashadi. **14.25**. Tekis yaqinlashadi. **14.26**. Tekis yaqinlashadi. **14.27**. Tekis yaqinlashadi. **14.28**. Tekis yaqinlashadi. **14.29**. Notekis yaqinlashadi.**14.30**. Tekis yaqinlashadi. **14.31**. Tekis yaqinlashadi. **14.32**. Tekis yaqinlashadi.

15- § *Parametrga bog'liq bo'lgan xosmas integrallarning funksional xossalari*

15.1. Integral belgisi ostida limitga o'tish. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;+\infty), y\in E\subset R\}$ to'plamda berilgan, y_0 nuqta E to'plamning limit nuqtasi bo'lsin.

15.1- teorema. Agar $f(x,y)$ funksiya:

1) y ning E dan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a,+\infty)$ da uzluksiz;

2) $y\to y_0$ da $\forall[a;t]$ $(a<t<\infty)$ da $\varphi(x)$ limit funksiyaga tekis yaqinlashuvchi;

3) $I(y)=\int_a^{+\infty} f(x,y)dx$ integral E to'plamda tekis yaqinlashuvchi

bo'lsa, $y\to y_0$ da $I(y)$ funksiya limitga ega va

$$\lim_{y\to y_0} I(y)=\lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\lim_{y\to y_0} f(x,y)=\int_a^{+\infty}\varphi(x)dx$$

tenglik o'rinli bo'ladi.

$f(x,y)$ funksiya $M_1=\{(x,y)\in R^2 : x\in[a;b),\ y\in E\subset R\}$ to'plamda berilgan bo'lib, y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida $x=b$ nuqta uning maxsus nuqtasi, y_0 esa E to'plamning limit nuqtasi bo'lsin.

15.2- teorema. Agar $f(x,y)$ funksiya:

1) y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a;b)$ da uzluksiz;

2) $y\to y_0$ da ixtiyoriy $[a,t]$ $(a<t<b)$ oraliqda $\varphi(x)$ limit funksiyaga tekis yaqinlashuvchi;

3) $I_1(y)=\int_a^{b)} f(x,y)dx$ integral E to'plamda tekis yaqinlashuvchi

bo'lsa, $y\to y_0$ da $I_1(y)$ funksiya limitga ega va

$$\lim_{y\to y_0} I_1(y)=\lim_{y\to y_0}\int_a^{b)} f(x,y)dx=\int_a^{b)}\lim_{y\to y_0} f(x,y)=\int_a^{b)}\varphi(x)dx$$

tenglik o'rinli bo'ladi.

15.1- misol. Ushbu tenglikning o'rinli ekanligini ko'rsating:

$$\lim_{y\to 1}\int_1^{+\infty} \operatorname{arctg}\frac{2y}{y^2+x^3}dx=\int_1^{+\infty}\operatorname{arctg}\frac{2}{1+x^3}dx.$$

Yechilishi. 15.1- teoremadan foydalanib, yuqoridagi tenglikning to'g'ri ekanligini ko'rsatamiz:

1) integral ostidagi $f(x,y)=\operatorname{arctg}\frac{2y}{y^2+x^3}$ funksiya $[1;+\infty)$ dan olingan har bir tayinlangan y uchun $[1;+\infty)$ da x ning funksiyasi sifatida uzluksiz;

2) $y\to 1$ da $\forall[1;t]$ $(1<t<\infty)$ da $f(x,y)\underset{\to}{\to}\operatorname{arctg}\frac{2}{1+x^3}$.

Haqiqatan ham, $\forall\varepsilon>0$ son olinganda, $\delta=\varepsilon$ deb olinsa, $\forall x\in[1;+\infty)$ va $\forall y\in[1;+\infty)$ uchun

$$\left|\operatorname{arctg}\frac{2y}{y^2+x^3}-\operatorname{arctg}\frac{2}{1+x^3}\right|\le\left|\frac{2y}{y^2+x^3}-\frac{2}{1+x^3}\right|\le\frac{2|1-y|(x^3+y^2)}{(y^2+x^3)(1+x^3)}<\delta=\varepsilon$$

tengsizlik bajariladi;

3) $\int_1^{+\infty}\operatorname{arctg}\frac{2y}{y^2+x^3}dx$ integral $E=[1;+\infty)$ to'plamda y parametr bo'yicha tekis yaqinlashuvchi (14.7- misolga qarang). Shunday qilib, 15.1-teoremaga ko'ra, yuqoridagi tenglik o'rinli.

15.2- misol. $\lim_{y\to 1}\int_0^{+\infty}\frac{y\operatorname{arctg}x}{x^2+y^2}dt=\frac{\pi^2}{4}$ unosabatning o'rinli ekanligini ko'rsating.

Yechilishi. Berilgan integralda, $x=ty$ $(t>0, y>0)$ almashtirishni bajaramiz:

$$\lim_{y\to 1}\int_0^{+\infty}\frac{y^2\cdot\operatorname{arctg}ty}{t^2y^2+y^2}dt=\lim_{y\to 1}\int_0^{+\infty}\frac{\operatorname{arctg}ty}{1+t^2}dt\,,$$

$$\left|\frac{\operatorname{arctg}}{1+t^2}\right|\le\frac{\pi}{2}\frac{1}{(1+t^2)},\quad\int_0^{+\infty}\frac{dt}{1+t^2}=\frac{\pi}{2}$$

bo‘lganligi uchun Veyershtrass teoremasiga asosan, $\int_0^{+\infty} \frac{y\operatorname{arctg} x}{x^2+y^2}dx$ tekis yaqinlashuvchi. $\frac{\operatorname{arctg} ty}{1+t^2}$ funksiya uzluksiz bo‘lganligi uchun

$$\frac{\operatorname{arctg} ty}{1+t^2} = \frac{\operatorname{arctg} t}{1+t^2}.$$

$\forall \varepsilon > 0$ ga ko‘ra, $\delta = \frac{\varepsilon}{2} > 0$ deb olinsa, $\forall t \in (a,b)$ uchun

$$\left|\frac{\operatorname{arctg} ty}{1+t^2} - \frac{\operatorname{arctg} t}{1+t^2}\right| \le \frac{t}{1+t^2}|y-1| < \frac{1}{2}\delta = \varepsilon$$

tengsizlik bajariladi. 15.1- teoremaga asosan,

$$\lim_{y\to 1}\int_0^{+\infty}\frac{\operatorname{arctg} ty}{1+t^2}dt = \int_0^{+\infty}\lim_{y\to 1}\frac{\operatorname{arctg} ty}{1+t^2}dt = \int_0^{+\infty}\frac{\operatorname{arctg} t}{1+t^2}dt = \frac{\pi^2}{4}.$$

15.2. Paramertga bog‘liq bo‘lgan xosmas integralning parametr bo‘yicha uzluksizligi. $f(x,y)$ funksiya $M = \{(x,y) \in R^2 : x \in [a;+\infty), y \in [c;d]\}$ to‘plamda berilgan bo‘lsin.

15.3- teorema. $f(x,y)$ funksiya M to‘plamda uzluksiz va

$$I(y) = \int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ oraliqda tekis yaqinlashuvchi bo‘lsin. U holda $I(y)$ funksiya $[c,d]$ oraliqda uzluksiz bo‘ladi.

$f(x,y)$ funksiya $M_1 = \{(x,y \in R^2 : x \in [a,b), y \in [c,d])\}$ to‘plamda berilgan. y o‘zgaruvchining $[c,d]$ oraliqdan olingan har bir tayin qiymatida $x=b$ nuqta uning uchun maxsus nuqta bo‘lsin.

15.4- teorema. $f(x,y)$ funksiya M_1 to‘plamda uzluksiz va

$$I_1(y) = \int_a^{b)} f(x,y)dx$$

integral $[c;d]$ da tekis yaqinlashuvchi bo‘lsin, u holda $I_1(y)$ funksiya $[c;d]$ oraliqda uzluksiz bo‘ladi.

15.3- misol. $F(y)=\int\limits_0^{+\infty}\sin(yx^2)dx$ funksiyaning $E=[1;+\infty)$ to‘plamda uzluksizligini ko‘rsating.

Yechilishi. Berilgan integralda $t=yx^2$ almashtirishni bajaramiz:

$$\int\limits_0^{+\infty}\sin(yx^2)dx=\frac{1}{2}\int\limits_0^{+\infty}\frac{\sin t}{\sqrt{ty}}dt=\frac{1}{2}\left(\int\limits_0^{1}\frac{\sin t}{\sqrt{ty}}dt+\int\limits_1^{+\infty}\frac{\sin t}{\sqrt{ty}}dt\right).$$

Bu integralni tekis yaqinlashishga tekshiramiz. $\frac{\sin t}{\sqrt{t}}$ funksiyaning $t=0$ dagi qiymatini 0 ga teng deb qabul qilsak, $f(t,y)=\frac{\sin t}{2\sqrt{ty}}$ funksiya $M=\{(x,y)\in R^2: 0\le x<+\infty, 1\le y<+\infty\}$ to‘plamda uzluksiz bo‘ladi. $\sin t$ funksiya $[0;+\infty)$ da chegaralangan boshlang‘ich funksiyaga ega, $g(t,y)=\frac{1}{2\sqrt{ty}}$ funksiya esa, $t\ge 1$ va $y\ge 1$ da

$$g'_t(t,y)=-\frac{1}{2\sqrt{ty}\cdot t}<0,\quad \frac{1}{2\sqrt{ty}}\le\frac{1}{2\sqrt{t}},\ \lim_{t\to\infty}\frac{1}{2\sqrt{t}}=0$$

bo‘lgani uchun $\int\limits_1^{+\infty}\frac{\sin t dt}{\sqrt{ty}}$ integral, Dirixle alomatiga ko‘ra, $[1;+\infty)$ to‘plamda tekis yaqinlashadi. $\forall y\in[1;+\infty)$ va $\forall t\in[0;1]$ uchun $\left|\frac{\sin t}{\sqrt{ty}}\right|\le\sqrt{t}$ tengsizlik o‘rinli; $\int\limits_0^1\sqrt{t}dt$ integral yaqinlashuvchi bo‘lganligi uchun $\int\limits_0^1\frac{\sin t}{\sqrt{ty}}dt$ integral Veyershtrass alomatiga ko‘ra tekis yaqinlashuvchi. Demak, 15.3- teoremaga asosan, $F(y)$ funksiyaning uzluksizligi kelib chiqadi.

