Foreword

This workbook is designed for high-school graduates who intend to enter an international higher education establishment. It provides the reader with a number of typical mathematics and logic questions that give an opportunity to refresh the reader's knowledge and prepares for university entrance examinations.

The workbook is intended as a resource for teachers of various learning centers, special courses that prepare students taking university entrance tests and independent learners. However, it is assumed that the reader is familiar with school level mathematics and wants to familiarize himself with the prevailing terminology and the various ways of expressing a solution in English.

The main content of the workbook has been developed over several years by the enthusiastic teacher and educator Farrukh Ataev at the Westminster International University in Tashkent. The effectiveness and reliability of the workbook may be seen by the great number of people who have successfully entered various universities and institutes over the last years, after using the material herein to refresh their knowledge.

The workbook consists of eleven chapters, each of which contains several topics. The structure of the workbook follows a logical sequence of mathematics topics, each of which can be referred to as a self contained seminar (practical session). The objectives of each topic are followed by the relevant terms, notation and formulae before the most typical problems in the area are introduced. In addition, the reader is provided with a set of homework problems, which serve to reinforce the gained knowledge and skills.

At the end of each topic, interesting and useful reference information from various mathematical fields is given, such as the history of mathematics, reference information on international standards and agreements, some classic problems and paradoxes, and many more.

Content

Chapter XI. Logic

Mathematics is the queen of the sciences and number theory the queen of mathematics.
Carl Friedrich Gauss (1777-1855), German mathematician

Like the crest of a peacock, so is mathematics at the head of all knowledge.
Indian Proverb

Chapter I. Numbers

I.1. Basic operations with numbers. HCD and LCM

Terms

1. **to add (subtract, multiply, divide)** – qo'shmoq (ayirmoq, ko'paytirmoq, bo'lmoq)|складывать (вычитать, умножать, делить);
2. **to compute, to calculate, to work out** – hisoblamoq|вычислять;
3. **a natural (whole, integer, rational, irrational, mixed, negative, positive, prime, composite) number** – natural (nomanfiy butun, butun, ratsional, irratsional, aralash, manfiy, musbat, tub, murakkab) son | натуральное (неотрицательное целое, целое, рациональное, иррациональное, смешанное, отрицательное, положительное, простое, составное) число;
4. **an ordinary (proper, improper, decimal) fraction** – oddiy (to'g'ri, noto'g'ri, o'nli) kasr|обыкновенная (правильная, неправильная, десятичная) дробь;
5. **factor** – ko'paytuvchi|множитель;

6. **prime factorization** – tub ko'paytuvchilarga ajratmoq|разложение на простые множители;
7. **the highest (greatest) common divisor (HCD)** – eng katta umumiy bo'luvchi (EKUB)|наибольший общий делитель (НОД);
8. **the lowest (least) common multiple (LCM)** – eng kichik umumiy ko'paytuvchi (EKUK)|наименьшее общее кратное (НОК);
9. **numerator (denominator)** – surat (mahraj)|числитель (знаменатель);
10. **numerical expression** – sonli ifoda|числовое выражение;
11. **remainder** – qoldiq|остаток;
12. **to simplify** – soddalashtirmoq|упрощать;
13. **product** – ko'paytma|произведение.

Learning Objectives:

– to have an understanding of arithmetic operations on numbers such as addition, subtraction, multiplication and division;

– to know how to compute expressions using the concepts of HCD and LCM.

Natural numbers

1, 2, 3, 4, 5, … (Used for counting)

Whole numbers

0, 1, 2, 3, 4, ...

Integers

..., – 2, – 1, 0, 1, 2, ...

Order of operations

BEDMAS – brackets, exponents, division, multiplication, addition, subtraction.

HCD (25, 35) = 5 (5 is the largest number that divides both numbers without remainder).

LCM (25, 35) = 175 (175 is the smallest number that can be divided by both numbers without remainder).

Number notation

1.4 and 1,023,145 (in American and British system) are written as 1,4 and 1.023.145 (in Uzbek, German and Russian system).

Also, 1.4 is pronounced as “one point four”.

Number Divisibility

For a number to be divisible by

2: the number must end with 0, 2, 4, 6 or 8.

3: the sum of digits of the number must be divisible by 3.

4: the last two digits must be divisible by 4.

5. the number must end with 0 or 5.

6: the number must be divisible by both 2 and 3.

8: the last three digits must be divisible by 8.

9: the sum of digits must be divisible by 9.

10: the number must end with 0.

11: the difference between the sums of the digits at odd and even places of the number is divisible by 11.

12: the number must be divisible by both 3 and 4.

Prime and composite numbers

2, 3, 5, 7, 11, 13, 17, 19, 23, ... are prime (divisible by 1 and itself).

4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, ... are composite (divisible by a number other than 1 and itself).

Expressions involving fractions

1) To add (or subtract) two fractions, we first need to bring them to a common denominator.

$$\frac{^{d/}a}{b}+\frac{^{b/}c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

2) To multiply two fractions, we multiply the numerator by the numerator and the denominator by denominator (if possible, cancel the numerator and the denominator).

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

3) To divide one fraction by another, we multiply the top fraction with the inverse of the bottom fraction.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Examples

1. What numbers between 1 and 12 divide 3,473,892?

Solution:

<u>The number is divisible by:</u>

a) 2, because the number ends with 2.

b) 3, because the sum of its digits, 36, is divisible by 3.

c) 4, because the number made up of the last two digits, 92, is divisible by 4.

d) 6, because the number is divisible by both 2 and 3.

e) 9, because the sum of its digits, 36, is divisible by 9.

f) 12, because the number is divisible by both 3 and 4.