15.4- misol. Ushbu integralni uzluksizlikka tekshiring:

$$F(a)=\int\limits_0^{\pi}\frac{\sin x}{x^a(\pi-x)^a}dx\quad(2>a>0).$$

Yechilishi. $0<\varepsilon\le a\le 2-\varepsilon<2$ bo'lsin. Berilgan integralni uch bo'lakka bo'lamiz va ularni baholaymiz:

$$\int_0^{\pi}\frac{\sin x}{x^a(\pi-x)^a}dx<\int_0^{1}\frac{xdx}{x^a(\pi-x)^a}+\int_1^{\pi-1}\frac{dx}{x^a(\pi-x)^a}+$$
$$+\int_{\pi-1}^{\pi}\frac{dx}{x^a(\pi-x)^{a-1}}\le\int_0^1\frac{dx}{x^{1-\varepsilon}}+\pi-2+\int_{\pi-1}^{\pi}\frac{dx}{(\pi-x)^{1-\varepsilon}}.$$

Tengsizlikning o'ng tomonidagi integrallar yaqinlashuvchi bo'lgani uchun, Veyershtrass alomatiga asosan, $\varepsilon\le a\le 2-\varepsilon$ bo'lganda berilgan integral tekis yaqinlashuvchi.

$f(x,a)=\dfrac{\sin x}{x^a(\pi-x)^a}$ funksiya $0<x<\pi$, $\varepsilon\le a\le 2-\varepsilon$ bo'lganda uzluksiz. U holda 15.3- teoremaga asosan, $F(a)$ funksiya $0<x<\pi$, $\varepsilon\le a\le 2-\varepsilon$ bo'lganda uzluksiz, *e* ning ixtiyoriyligini hisobga olganda, $F(a)$ funksiyaning $0<a<2$ oraliqda uzluksizligi kelib chiqadi.

15.5- misol. Ushbu tenglik o'rinli ekanligini isbotlang:

$$\lim_{y\to 0}\int_0^{+\infty}e^{-xy}\frac{\sin x}{x}dx=\int_0^{+\infty}\frac{\sin x}{x}dx. \qquad (15.1)$$

Isboti. Agar $\dfrac{\sin x}{x}$ funksiyani $x=0$ nuqtada uzluksizlikka qayta aniqlasak, ya'ni $\dfrac{\sin x}{x}$ funksiyaning qiymatini $x=0$ nuqtadagi qiymatiga teng deb qabul qilsak, $f(x,y)=e^{-xy}\dfrac{\sin x}{x}$ funksiya $\{(x,y)\in R^2: x\ge 0,\ y\ge 0\}$ to'plamda uzluksiz bo'ladi.

$\sin x$ funksiya chegaralangan boshlang'ich funksiyaga ega, $e^{-xy}\dfrac{\sin x}{x}$ funksiya $x\ge 0$, $y\ge 0$ da

$$\frac{\partial}{\partial x}\left(\frac{e^{-xy}}{x}\right)=-\frac{e^{-xy}}{x}(1+xy)<0,\ \frac{e^{-xy}}{x}\le\frac{1}{x},\ \lim_{x\to\infty}\frac{1}{x}=0$$

bo‘lgani uchun $\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx$ integral, Dirixle alomatiga ko‘ra, $[0,+\infty)$ oraliqda y bo‘yicha tekis yaqinlashuvchi. 15.3- teoremaga asosan,

$$I(y)=\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx$$

funksiya ixtiyoriy $[0;b]$ oraliqda uzluksiz. Xususiy holda, y=0 nuqtada ham uzluksiz, ya’ni $\lim_{y\to 0} I(y)=I(0)$. Shuning uchun (15.1) tenglik o‘rinli bo‘ladi.

15.6- misol. $F(\alpha)=\int_0^1 \frac{\sin\frac{\alpha}{x}}{x^{\alpha}}dx$ funksiyaning $-\infty<\alpha<2$ oraliqda uzluksizligini ko‘rsating.

Yechilishi. Integral ostida $x=\frac{1}{t}$ $(t>0)$ almashtirishni bajarib, $F(\alpha)=\int_1^{+\infty}\frac{\sin\alpha t}{t^{2-\alpha}}dt$ ni hosil qilamiz. $-\infty<\alpha<\frac{1}{2}$ bo‘lsin. U holda $\forall t\in[1,+\infty)$ uchun $\frac{|\sin\alpha t|}{t^{2-\alpha}}\le\frac{1}{\sqrt{t^3}}$ tengsizlik o‘rinli va $\int_1^{+\infty}\frac{dt}{\sqrt{t^3}}$ integral yaqinlashuvchi. Demak, Veyershtrass alomatiga ko‘ra, $\int_1^{+\infty}\frac{\sin\alpha t}{t^{2-\alpha}}dt$ integral $-\infty<\alpha<\frac{1}{2}$ da tekis yaqinlashuvchi.

$f(x,\alpha)=\frac{\sin\alpha t}{t^{2-\alpha}}$ funksiya $t\ge 1$ bo‘lganda uzluksiz. 15.3- teoremaga asosan, $F(a)$ funksiya $(-\infty;\ \frac{1}{2}]$ da uzluksiz bo‘ladi.

Endi $\frac{1}{2} \le \alpha \le 2-\varepsilon$ bo‘lsin, $\varepsilon > 0$. U holda $\left|\int\limits_1^x \sin\alpha t dt\right| \le \frac{2}{\alpha} \le 4$.

$\frac{1}{t^{2-\alpha}}$ funksiya α ning belgilangan qiymatida $t \to +\infty$ da monoton ravishda nolga intiladi, chunki $\frac{1}{t^{2-\alpha}} \le \frac{1}{t^{\varepsilon}}$ tengsizlik o‘rinli. Demak, Dirixle alomatiga asosan, berilgan integral tekis yaqinlashuvchi. Integral ostidagi funksiya $\forall t \ge 1$ va $\frac{1}{2} \le \alpha \le 2-\varepsilon$ da uzluksiz. Bularni e’tiborga olsak, *F(a)* funksiyaning $\frac{1}{2} \le \alpha \le 2-\varepsilon$ oraliqda uzluksiz ekanligi kelib chiqadi.

Shunday qilib, *F(a)* funksiya $-\infty < \alpha \le 2-\varepsilon$ oraliqda uzluksiz ekan. $\varepsilon > 0$ sonning ixtiyoriyligini hisobga olsak, bundan *F(a)* funksiyaning $-\infty < \alpha \le 2$ oraliqda uzluksizligi kelib chiqadi.

15.3. Parametrga bog‘liq bo‘lgan xosmas integrallarni parametr bo‘yicha differensiallash. $f(x,y)$ funksiya $M = \{(x.,y) \in R^2 :$ $: x \in [a;+\infty),\ y \in [c;d]\}$ to‘plamda berilgan bo‘lsin.

15.5- teorema. $f(x,y)$ funksiya M to‘plamda uzluksiz, $f'_y(x,y)$ uzluksiz xususiy hosilaga ega va y o‘zgaruvchining $[c;d]$ dan olingan har bir tayin qiymatida

$$I(y) = \int\limits_a^{+\infty} f(x,y)dx$$

integral yaqinlashuvchi bo‘lsin. Agar $\int\limits_a^{+\infty} f'_y(x,y)dx$ integral $[c;d]$ da tekis yaqinlashuvchi bo‘lsa, $I(y)$ funksiya ham $[c;d]$ oraliqda $I'_y(y)$ hosilaga ega bo‘ladi va

$$I'(y)=\int_a^{+\infty} f'_y(x,y)dx$$

munosabat o'rinli bo'ladi.

$f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a;+\infty),\ y\in[c;d]\}$ to'plamda berilgan, y o'zgaruvchining $[c;d]$ dan olingan har bir tayin qiymatida $x=b$ nuqta uning maxsus nuqtasi bo'lsin.

15.6- teorema. $f(x,y)$ funksiya M_1 to'plamda uzluksiz va $f'_y(x,y)$ uzluksiz xususiy hosilaga ega hamda y o'zgaruvchining $[c;d]$ dan olingan har bir tayin qiymatida

$$I_1(y)=\int_a^{b)} f(x,y)dx$$

integral yaqinlashuvchi bo'lsin. Agar $\int_a^{b)} f'_y(x,y)dx$ integral $[c;d]$ da tekis yaqinlashuvchi bo'lsa, $I_1(y)$ funksiya ham $[c;d]$ oraliqda $I'_1(y)$ hosilaga ega bo'ladi va

$$I'_1(y)=\int_a^{+\infty} f'_y(x,y)dx$$

munosabat o'rinlidir.

Parametrga bog'liq bo'lgan xosmas integrallarni hisoblashda quyidagi integrallardan foydalanamiz:

$$I_1=\int_0^{+\infty} e^{-\alpha x}\cos\beta x dx=\frac{\alpha}{\alpha^2+\beta^2},\alpha>0,\ \beta\in R, \qquad (1)$$

$$I_2=\int_0^{+\infty} e^{-\alpha x}\sin\beta x dx=\frac{\beta}{\alpha^2+\beta^2},\alpha>0,\ \beta\in R. \qquad (2)$$

(1), (2) formulalar I_1, I_2 integrallarni ikki marta bo'laklab integrallash natijasida hosil qilinadi.

15.7- misol. Ushbu integralni parametr bo'yicha differensiallash teoremasidan foydalanib, hisoblang:

$$I(\alpha,\beta)=\int_0^{+\infty}\frac{1-\cos\alpha x}{x}e^{-\beta x}dx,\ \beta>0\,.$$

Yechilishi. $\frac{1-\cos\alpha x}{x}$ funksiyaning x=0 nuqtadagi qiymatini 0 ga teng deb qabul qilsak, $f(x,\alpha)=\frac{1-\cos\alpha x}{x}e^{-\beta x}$, $f'_\alpha(x,\alpha)=\sin\alpha xe^{-\beta x}$ funksiyalar $M=\{(x;\alpha)\in R^2:0\le X<+\infty,\quad \alpha\in R\}$ to'plamda uzluksiz bo'ladi.

Berilgan integralni yaqinlashishga tekshiramiz:

$$\int_0^{+\infty}\frac{1-\cos\alpha x}{x}e^{-\beta x}dx=2\int_0^1\sin^2\frac{\alpha x}{2}\frac{e^{-\beta x}}{x}dx+2\int_1^{+\infty}\sin^2\frac{\alpha x}{2}\frac{e^{-\beta x}}{x}dx=I_1+I_2,$$

$$x\to+0\text{ da }\frac{2\sin^2\frac{\alpha x}{2}\cdot e^{-\beta x}}{x}=O\left(\frac{\alpha^2 x}{4}\right),$$

$$x\to+\infty\text{ da esa }2\sin^2\frac{\alpha x}{2}\cdot\frac{e^{-\beta x}}{x}=o\left(\frac{1}{x^2}\right)(\beta>1)$$

bo'lganligi sababli, taqqoslash teoremasiga asosan, I_1 va I_2 integrallar yaqinlashuvchi bo'ladi.

Endi $\int_0^{+\infty}\sin\alpha xe^{-\beta x}dx$ integralni $\alpha\in R$ da tekis yaqinlashishga tekshiramiz: $\forall\beta>0$ uchun

$$\left|e^{-\beta x}\sin\alpha x\right|\le e^{-\beta x},\quad \int_0^{+\infty}e^{-\beta x}dx$$

integral yaqinlashuvchi bo'lganligidan, Veyershtrass teoremasiga ko'ra $\int_0^{\infty}e^{-\beta x}\sin\alpha xdx$ integral $\alpha\in R$ to'plamda tekis yaqinlashuvchi. Demak, 15.5- teoremaga asosan,

$$J'_\alpha(\alpha,\beta)=\int_0^{+\infty}e^{-\beta x}\sin\alpha xdx$$

bo'ladi. (2) formulani e'tiborga olsak,

$$J'_{\alpha}(\alpha,\beta)=\frac{\alpha}{\alpha^2+\beta^2}.$$

Bu yerdan

$$J(\alpha,\beta)=\frac{1}{2}\ln\left|\alpha^2+\beta^2\right|+C(\beta),\quad J(0,\beta)=0$$

bo'lganligi uchun $C(\beta)=-\ln\beta$ bo'lib, $J(\alpha,\beta)=\frac{1}{2}\ln\left(1+\frac{\alpha^2}{\beta^2}\right)$

bo'ladi.

15.7- misol. Ushbu

$$I(\alpha)=\int_0^{+\infty}\frac{\sin\alpha x}{x}dx,\quad \alpha\geq 0 \tag{4}$$

Dirixle integralini hisoblang.

Yechilishi. Berilgan integralni hisoblash uchun quyidagi integralni qaraymiz:

$$F(\alpha,\beta)=\int_0^{+\infty}e^{-\beta x}\frac{\sin\alpha x}{x}dx,\quad \beta>0. \tag{5}$$

$\frac{1}{x}e^{-\beta x}$ funksiya $(0;+\infty)$ oraliqda monoton kamayuvchi, $\sin\alpha x$ funksiya esa $\alpha\neq 0$ bo'lganda chegaralangan boshlang'ich funksiyaga ega, ya'ni

$$\int_0^x\sin\alpha t dt=\frac{1-\cos\alpha x}{\alpha}.$$

$\alpha=0$ bo'lganda (5) integral nolga teng. Dirixle alomatiga asosan, belgilangan $\beta>0$ va ixtiyoriy $\alpha\neq 0$ uchun (5) integral yaqinlashuvchi.

Integral ostidagi funksiya va uning α bo'yicha hosilasi x va α bo'yicha $M=\left\{(x,\alpha)\in R^2 : x\in[0;+\infty),\alpha\in[0;+\infty)\right\}$ da uzluksiz.

$$\int_0^{+\infty}e^{-\beta x}\cos\alpha x dx=\frac{\beta}{\alpha^2+\beta^2}$$

integral α ga nisbatan tekis yaqinlashuvchi, chunki

$$\left|e^{-\beta x}\cos\alpha x\right|\le e^{-\beta x},\ \int_0^{+\infty} e^{-\beta x}dx$$

integral yaqinlashuvchi. Demak, 13.5- teoremaga asosan, (5) integralni α parametr bo'yicha differensiallash mumkin: $\alpha\ge 0$ uchun

$$F'_\alpha(\alpha,b)=\frac{\beta}{a^2+\beta^2}$$

munosabatga ega bo'lamiz. Buni $[0;\alpha]$ oraliqda integrallab,

$$F(\alpha,\beta)-F(0,\beta)=\beta\int_0^\alpha\frac{dt}{t^2+\beta^2}=\operatorname{arctg}\frac{\alpha}{\beta}$$

ifodani hosil qilamiz, bunda $F(0,\beta)=0$ bo'lgani uchun $F(\alpha,\beta)=\operatorname{arctg}\frac{\alpha}{\beta}$ ifodani hosil qilamiz. Shunday qilib,

$$F(\alpha,\beta)=\int_0^{+\infty}e^{-\beta\cdot x}\frac{\sin\alpha x}{x}dx=\operatorname{arctg}\frac{\alpha}{\beta}.$$

Endi $\alpha>0$ deb, (4) integralni hisoblaymiz. Belgilangan $\alpha>0$ larda (5) integral β bo'yicha $[0;1]$ oraliqda tekis yaqinlashuvchi, chunki $\sin\alpha x$ ($\alpha>0$ — belgilangan) chegaralangan boshlang'ich funksiyaga ega, $\frac{e^{-\beta\cdot x}}{x}$ funksiya esa monoton kamayuvchi $\left(\left(\frac{e^{-\beta\cdot x}}{x}\right)'_x<0,\beta\ge 0\right)$ va $[0;1]$ da $\frac{e^{-\beta\cdot x}}{x}\underset{x\to\infty}{\to}0$. Dirixle alomatiga asosan, (5) integral $[0,1]$ oraliqda β bo'yicha tekis yaqinlashuvchi, hamda $e^{-\beta x}\frac{\sin\alpha x}{x}$ funksiya $\{(x,\beta)\in R^2: x\in[0;+\infty),\beta\in[0;1]\}$ to'plamda uzluksiz bo'lgani uchun, $F(\alpha,\beta)$ funksiya $[0;1]$ da β bo'yicha uzluksiz. 15.1- teoremaga asosan, (5) integralda integral ostida $\beta\to+0$ da limitga o'tish mumkin:

$$\lim_{\beta\to+0}\int_0^{+\infty} e^{-\beta x}\frac{\sin\alpha x}{x}dx=\int_0^{+\infty}\frac{\sin\alpha x}{x}dx=\lim_{\beta\to+0}\operatorname{arctg}\frac{\alpha}{\beta}=\frac{\pi}{2}.$$

$\frac{\sin\alpha x}{x}$ funksiya α bo'yicha toq funksiya bo'lganligi uchun

$$\int_0^{+\infty}\frac{\sin\alpha x}{x}dx=\frac{\pi}{2}\operatorname{sign}\alpha,\ \alpha\in R. \tag{7}$$

Xususiy holda, $\alpha=1$ bo'lganda

$$\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}.$$

15.4. Parametrga bog'liq bo'lgan xosmas integrallarni parametr bo'yicha integrallash. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:$ $: x\in[a;+\infty), y\in[c;d]\}$ to'plamda berilgan bo'lsin.

15.7- teorema. Agar $f(x,y)$ funksiya M to'pldamda uzluksiz va $I(y)=\int_a^{+\infty}f(x,y)dx$ integral $[c;d]$ oraliqda tekis yaqinlashuvchi bo'lsa, $I(y)$ funksiya $[c;d]$ da integrallanuvchi va

$$\int_c^d I(y)dy=\int_c^d\left[\int_a^{+\infty}f(x,y)dx\right]dy=\int_a^{+\infty}\left[\int_c^d f(x,y)dy\right]dx$$

munosabat o'rinli.

$f(x,y)$ funksiya $M_1=\{(x,y)\in R^2: x\in[a;+\infty), y\in[c;+\infty)\}$ to'plamda aniqlangan bo'lsin.

15.8- teorema. $f(x,y)$ funksiya M_1 to'plamda uzluksiz hamda

$$\int_a^{+\infty}f(x,y)dx,\quad \int_c^{+\infty}f(x,y)dy$$

integrallar, mos ravishda, $[a;+\infty)$, $[c;+\infty)$ oraliqda tekis yaqinlashuvchi bo'lsin.

Agar

$$\int_a^{+\infty} dx \left|\int_c^{+\infty} |f(x,y)|dy\right|, \quad \int_c^{+\infty} dy \left|\int_a^{+\infty} |f(x,y)|dx\right|$$

integrallarning hech bo‘lmaganda bittasi yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty} dx \left[\int_c^{+\infty} f(x,y)dy\right], \quad \int_c^{+\infty} dy \left[\int_a^{+\infty} f(x,y)dx\right]$$

integrallar ham yaqinlashuvchi va ular o‘zaro teng bo‘ladi.

$f(x,y)$ funksiya $M_2 = \{(x,y) \in R^2 : x \in [a;b), y \in [c;d]\}$ to‘plamda berilgan, y ning $[c;d]$ dan olingan har bir tayin qiymatida $x=b$ nuqta uning maxsus nuqtasi bo‘lsin.

15.9- teorema. $f(x,y)$ funksiya M_2 to‘plamda aniqlangan, uzluksiz va

$$I_1(y) = \int_a^{b)} f(x,y)dx$$

integral $[c;d]$ oraliqda tekis yaqinlashuvchi bo‘lsa, u holda $I_1(y)$ funksiya $[c;d]$ oraliqda integrallanuvchi va

$$\int_c^d I_1(y)dy = \int_c^d dy \int_a^{b)} f(x,y)dx = \int_a^{b)} dx \int_c^d f(x,y)dy$$

munosabat o‘rinli.

15.8- misol. $\int_0^{+\infty} \frac{\sin^5 x}{x} dx$ integralni hisoblang.

Yechilishi. Integral ostidagi $\sin^5 x$ funksiyani quyidagi ko‘rinishda ifodalaymiz:

$$\sin^5 x = \sin x \sin^4 x = (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x)\sin x =$$

$$= \frac{5}{8}\sin x - \frac{5}{16}\sin 3x + \frac{1}{16}\sin 5x.$$

Bu tenglikni e’tiborga olgan holda, berilgan integralni

$$\int_0^{+\infty}\frac{\sin^5 x}{x}dx=\frac{5}{8}\int_0^{+\infty}\frac{\sin x}{x}dx-\frac{5}{16}\int_0^{+\infty}\frac{\sin 3x}{x}dx+\frac{1}{6}\int_0^{+\infty}\frac{\sin 5x}{x}dx=\frac{3}{8}\int_0^{+\infty}\frac{\sin x}{x}dx$$

ko'rinishga keltiramiz. Endi, $J=\int_0^{+\infty}\frac{\sin x}{x}dx$ integralni hisoblash uchun $\frac{1}{x}=\int_0^{\infty}e^{-xy}dy$ munosabatdan foydalanamiz:

$$J=\int_0^{+\infty}\sin x dx\int_0^{+\infty}e^{-xy}dy.$$

Bu yerda integrallash tartibini almashtirish mumkin, deb faraz qilib, (2) formulani e'tiborga olgan holda,

$$J=\int_0^{+\infty}dy\int_0^{+\infty}e^{-xy}\sin x dy=\int_0^{+\infty}\frac{dy}{1+y^2}=\frac{\pi}{2}$$

munosabatga ega bo'lamiz. Endi $J=\int_0^{+\infty}\sin x dx\int_0^{+\infty}e^{-xy}dy$ integralda integrallash tartibini o'zgartirishning mumkin ekanligini asoslaymiz, ya'ni 15.8- teoremaning shartlarini tekshiramiz. $f(x,y)=e^{-xy}\sin x$ funksiya $M_1=\{(x,y)\in R^2 : x\in[0;+\infty)\ y\in[0;+\infty)\}$ to'plamda uzluksiz. $0<a<A<+\infty$ deb olamiz. Unda

$$\int_a^A\frac{\sin x}{x}dx=\int_a^A\sin x dx\int_0^{+\infty}e^{-xy}dy=\int_0^{+\infty}dy\int_a^A e^{-xy}\sin x dx=$$

$$=\int_0^{+\infty}\{\frac{y\sin a+\cos a}{1+y^2}e^{-ay}-\frac{y\sin A+\cos A}{1+y^2}e^{-Ay}\}dy=$$

$$=\sin a\cdot\int_0^{+\infty}\frac{y}{1+y^2}e^{-ay}dy+\cos a\cdot\int_0^{+\infty}\frac{y}{1+y^2}e^{-ay}dy-$$

$$-\sin A\cdot\int_0^{\infty}\frac{y}{1+y^2}e^{-Ay}dy-\cos A\cdot\int_0^{\infty}\frac{1}{1+y^2}e^{-Ay}dy$$

bo‘ladi. Keyingi ikki integral A $(A \geq A_0 > 0)$ ga nisbatan tekis yaqinlashuvchi. Shuning uchun $A \to +\infty$ da integral ostida limitga o‘tsak, ularning nolga intilishini ko‘rish qiyin emas.

Ikkinchi integral a $(a \geq 0)$ ga nisbatan tekis yaqinlashuvchi, $a \to 0$ da $y \to \frac{\pi}{2}$. $\sin a \int_0^{\infty} \frac{y}{1+y^2} e^{-ay} dy$ integralning $a \to 0$ da nolga intilishini ko‘rsatamiz:

$$\int_0^{+\infty} \frac{y}{1+y^2} e^{-ay} dy = \int_0^{+\infty} \frac{t}{a^2+t^2} e^{-t} dt = \int_0^1 \frac{t}{a^2+t^2} e^{-t} dt + \int_1^{+\infty} \frac{t}{a^2+t^2} e^{-t} dt,$$

$$\int_0^1 \frac{t}{a^2+t^2} e^{-t} dt < \int_0^1 \frac{t}{a^2+t^2} dt = \frac{1}{2} \ln\left(1+a^2\right) - \ln a,$$

$$\lim_{a \to 0} \sin a \left(\frac{1}{2} \ln\left(1+a^2\right) - \ln a \right) = 0, \int_1^{+\infty} \frac{t}{a^2+t^2} e^{-t} dt \leq \int_1^{+\infty} \frac{dt}{te^t} = c,$$

$$\lim_{a \to 0} c \sin a = 0.$$

Demak, $\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ ekanligi kelib chiqadi. Shunday qilib,

$$\int_0^{+\infty} \frac{\sin^5 x}{x} dx = \frac{3\pi}{16}.$$

15.9- misol. Ushbu tenglik o‘rinli bo‘ladimi:

$$\int_1^{+\infty} dy \int_1^{+\infty} \frac{y^2-x^2}{(x^2+y^2)^2} dx = \int_1^{+\infty} dx \int_1^{+\infty} \frac{y^2-x^2}{(x^2+y^2)^2} dy \ ?$$

Yechilishi. Ravshanki,

$$\int_1^{+\infty} \frac{y^2-x^2}{\left(x^2+y^2\right)^2} dx = \frac{x}{x^2+y^2} \Bigg|_1^{+\infty} = -\frac{1}{1+y^2}, \ -\int_1^{+\infty} \frac{dy}{1+y^2} = -\frac{\pi}{4};$$

$$\int_1^{+\infty} \frac{y^2-x^2}{(x^2+y^2)^2} dy = -\frac{y}{x^2+y^2} \Bigg|_1^{+\infty} = \frac{1}{1+x^2}, \ \int_1^{+\infty} \frac{1}{1+x^2} dx = \frac{\pi}{4}.$$

Demak, berilgan integrallar teng emas, ya'ni berilgan integralda integrallash tartibini almashtirish mumkin emas, chunki 15.8-teoremaning hamma shartlari bajarilmaydi:

$f(x,y)=\dfrac{y^2-x^2}{(x^2+y^2)^2}$ funksiya

$M_1=\{(x,y)\in R^2:1\le x<+\infty,1\le y<+\infty\}$ to'plamda uzluksiz,

$$\int_1^{+\infty}\frac{y^2-x^2}{(x^2+y^2)^2}dx,\quad \int_1^{+\infty}\frac{y^2-x^2}{(x^2+y^2)^2}dy$$

integrallar, mos ravishda, $y\ge 1$, $x\ge 1$ bo'lganda y va x parametrlarga nisbatan tekis yaqinlashuvchi, lekin

$$\int_1^{+\infty}dx\int_1^{+\infty}\frac{|y^2-x^2|}{\left(x^2+y^2\right)^2}dy,\quad \int_1^{+\infty}dy\int_1^{+\infty}\frac{|y^2-x^2|dx}{\left(x^2+y^2\right)^2}$$

integrallar uzoqlashuvchi. Haqiqatan ham,

$$\int_1^{+\infty}dx\int_1^{+\infty}\frac{|y^2-x^2|}{\left(x^2+y^2\right)^2}dy=\int_1^{+\infty}dx\left(\int_1^{x}\frac{x^2-y^2}{\left(x^2+y^2\right)^2}dy+\int_x^{+\infty}\frac{y^2-x^2}{(x^2+y^2)^2}dy\right)=$$

$$=\int_1^{+\infty}\left(\frac{y}{x^2+y^2}\Bigg|_1^x-\frac{y}{x^2+y^2}\Bigg|_x^{+\infty}\right)dx=\int_1^{+\infty}\left(\frac{1}{x}-\frac{1}{1+x^2}\right)dx=$$

$$=\left(\ln x-\mathrm{arctg}x\right)\Bigg|_1^{+\infty}=+\infty.$$

Shunga o'xshash, ikkinchi integralning ham uzoqlashuvchi ekanligini ko'rsatish mumkin.

Shunday qilib, 15.8- teoremani qo'llash mumkin emas, ya'ni tenglik o'rinli emas.

15.10- misol. $I=\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^2}}\int_0^1\frac{dy}{1+x^2y^2}$ integralni, integrallash tartibini almashtirish haqidagi teoremadan foydalanib, hisoblang.

Yechilishi. $f(x,y)=\dfrac{1}{\sqrt{1-x^2}\left(1+x^2y^2\right)}$ funksiya

$$M_2=\{(x,y)\in R^2: 0\le x<1,\ 0\le y\le 1\}$$

to'plamda uzluksiz. $\int\limits_0^1 \dfrac{1}{\sqrt{1-x^2}\left(1+x^2y^2\right)}dx$ integral y bo'yicha $[0;1]$ da tekis yaqinlashuvchi.

Haqiqatan ham, $\forall x\in[0;1)$ va tayinlangan $y\in[0;1]$ uchun $\dfrac{1}{\sqrt{1-x^2}\left(1+x^2y^2\right)}\le\dfrac{1}{\sqrt{1-x^2}}$ tengsizlik o'rinli, $\int\limits_0^1\dfrac{dx}{\sqrt{1-x^2}}$ integral yaqinlashuvchi.

Demak, $\int\limits_0^1\dfrac{1}{\sqrt{1-x^2}(1+x^2y^2)}dx$ integral, Veyershtrass teoremasiga asosan, tekis yaqinlashuvchi. 15.9- teoremaga asosan,

$$I=\int\limits_0^1\frac{dx}{\sqrt{1-x^2}}\int\limits_0^1\frac{dy}{1+x^2y^2}=\int\limits_0^1 dy\int\limits_0^1\frac{dx}{\sqrt{1-x^2}\left(1+x^2y^2\right)}$$

tenglik o'rinli bo'lgani uchun, x=cosφ almashtirishni bajarib, quyidagiga ega bo'lamiz:

$$I=\frac{\pi}{2}\int\limits_0^1\frac{dy}{\sqrt{1+y^2}}=\frac{\pi}{2}\ln\left(1+\sqrt{2}\right).$$

15.5. Ba'zi bir muhim aniq integrallarni hisoblashda parametrga bog'liq integrallardan foydalanish. Biz shu vaqtgacha aniq integrallarni hisoblashda ikki muhim usuldan foydalanib keldik: bulardan birinchisida aniq integralni integral yig'indining limiti sifatida qarab, ikkinchi usulda esa integral ostidagi funksiyaning boshlang'ich funksiyasini topib, Nyuton — Leybnis formulasidan foydalanib hisobladik. Lekin, ba'zi bir hollarda integral ostidagi funksiyaning boshlang'ich funksiyasini elementar funksiyalar yordamida ifodalab bo'lmaydi. Bunday integrallarni hisoblashda parametrga bog'liq integrallar nazariyasidan foydalanish muhim ahamiyatga ega.

Frullani integrallari

1. Agar $f(x)$ funksiya $[0;+\infty)$ da uzluksiz bo'lib, chekli $\lim\limits_{x\to+\infty} f(x)=f(+\infty)$ mavjud bo'lsa, $\forall a>0, b>0$ lar uchun

$$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=[f(0)-f(+\infty)]\ln\frac{b}{a} \qquad (1)$$

Frullani formulasi o'rinli.

Haqiqatan ham, $f(x)$ funksiya uzluksiz bo'lganligi uchun $\forall[A;B]$ $(0<A<B<+\infty)$ da

$$\int_A^B\frac{f(ax)-f(bx)}{x}dx=\int_A^B\frac{f(ax)}{x}dx-\int_A^B\frac{f(bx)}{x}dx=$$

$$=\int_{aA}^{aB}\frac{f(z)}{z}dz-\int_{bA}^{bB}\frac{f(z)}{z}dz=\int_{aA}^{bA}\frac{f(z)}{z}dz-\int_{aB}^{bB}\frac{f(z)}{z}dz$$

integral mavjud. Ma'lumki, berilgan integral quyidagicha aniqlanadi:

$$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=\lim_{A\to0}\int_{aA}^{bA}\frac{f(z)}{z}dz-\lim_{B\to+\infty}\int_{aB}^{bB}\frac{f(z)}{z}dz.$$

Keyingi ikki integralga umumlashgan o'rta qiymat haqidagi teoremani qo'llab, quyidagilarga ega bo'lamiz:

$$\int_{aA}^{bA}\frac{f(z)}{z}dz=f(\xi)\int_{aA}^{bA}\frac{dz}{z}=f(\xi)\ln\frac{b}{a} \quad (aA\le\xi\le bA),$$

$$\int_{aB}^{bB}\frac{f(z)}{z}dz=f(\eta)\int_{aB}^{bB}\frac{dz}{z}=f(\eta)\cdot\ln\frac{b}{a} \quad (aB\le\eta\le bB).$$

$A\to0$ da $\xi\to0$, $B\to+\infty$ da esa $\eta\to+\infty$ ekanligini e'tiborga olgan holda, yuqoridagi mulohazalardan

$$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx=[f(0)-f(+\infty)]\ln\frac{b}{a}$$

tenglikning o'rinli bo'lishi kelib chiqadi.

2. Agar $x \to +\infty$ da $f(x)$ funksiyaning chekli limiti mavjud bo‘lmasa, lekin $\forall A > 0$ uchun $\int_A^{+\infty} \frac{f(z)}{z} dz$ integral mavjud bo‘lsa, u holda

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \cdot \ln \frac{b}{a} \qquad (1')$$

formula o‘rinli.

3. Agar $x=0$ nuqtada $f(x)$ funksiyaning uzluksizligi buzilib, $\int_0^A \frac{f(z)}{z} dz \; (A < +\infty)$ integral mavjud bo‘lsa, u holda ushbu formula o‘rinli:

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = f(+\infty) \ln \frac{b}{a} . \qquad (1'')$$

15.11- misol. Integralni (1) formuladan foydalanib hisoblang:

$$\int_0^{+\infty} \lg \frac{7 + 3e^{-ax}}{7 + 3e^{-bx}} \cdot \frac{dx}{x}, \quad a > 0, \; b > 0 .$$

Yechilishi. Berilgan integralda $f(x) = \lg(7 + 3e^{-x})$. Bu funksiya $[0; +\infty)$ da uzluksiz, $\lim_{x \to 0} f(x) = f(0) = 1$, $\exists \lim_{x \to +\infty} f(x) = f(+\infty) = \lg 7$.

Demak, (1) formulaga asosan, $\int_0^{+\infty} \lg \frac{7 + 3e^{-ax}}{7 + 3e^{-bx}} dx = \lg(1 + \frac{3}{7}) \cdot \ln \frac{b}{a}$.

15.12- misol. $\int_0^{+\infty} \frac{\sin \alpha x - \sin \beta x}{x} dx, \; \alpha > 0, \; \beta > 0$ integralni, (1′) formuladan foydalanib, hisoblang.

Yechilishi. Berilgan integralda $f(x) = \sin x$ bo‘lib, $f(0) = 0$ va $x \to +\infty$ da $\sin x$ funksiyaning limiti mavjud emas. $\forall A > 0$ uchun $\int_A^{+\infty} \frac{\sin x}{x} dx$ integral Dirixle alomatiga asosan yaqinlashuvchi. Demak, (1′) formulaga asosan, integral yaqinlashuvchi va

$$\int_0^{+\infty} \frac{\sin\alpha x - \sin\beta x}{x} dx = 0.$$

15.13- misol. Ushbu integralni, (1″) formuladan foydalanib hisoblang:

$$\int_0^{+\infty} \frac{\sin\frac{1}{ax} - \sin\frac{1}{bx}}{x} dx, \ a > 0, \ b > 0 .$$

Yechilishi. Ravshanki, $f(x) = \sin\frac{1}{x}$ bo‘lib, x=0 nuqtada uzilishga ega, $f(+\infty) = 0$. $\forall A > 0$ uchun $\int_0^{A} \frac{\sin\frac{1}{x}}{x} dx$ integral yaqinlashuvchi. Demak, (1′) formulaga ko‘ra,

$$\int_0^{+\infty} \frac{\sin\frac{1}{ax} - \sin\frac{1}{bx}}{x} dx = f(+\infty) \cdot \ln\frac{b}{a} = 0.$$

Mustaqil yechish uchun misollar

15.1. Agar $f(x)$ funksiya $(0;+\infty)$ da absolut integrallanuvchi bo‘lsa,

$$\lim_{n\to\infty} \int_0^{+\infty} f(x)\sin nx dx = 0, \quad \lim_{n\to\infty} \int_0^{+\infty} f(x)\cos nx dx = 0$$

ekanligini isbotlang.

15.2. Ushbu tenglikni isbotlang:

$$\lim_{\alpha\to 0} \frac{2}{\pi} \int_0^{+\infty} \frac{\alpha\cos x}{x^2 + \alpha^2} dx = 1 .$$

15.3. $\int_0^{+\infty} \alpha e^{-\alpha x} dx$ integralda $\alpha \to +0$ da integral ostida limitga o‘tish mumkinmi?

15.4. Agar $f(x)$ funksiya $[0;+\infty)$ da uzluksiz, chegaralangan va $f(0) = 0$, $g(x)$ funksiya esa $[0;+\infty)$ oraliqda absolut integrallanuvchi bo‘lsa, u holda

$$\lim_{\alpha\to+0}\int_0^{+\infty} f\left(\frac{\alpha}{x}\right)g(x)dx=0$$

ekanligini isbotlang.

15.5. Agar $f(x)$ funksiya $(0;+\infty)$ da integrallanuvchi bo'lsa,

$$\lim_{\alpha\to+0}\int_0^{+\infty} e^{-\alpha x}f(x)dx=\int_0^{+\infty} f(x)dx$$

ekanligini isbotlang.

15.6. $F(\alpha)=\int_0^{+\infty}\sin(\alpha x^2)dx$ funksiyaning $E=[1;\ +\infty)$ to'plamda uzluksizligini isbotlang.

15.7. $F(\alpha)=\int_0^{+\infty}\frac{\cos\alpha x}{1+x^2}dx$ funksiyaning $E=R$ to'plamda uzluksizligini isbotlang.

15.8. $F(\alpha)=\int_0^{1}\frac{\sin\frac{\alpha}{x}}{x^\alpha}dx$ funksiyaning $E=(0;1)$ to'plamda uzluksizligini isbotlang.

15.9. $F(\alpha)=\int_0^{+\infty}\frac{xdx}{2+x^\alpha}$ funksiyaning $E=(2;+\infty)$ to'plamda uzluksizligini isbotlang.

15.10. $F(\alpha)=\int_1^{+\infty}\frac{\cos x}{x^\alpha}dx$ funksiyaning $E=(0;+\infty)$ to'plamda uzluksizligini isbotlang.

Funksiyalarni ko'rsatilgan oraliqda uzluksizlikka tekshiring:

15. 11. $F(\alpha)=\int_0^{+\infty}\frac{\sin x}{x}e^{-\alpha x}dx,\ E=[0;\ +\infty)$.

15.12. $F(\alpha)=\int_0^{1}\frac{\ln x}{(x-\alpha)^2+4}dx\ \ E=R$.

15.13. $F(\alpha)=\int\limits_0^{\pi}\frac{dx}{\sin^{\alpha}x}, \quad E=[0;1)$.

15.14. $F(\alpha)=\int\limits_0^{+\infty}\frac{\sin(1-\alpha^2)x}{x}dx, \ E=R.$

15.15. $F(\alpha)=\int\limits_0^{+\infty}\alpha e^{-\alpha x^2}dx, \ E=R.$

15.16. $F(\alpha)=\int\limits_0^{+\infty}e^{-\alpha x}\cos x^2 dx, \ E=[0;+\infty).$

Dirixle integralidan foydalanib, integrallarni hisoblang:

15.17. $\int\limits_0^{+\infty}\frac{1-\cos\alpha x}{x^2}dx$. **15.18.** $\int\limits_0^{+\infty}\left(\frac{\sin x}{x}\right)^2 dx$. **15.19.** $\int\limits_0^{+\infty}\frac{\sin x^2}{x}dx$.

15.20. $\int\limits_0^{+\infty}\frac{\sin x^3}{x}dx$. **15.21.** $\int\limits_0^{+\infty}\frac{\sin x - x\cos x}{x^3}dx$.

15.22. $\int\limits_0^{+\infty}\frac{\sin^3\alpha x}{x}dx$. **15.23.** $\int\limits_0^{+\infty}\frac{\sin x\cos^4 x}{x}dx$.

15.24. $\int\limits_0^{+\infty}\frac{\cos\alpha x-\cos\beta x}{x^2}dx$.

Dirixle yoki Frullani integrallaridan foydalanib, integrallarni hisoblang:

15.25. $\int\limits_0^{+\infty}\frac{\alpha\sin\alpha x-\sin\alpha x}{x^2}dx, \quad \alpha>0$.

15.26. $\int\limits_0^{+\infty}\frac{\sin^4\alpha x-\sin^4\beta x}{x}dx, \quad \alpha>0, \ \beta>0.$

15.27. $\int\limits_0^{+\infty}\frac{\sin\alpha x\sin\beta x}{x}dx, \ \alpha>0, \ \beta>0, \ \alpha\neq\beta.$

15.28. $\int\limits_0^{+\infty}\frac{\sin^3 x\cos\alpha x}{x^3}dx, \quad \alpha>3.$

Parametr bo'yicha integral ostida differensiallab, integrallarni hisoblang:

15.29. $I(\alpha)=\int_0^{+\infty}\frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}dx$, $(\alpha>0,\ \beta>0)$ integralni hisoblang.

15.30. $I(\alpha)=\int_0^{+\infty}\frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin mx dx$, $(\alpha>0,\ \beta>0,\ m\neq 0)$ integralni hisoblang.

15.31. $\int_0^{\infty}\frac{e^{-\alpha x}-e^{-\beta x}}{x}\cos\lambda x dx$, $\alpha>0$, $\beta>0$ integralni hisoblang.

15.32. $\int_0^1\frac{\operatorname{arctg}\alpha x}{x\sqrt{1-x^2}}dx$ integralni hisoblang.

15.33. $\int_0^1\frac{\operatorname{arctg}\alpha x}{x(1-x^2)}dx=\frac{\pi}{2}\ln(1+\alpha)$ $(\alpha>0)$ ekanligini isbotlang.

15.34. 15.33- misolning natijasidan foydalanib, $\int_0^{\frac{\pi}{2}}\frac{x}{\operatorname{tg}x}dx=\frac{\pi}{2}\ln 2$ ekanligini isbotlang.

15.35. $\int_0^1 x^{\alpha-1}dx=\frac{1}{\alpha}$, $\alpha>0$ ekanligidan foydalanib, $\int_0^1 x^{\alpha-1}\cdot\ln x^m dx$, $m\in N$ integralni hisoblang.

15.36. $I(b)=\int_0^{+\infty}e^{-\alpha x^2}\cos bx dx$ $(b>0)$ integralni hisoblang.

15.37. $\int_0^{+\infty}x^{2n}e^{-x^2}\cos 2bx dx$ $(n\in N)$ integralni hisoblang.

15.38. $\int_0^{+\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ $(a>0)$ formuladan foydalanib, $\int_0^{+\infty} \frac{dx}{\left(x^2+a^2\right)^{n+1}}$, $(n \in N)$ integralni hisoblang.

15.39. Ushbu $J(\alpha) = \int_0^{+\infty} \frac{\cos \alpha x}{1+x^2} dx$, $K(\alpha) = \int_0^{+\infty} \frac{x \sin \alpha x}{1+x^2} dx$ Laplas integrallarini hisoblang:

15.40. Ushbu Frenel integrallarini hisoblang:

$$J = \int_0^{+\infty} \sin x^2 dx, \quad J_1 = \int_0^{+\infty} \cos x^2 dx.$$

15.41. Ushbu $\int_0^{+\infty} e^{-x^2} dx$ Eyler — Puasson integralini hisoblang.

15.42. $\int_0^{+\infty} \frac{\sin \alpha\, x}{x} = \frac{\pi}{2}$, $\alpha > 0$ dan foydalanib,

$$\int_0^{+\infty} \frac{\cos \alpha x - \cos bx}{x^2} dx = \frac{\pi(b-\alpha)}{2}$$

bo‘lishini isbotlang.

15.43. $\int_0^{+\infty} e^{-\alpha x} dx = \frac{1}{\alpha}$, $\alpha > 0$ munosabatdan foydalanib, $\int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx = \ln \frac{b}{\alpha}$, $\alpha > 0,\ b > 0$ bo‘lishini isbotlang.

15.44. $\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ Eyler — Puasson integralidan foydalanib, quyidagi munosabatlarni isbotlang:

1) $\int_{-\infty}^{+\infty} e^{-(\alpha x^2 + 2\beta x)} dx = \sqrt{\frac{\pi}{\alpha}} \cdot e^{\frac{\beta^2}{\alpha}}, \alpha > 0;$

2) $\int\limits_{-\infty}^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx = 2\sqrt{\pi}(\sqrt{\beta} - \sqrt{\alpha}),\ \alpha > 0, \beta > 0$;

3) $\int\limits_{0}^{+\infty} e^{-(x^2 + \frac{\alpha^2}{x^2})} dx = \frac{\sqrt{\pi}}{2} e^{-2\alpha},\ \alpha > 0.$

15.45. Integrallash tartibini almashtirish mumkinmi:

$$\int\limits_{1}^{+\infty} dx \int\limits_{0}^{1} \frac{y - x}{(x + y)^2} dy\ ?$$

Misollarning javoblari

15.11. Uzluksiz. **15.12.** Uzluksiz. **15.13.** Uzluksiz. **15.14.** $\alpha \neq \pm 1$ da uzluksiz; $\alpha = -1$ va $\alpha = 1$ da uzilishiga ega. **15.15.** $\alpha \neq 0$ da uzluksiz; $\alpha = 0$ da uzilishga ega. **15.16.** Uzluksiz. **15.17.** $\frac{\pi|\alpha|}{2}$.

15.18. $\frac{\pi}{2}$. **15.19.** $\frac{\pi}{4}$. **15.20.** $\frac{\pi}{6}$. **15.21.** $\frac{\pi}{4}$. **15.22.** $\frac{\pi}{4}\operatorname{sign}a$. **15.23.** $\frac{3\pi}{16}$.

15.24. $(|\beta| - |\alpha|)\frac{\pi}{2}$. **15.25.** $\alpha \ln \alpha$. **15.26.** $\frac{3}{8}\ln(\frac{\alpha}{\beta})$.

15.27. $\frac{1}{2}\ln(\left|\frac{\alpha + \beta}{\alpha - \beta}\right|)$. **15.28.** 0. **15.29.** $I(\alpha) = \frac{1}{2}\ln\frac{\beta}{\alpha}\ (\alpha > 0,\ \beta > 0)$.

15.30. $I(m) = \operatorname{arctg}\frac{\beta}{m} - \operatorname{arctg}\frac{\alpha}{m}$. **15.31.** $\frac{1}{2}\ln\frac{b^2 + \lambda}{\alpha^2 + \lambda^2}$.

15.32. $\frac{\pi}{2}\ln(\alpha + \sqrt{1 + \alpha^2})$. **15.35.** $\frac{(-1)^m m!}{\alpha^{m+1}}$. **15.36.** $I(b) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{b^2}{4\alpha}}$.

15.37. $(-1)^n \frac{\sqrt[n]{\pi}}{2^{2n+1}} \frac{d^{2n}}{db^{2n}}(e^{-b^2})$. **15.38.** $\frac{\pi(2n-1)!!}{2(2n)!!\alpha^{2n+1}}$.

15.39. $I(a) = \frac{\pi}{2}e^{-|\alpha|},\ \ \alpha \in R, K(a) = \frac{\pi}{2}\operatorname{sign}\alpha \cdot e^{-|\alpha|},\ \ \alpha \in R.$

15.40. $I(a) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. $I_1(a) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. **15.41.** $\frac{\sqrt{\pi}}{2}$. **15.45.** Yo‘q.

16- §. Eyler integrallari

16.1. Beta funksiya va uning xossalari. Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx \quad (a>0, b>0) \tag{16.1}$$

xosmas integral *Beta funksiya* yoki *1- tur Eyler integrali* deyiladi va $B(a,b)$ kabi belgilanadi, ya'ni

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx. \tag{16.2}$$

Integral ostidagi funksiya uchun:

1) $a<1, \quad b\geq 1$ bo'lganda x=0 nuqta;

2) $a\geq 1, \quad b<1$ bo'lganda x=1 nuqta;

3) $a<1, \quad b<1$ bo'lganda x=0 va x=1 nuqtalar maxsus nuqtalar bo'ladi.

Demak, (16.1) integral parametrga bog'liq bo'lgan xosmas integraldir.

Beta funksiya quyidagi xossalarga ega.

1- xossa. (16.1) integral

$$M=\{(a,b)\in R^2: \quad a\in(0;+\infty), \quad b\in(0;+\infty)\}$$

to'plamda yaqinlashuvchi.

2- xossa. (16.1) integral $M_0=\{(a,b)\in R^2: \quad a\in[a_0;+\infty),$ $b\in[b_0;+\infty)\}, \; a_0>0, \quad b_0>0$ to'plamda tekis yaqinlashuvchi, lekin M to'plamda esa notekis yaqinlashuvchi.

3- xossa. $B(a,b)$ funksiya M to'plamda uzluksiz funksiyadir.

4- xossa. $\forall(a,b)\in M$ uchun $B(a,b)=B(b,a)$ (simmetrik) bo'ladi.

5- xossa. $B(a,b)$ funksiya quyidagicha ham ifodalanadi:

$$B(a,b)=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

6- xossa. $\forall(a,b)\in M_1=\{(a,b)\in R^2: a\in(0;+\infty), b\in(1;+\infty)\}$ uchun

$$B(a,b)=\frac{b-1}{a+b-1}B(a,b-1).$$

7- xossa. $b=n$ bo'lganda,

$$B(a,n)=\frac{n-1}{a+n-1}\frac{n-2}{a+n-2}\cdots\frac{1}{n+1}B(a,1),\ B(a,1)=\frac{1}{a}.$$

8- xossa. $B(m,n)=\frac{(n-1)!(m-1)!}{(n+m-1)!}\quad (m,n\in N)$.

9- xossa. $B(a,1-a)=\frac{\pi}{\sin a\pi}(0<a<1)$, xususiy holda $a=\frac{1}{2}$ bo'lganda, $B=\left(\frac{1}{2},\frac{1}{2}\right)=\pi$.

16.2. Gamma funksiya va uning xossalari. Ushbu

$$\int_0^{+\infty} x^{a-1}e^{-x}dx\ (a>0) \tag{16.3}$$

xosmas integral *Gamma funksiya* yoki *2-tur Eyler integrali* deyiladi va kabi belgilanadi, ya'ni

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx\ (a>0). \tag{16.4}$$

Integral ostidagi funksiya uchun:
1) $a<1$ bo'lganda $x=0$ nuqta maxsus nuqta;
2) $a>1$ bo'lganda (16.3) integral yaqinlashuvchi;
3) $a\le 0$ bo'lganda (16.3) integral uzoqlashuvchi;
Gamma funksiya quyidagi xossalarga ega.

1- xossa. $\Gamma(a)=\lim_{n\to\infty} n^a\frac{1\cdot 2\cdot 3\cdots(n-1)}{a(a+1)\cdots(a+n+1)}$. (16.5)

(16.5) formula *Eyler – Gauss* formulasi deyiladi.

2- xossa. (16.3) integral $\forall a\in[a_0,b_0]$ $(0<a_0<b_0<+\infty)$ oraliqda tekis yaqinlashuvchi, $a\in(0;+\infty)$ da esa notekis yaqinlashuvchi.

3- xossa. $\Gamma(a)$ funksiya $(0,+\infty)$ oraliqda uzluksiz va barcha tartibdagi uzluksiz hosilalarga ega, ya'ni

$$\Gamma^{(n)}(a)=\int_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx \quad (n\in N).$$

4- xossa. $\Gamma(a+1)=a\Gamma(a)\,(a>0)$.

5- xossa. $\Gamma(n+1)=n!$.

6- xossa. $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

7- xossa. $\Gamma(a)\Gamma(1-a)=B(a,1-a)=\dfrac{\pi}{\sin a\pi}, \quad 0<a<1$.

8- xossa. $\Gamma\left(n+\dfrac{1}{2}\right)=\dfrac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad n\in N$.

9- xossa. Lejandr formulasi: $\Gamma(a)\Gamma\left(a+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a)$.

16.1- misol. Ushbu integralni Beta funksiya orqali ifodalang:

$$\int_0^a x^{\alpha-1}(a^n-x^n)^{\beta-1}dx, \quad \alpha>0,\ a>0,\ \beta>0,\ n>0.$$

Yechilishi. Berilgan integralda $x=at^{\frac{1}{n}}$ $(t>0)$ almashtirish bajarib,

$$\int_0^a x^{\alpha-1}(a^n-x^n)^{\beta-1}dx=\frac{a^{\alpha+n\beta-n}}{n}\int_0^1 t^{\frac{\beta}{n}-1}(1-t)^{\beta-1}dt$$

ni hosil qilamiz. (16.2) formulaga ko'ra

$$\int_0^a x^{\alpha-1}(a^n-x^n)^{\beta-1}dx=\frac{a^{\alpha+n\beta-n}}{n}B\left(\frac{\beta}{n},\beta\right).$$

16.2- misol. $\int_0^a x^2\sqrt{a^2-x^2}\,dx$ $(a>0)$ integralni hisoblang.

Yechilishi. Xususiy holda, 16.1- misolda $n=2$, $\beta=\dfrac{3}{2}$, $\alpha=3$ deb olsak,

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx=\frac{a^4}{2}B\left(\frac{3}{2},\frac{3}{2}\right)$$

munosabatga ega bo'lamiz. Gamma funksiyaning 5-, 6- va 8-xossalariga asosan,

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx=\frac{a^4}{2}\frac{\Gamma^2\left(\frac{3}{2}\right)}{\Gamma(3)}=\frac{\pi a^4}{16}$$

ekanligini topamiz.

16.3- misol. $I=\int_0^1\sqrt{\frac{1-x}{x}}\frac{dx}{(x+2)^2}$ integralni Eyler integrallari yordamida hisoblang.

Yechilishi. Berilgan integralda $\frac{3x}{2+x}=t$ almashtirish bajaramiz va $x=\frac{2t}{3-t}$, $\frac{dx}{(x+2)^2}=\frac{dt}{6}$ ekanligini e'tiborga olib,

$$I=\frac{1}{2\sqrt{6}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt$$

ifodani hosil qilamiz. Beta funksiyaning ta'rifi va Gamma funksiyaning 4-, 6-, 7- xossalaridan foydalanib, Γ(2)=1 ekanligini e'tiborga olgan holda,

$$I=\frac{1}{2\sqrt{6}}B\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\pi}{4\sqrt{6}}$$

bo'lishini topamiz.

16.4- misol. $\int_0^1\left(\ln\frac{1}{x}\right)^{\alpha-1}x^{\beta-1}dx,\ \alpha,\ \beta>0$ integralni hisoblang.

Yechilishi. Berilgan integralda $\ln\frac{1}{x}=t,\quad (t>0)$ almashtirish bajarib, so'ngra Gamma funksiyaning ta'rifidan foydalangan holda,

$$\int_0^1\left(\ln\frac{1}{x}\right)^{\alpha-1}x^{\beta-1}dx=\int_0^{+\infty}t^{\alpha-1}e^{-\beta t}dt=\int_0^{+\infty}\frac{y^{\alpha-1}}{\beta^{\alpha-1}}e^{-y}\frac{dy}{\beta}=\frac{1}{\beta^{\alpha}}\Gamma(a)$$

ga ega bo'lamiz.

16.5- misol. $\int\limits_0^{+\infty} \frac{x^m dx}{(a+bx^n)^p}$ $(a>0, b>0, n>0)$ integralni hisoblang.

Yechilishi. Berilgan integralni hisoblash uchun $x=\sqrt[n]{\frac{b}{a}t}$, $(t>0)$ almashtirish bajaramiz, so'ngra Beta funksiyaning 5- xossasidan foydalangan holda,

$$\int\limits_0^{+\infty} \frac{x^m dx}{(a+bx^n)^p} = \frac{\left(\frac{a^{\frac{m+1}{n}}}{b}\right)}{na^p}\int\limits_0^{+\infty} \frac{t^{\frac{m+1}{n}-1}}{(1+t)^p} dt = \left(\frac{a}{b}\right)^{\frac{m+1}{n}} \frac{1}{na^p} B\left(\frac{m+1}{n}, p-\frac{m+1}{n}\right)$$

ifodani hosil qilamiz.

Berilgan integral $0<\frac{m+1}{n}<p$ bo'lganda yaqinlashuvchi.

16.6- misol. Ushbu integralni hisoblang:

$$I=\int\limits_a^b \frac{(x-a)^m (b-x)^n}{(x+c)^{m+n+2}} dx \quad (0<a<b, c>0).$$

Yechilish. Bu integralda $\frac{x-a}{x+c}=\frac{b-a}{b+c}t$ almashtirish bajarib, uni

$$I=\frac{(b-a)^{m+n+1}}{(b+c)^{m+1}(a+c)^{n+1}}\int\limits_0^1 t^m (1-t)^n dt$$

shaklga keltiramiz. So'ngra $m>-1$, $n>-1$ deb, Beta funksiyaning ta'rifini e'tiborga olgan holda,

$$I=\frac{(b-a)^{m+n+1}}{(b+c)^{m+1}(a+c)^{n+1}} B(m+1, \quad n+1)$$

munosabatni olamiz.

16.7- misol. $I=\int\limits_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx \quad (m>-1, n>-1)$ integralni hisoblang.

Yechilishi. $\sin x=\sqrt{t}$ $(t>0)$ almashtirish bajarib, berilgan integralni ushbu ko'rinishga keltiramiz:

$$I=\frac{1}{2}\int_0^1 t^{\frac{m-1}{2}}(1-t)^{\frac{n-1}{2}}\,dt.$$

Hosil bo'lgan integralga Beta funksiyaning ta'rifini qo'llab,

$$I=\frac{1}{2}B\left(\frac{m+1}{2},\ \frac{n+1}{2}\right)$$

ifodani olamiz.

16.8- misol. $I=\int_0^{\frac{\pi}{2}}\frac{\mathrm{tg}^{\alpha}x}{(\sin x+\cos x)^2}dx,\quad 0<\alpha<1$ integralni hisoblang.

Yechilishi. Berilgan integralda $\mathrm{tg}x=t\quad (t>0)$ almashtirish bajarib,

$$I=\int_0^{\frac{\pi}{2}}\frac{\mathrm{tg}^{\alpha}x\,dx}{\cos^2x\,(1+\mathrm{tg}x)^2}=\int_0^{\infty}\frac{t^{\alpha}dt}{(1+t)^2}$$

munosabatga ega bo'lamiz.

$0<\alpha<1$ shartda berilgan integral yaqinlashuvchi. So'ngra keyingi hosil bo'lgan integralga Beta funksiyaning 5-xossasini, Gamma funksiyaning 4- va 7- xossalarini qo'llash natijasida

$$I=B(1+\alpha,\ 1-\alpha)=\alpha\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi\alpha}{\sin\alpha\pi}$$

bo'lishini topamiz.

16.9- misol. $I=\int_0^{\pi}\frac{\sin^{n-1}x\,dx}{(1+k\cos x)^n}\quad (0<|k|<1)$ integralni hisoblang.

Yechilishi. Bu integralda $t=\mathrm{tg}\frac{x}{2}$ almashtirish bajaramiz. Natijada

$$I=\frac{2^n}{(1+k)^n}\int_0^{+\infty}\frac{t^{n-1}dt}{(1+\alpha^2t^2)^n}\quad\left(\alpha=\sqrt{\frac{1-k}{1+k}}\right)$$

ifodani hosil qilamiz. Bu integralda $\alpha t=\sqrt{z}$ almashtirish bajarib, Beta funksiyaning ta'rifiga asosan,

$$I=\frac{2^{n-1}}{(1-k^2)^{\frac{n}{2}}}B\left(\frac{n}{2},\frac{n}{2}\right)\quad (n>0)$$

ifodani topamiz.

16.10- misol. $I=\int_0^1\left(\ln\frac{1}{x}\right)^p dx\quad (p>-1)$ integralni hisoblang.

Yechilishi. Bu integralda $t=\ln\frac{1}{x}$ almashtirish bajarib,

$$I=\int_0^1\left(\ln\frac{1}{x}\right)^p dx=\int_0^{+\infty}t^p e^{-t}dt=\Gamma(p+1)$$

ega bo'lamiz.

16.11- misol. $I=\int_0^{+\infty}x^p e^{-ax}\ln x\,dx\quad (a>0)$ integralni hisoblang.

Yechilishi. Berilgan integralda $t=ax$ almashtirish bajarib,

$$I=\frac{1}{a^{1+p}}\int_0^{+\infty}t^p e^{-t}\ln t\,dt-\frac{\ln a}{a^{1+p}}\int_0^{+\infty}t^p e^{-t}dt$$

munosabatni hosil qilamiz. Bunda birinchi integral $p+1>0$ argumentli Gamma funksiyaning hosilasini beradi (Gamma funksiya $p+1>0$ da istalgan tartibdagi hosilaga ega), ikkinchi integral esa $\Gamma(p+1)$ ni ifodalaydi. Shunday qilib,

$$I=\frac{\Gamma'(p+1)}{a^{p+1}}-\frac{\ln a}{a^{1+p}}\Gamma(p+1)=\frac{d}{dp}\left(\frac{\Gamma(p+1)}{a^{p+1}}\right)$$

bo'ladi.

Mustaqil yechish uchun misollar

Integrallarni Beta va Gamma funksiyalar orqali ifodalang:

16.1. $\int_0^{+\infty}x^{p-1}e^{-\alpha x}dx,\quad p>0,\quad \alpha>0$.

16.2. $\int_0^{+\infty}x^{p-1}\cos\alpha x\,dx,\quad 1>p>0$.

16.3. $\int_0^{+\infty} \frac{x^{p-\frac{3}{2}}}{(x^2+ax+b)^p}dx, \quad p>\frac{1}{2}, \quad b>0 \quad a+2\sqrt{b}>0$.

16.4. $\int_0^{+\infty} \frac{e^{-\frac{\alpha}{2x^2}}}{x^{n+1}}dx, \quad n\in N, \quad \alpha>0$.

16.5. $\int_0^{+\infty} \frac{x^{\alpha-1}dx}{1+x}, \quad 0<\alpha<1$.

16.6. $\int_{-\infty}^{+\infty} e^{-e^x}e^{px}dx, \quad p>0$. 16.7. $\int_0^{+\infty} \frac{dx}{1+x^4}$. 16.8. $\int_0^1 \frac{dx}{\sqrt[n]{1-x^\alpha}}, \quad \alpha>0$.

16.9. $\int_0^1 x^\alpha(1-x^\beta)^\nu dx, \quad \beta>0, \quad \alpha>-1, \quad \nu>-1$.

16.10. $\int_0^{+\infty} \frac{x^{\alpha-1}dx}{(1+x)^\beta}, \quad \beta>\alpha>0$. 16.11. $\int_0^1 \frac{x^{3\alpha}}{\sqrt[3]{1-x^3}}dx, \quad \alpha>-\frac{1}{3}$.

16.12. $\int_0^\pi \frac{\sin^p x}{1+\cos x}dx, \quad p>1$.

16.13. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\alpha-1}x\cos^{\beta-1}x}{(\sin x+\cos x)^{\alpha+\beta}}dx, \quad \beta, \quad \alpha>0$.

16.14. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\alpha-1}x\cos^{\alpha-1}x}{(a^2\sin^2 x+b^2\cos^2 x)^\alpha}dx, \quad 0<ab, \quad 0<\alpha$.

16.15. $\int_0^\pi \frac{\sin^{\frac{1}{2}}x}{(1+\beta\cos x)^{\frac{3}{2}}}dx, \quad 0<|\beta|<1$.

16.16. $\int_a^b \frac{(x-a)^{\frac{1}{2}}(b-x)^{\frac{1}{2}}}{(x+c)^3}dx, \quad 0<a<b, \ 0<c$.

Eyler integrallaridan foydalanib, integrallarni hisoblang:

16.17. $\int_0^2 x^2\sqrt{4-x^2}dx$. 16.18. $\int_0^{+\infty} \frac{\sqrt[4]{x}}{(1+x)^2}dx$.

16.19. $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}}, \quad n>1$.

16.20. $\int_0^2 \frac{dx}{\sqrt[3]{x^2(2-x)}}$. 16.21. $\int_{-1}^2 \frac{dx}{\sqrt[4]{(2-x)(1+x)^3}}$.

16.22. $\int_1^2 \sqrt[3]{(2-x)^2(x-1)}dx$. 16.23. $\int_1^2 \sqrt{\frac{x-1}{2-x}}\frac{dx}{(1+x)^3}$.

16.24. $\int_0^1 \frac{x\,dx}{(2-x)\sqrt[3]{x^2(1-x)}}$. 16.25. $\int_{-1}^2 \frac{\sqrt[4]{x(1-x)^3}}{(x+1)^3}dx$.

16.26. $\int_0^{+\infty} \frac{\sqrt[4]{x^3}}{(1+x)^3}dx$. 16.27. $\int_{-\infty}^{+\infty} \frac{xe^{x/2}dx}{e^{2x}+1}$.

16.28. $\int_0^1 \frac{x^{1-\alpha}(1-x)^\alpha}{(1+x)^3}dx, \; -1<\alpha<2$.

16.29. $\int_0^{+\infty} \frac{\sqrt{x}\ln x}{1+x}dx$. 16.30. $\int_0^{+\infty} \frac{x^\alpha \ln x}{1+x^2}dx, \quad |\alpha|<1$.

16.31. $\int_{-1}^1 \ln\frac{1+x}{1-x}\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}}$. 16.32. $\int_1^{+\infty} \frac{\sqrt{x-1}\ln(x-1)}{x^2+3x}dx$.

16.33. $\int_0^{+\infty} \frac{\ln^2 x dx}{x^2+a^2}, \quad a\neq 0$.

16.34. $\int_0^{+\infty} \frac{x^{\alpha-1}\ln x}{1+x}dx \quad ,0<\alpha<1$. 16.35. $\int_0^{+\infty} \frac{x^{\alpha-1}\ln^2 x}{1+x}dx$.

16.36. $\int_0^{+\infty} \frac{x^{p+1}dx}{(1+x^2)^2}, \quad |p|<2$.

16.37. $\int_0^{+\infty} \frac{\mathrm{tg}^\alpha x\,dx}{(\sin x+\cos x)^2}dx, \quad 0<\alpha<1$.

16.38. $\int\limits_0^{+\infty} \frac{x^{2\alpha-1}}{1+x^2}dx, \quad 0<\alpha<1$.

16.39. $\int\limits_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x\, dx$.

Tengliklarni isbotlang:

16.40. $B(p;q)=\int\limits_1^{+\infty} \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx, \quad p>0, \quad q>0$.

16.41. $\int\limits_0^{\frac{\pi}{2}} \mathrm{tg}^{3\alpha} x dx = \frac{1}{2}\frac{\pi}{\cos\frac{\alpha\pi}{2}}, |\alpha|<\frac{1}{3}$.

16.42. $B(p;p)=\frac{1}{2^{2p-1}}B\left(\frac{1}{2};p\right), \quad p>0$.

16.43. $B(p;q)=\int\limits_0^{1} \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx, \quad p>0, \quad q>0$.

16.44. $\int\limits_0^{+\infty} e^{-x^4}dx \int\limits_0^{+\infty} x^2 e^{-x^4}dx = \frac{\pi}{8\sqrt{2}}$.

16.45. $\int\limits_0^{1} \frac{dx}{\sqrt{1-x^4}} \int\limits_0^{1} \frac{x^2 dx}{\sqrt{1-x^4}} dx = \frac{\pi}{4}$.

16.46. $\int\limits_0^{+\infty} x^{p-1}\sin ax dx = \frac{1}{a^p}\Gamma(p)\sin\frac{\pi p}{2}, \quad |p|<1$.

16.47. $\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_a^{x} \frac{(t-a)^{-\mu}}{(x-t)^{\alpha}}dt = \frac{\Gamma(1-\mu)\cdot(x-a)^{-\mu-\alpha}}{\Gamma(1-\alpha-\mu)}$,

$0<\alpha<1, \quad 0<\mu<1$.

16.48. $\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_a^{x} \frac{(t-a)^{\alpha-1}}{(x-t)^{\alpha}}dt = 0, \quad 0<\alpha<1$.

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16.1. $\frac{\Gamma(p)}{\alpha^p}$. **16.2.** $\frac{1}{\alpha^p}\Gamma(p)\cos\frac{\pi p}{2}$.

16.3. $\frac{\sqrt{\pi}}{\sqrt{a}(a+2\sqrt{b})}\frac{\Gamma\left(p-\frac{1}{2}\right)}{\Gamma(p)}$.

16.4. $2^{\frac{n}{2}-1}a^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)$. **16.5.** $\frac{\alpha}{\sin\alpha\pi}$ **16.6.** $\Gamma(p)$. **16.7.** $\frac{\pi}{2\sqrt{2}}$.

16.8. $\frac{1}{\alpha}B\left(\frac{1}{\alpha},1-\frac{1}{n}\right)$. **16.9.** $\frac{1}{\beta}B\left(\frac{\alpha+1}{\beta},v+1\right)$.

16.10. $B(\beta-\alpha,\ \alpha)$.

16.11. $\frac{1}{3}B\left(\alpha+\frac{1}{3},\ \frac{2}{3}\right)$. **16.12.** $2^{p-1}B\left(\frac{p-1}{2},\ \frac{p+1}{2}\right)$.

16.13. $B(\alpha,\ \ \beta)$. **16.14.** $\frac{B\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)}{2(a,b)^{\alpha}}$. **16.15.** $\frac{\sqrt{2}}{(1-\beta^2)^{\frac{3}{4}}}B(\frac{3}{4},\frac{3}{4})$.

16.16. $\frac{\pi(b-a)^2}{8(a+c)^{\frac{3}{2}}(b+c)^{\frac{3}{2}}}$. **16.17.** p. **16.18.** $\frac{\pi}{2\sqrt{2}}$. **16.19.** $\frac{\pi}{n\sin\frac{\pi}{n}}$.

16.20. $\frac{2\pi}{\sqrt{3}}$. **16.21.** $\pi\sqrt{2}$. **16.22.** $\frac{2\pi}{9\sqrt{3}}$. **16.23.** $\pi\frac{\sqrt{5}}{100}$.

16.24. $\frac{2\pi}{\sqrt{3}}\left(\sqrt[3]{2}-1\right)$. **16.25.** $3\pi 2^{-\frac{23}{4}}$. **16.26.** $B(\frac{7}{4},\ \frac{5}{4})$.

16.27. $-\pi^2\frac{\sqrt{2}}{4}$. **16.28.** $2^{\alpha-3}\pi\alpha\frac{1-\alpha}{\sin\alpha\pi}$. **16.29.** 0.

16.30. $\pi^2\frac{\sin\frac{\alpha\pi}{2}}{4\cos^2\frac{\alpha\pi}{2}}$. **16.31.** $2\frac{\pi^2}{3}$. **16.32.** $\frac{4\pi\ln 2}{3}$.

16.33. $\frac{\pi}{2|a|}\left(\ln^2|a|+\frac{\pi^2}{4}\right)$. **16.34.** $-\frac{\pi^2\cos\alpha\pi}{\sin^2\alpha\pi}$.

16.35. $\pi^2\frac{(1+\cos^2\alpha\pi)}{\sin^3\alpha\pi}$. **16.36.** $\pi\frac{(1-\alpha)}{\sin\alpha\pi}$. **16.37.** $\pi\frac{\alpha}{\sin\alpha\pi}$.

16.38. $\frac{\pi}{2\sin\alpha\pi}$. **16.39.** $\frac{3\pi}{512}$.

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MUNDARIJA

III BOB

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