<u>The number is not divisible by:</u>

g) 5, because the number does not end with 0 or 5.

h) 8, because the number made up of the last 3 digits, 892, is not divisible by 8.

i) 11, because the difference, 4, between the sums of digits at odd (20) and even (16) places is not divisible by 11. ■

2. Do prime factorization of the number 462,000.

Solution: Doing prime factorization means representing the given number as a product of prime numbers. To achieve this we divide the number continuously by prime numbers starting from the smallest possible on.

462,000	**2**
231,000	**2**
115,500	**2**
57,750	**2**
28,875	**3**
9,625	**5**
1,925	**5**
385	**5**
77	**7**
11	**11**
1	

Thus, $462{,}000 = 2^4 \cdot 3 \cdot 5^3 \cdot 7 \cdot 11$. ■

3. Evaluate $338 - 34 \cdot (45:9 + 2 \cdot 4^2 - 17):10 + 34$.

Solution: To calculate the value of the numerical expression above we follow the order of operations (BEDMAS) rule, once for the bracket and then for the whole expression:

1) Brackets. $45:9 + 2 \cdot 4^2 - 17$.

   a) Exponent. $4^2 = 16$.

   b) Division. $45:9 = 5$.

   c) Multiplication. $2 \cdot 16 = 32$.

   d) Addition. $5 + 32 = 37$.

   e) Subtraction. $37 - 17 = 20$.

2) Division. $20:10 = 2$.

3) Multiplication. $34 \cdot 2 = 68$.

4) Addition. $-68 + 34 = -34$.

5) Subtraction. $338 - 34 = 304$. ■

4. Find HCD and LCM of the numbers 60 and 270.

Solution: To find HCD and LCM of two or more numbers one should do prime factorization of each number. $60 = 2^2 \cdot 3 \cdot 5$. $270 = 2 \cdot 3^3 \cdot 5$.

HCD of the numbers will be the product of common prir factors in smallest powers.

Hence, HCD (60 and 270) = $2\cdot3\cdot5=30$.

LCM of the numbers will be the product of common prir factors in largest powers.

Hence, LCM (60 and 270) = $2^2\cdot3^3\cdot5=540$. ■

5. Evaluate the expression $\left(\frac{1}{6}+\frac{2}{5}\cdot\frac{25}{16}\right)-\frac{4}{35}:\frac{2}{15}$.

Solution:

a) $\frac{2}{5}\cdot\frac{25}{16}=\frac{2\cdot25}{5\cdot16}=\frac{50}{80}=\frac{50:10}{80:10}=\frac{5}{8}$;

b) $\frac{1}{6}+\frac{5}{8}=\frac{^{4/}1}{6}+\frac{^{3/}5}{8}=\frac{4}{24}+\frac{15}{24}=\frac{4+15}{24}=\frac{19}{24}$;

c) $\frac{4}{35}:\frac{2}{15}=\frac{^{2}\cancel{4}}{_{7}\cancel{35}}\cdot\frac{\cancel{15}^{3}}{\cancel{2}_{1}}=\frac{2\cdot3}{7\cdot1}=\frac{6}{7}$;

d) $\frac{19}{24}-\frac{6}{7}=\frac{^{7/}19}{24}-\frac{^{24/}6}{7}=\frac{133-144}{168}=-\frac{11}{168}$. ■

Exercises

1. Evaluate

a) 53 + 81; b) 124 + 529; c) 79 + 253 + 1395;

d) 245 – 123; e) 1,249 – 398; f) 45 + (–34);

g) 13 + (–28) + 55 – 13; h) add two thousand nine hundred and five and thousand and eighteen.

2. Do the following multiplications

a) $8 \cdot 6$; b) $12 \cdot 7$; c) $27 \cdot 13$;

d) $231 \cdot 92$; e) $345 \cdot 103$.

3. Do the following divisions

a) 45:9; b) 124:4; c) 4239:9;

d) 345:15; e) 21,252:92.

4. Malika had 10,000 soums to spend at a store. She bought toothpaste for 1,200 soums, a toothbrush for 800 soums and two soaps for 600 soums each. How much money did she spend? How much money did she have when she came out of the store?

5. Evaluate (using the order of operations)

a) $52+3\cdot7-43$; b) $43-84:21\cdot2+(-17)$;

c) $9+18:2-2\cdot(13-3\cdot5+7)+24:(2+2\cdot3)$;

d) $12+3\cdot[37-4\cdot(7-3)]+56:(13-3\cdot5)$.

6. Tell if the numbers are divisible

a) 46 by 2; b) 45 by 3; c) 224 by 4;

d) 120 by 5; e) 246 by 6; f) 1,344 by 8;

g) 130 by 10; h) 124,440 by 2, 3, 4, 5, 6, 8, 9 and 10.

7. Identify prime numbers: 2, 4, 10, 13, 25, 19, 111, 39, 25, 41, 37.

8. Do prime factorization for the numbers: 100 and 180.

9. Write all divisors of 28 and 36. Point out the greatest common divisor.

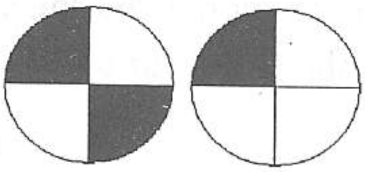
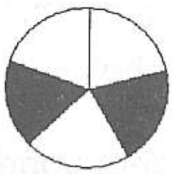
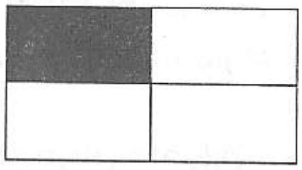
10. Write all two-digit numbers that are multiples to 12 and 15. Show the lowest common multiple.

11. Find

a) HCD (28 and 36); b) HCD (12, 18 and 30);

c) HCD (15, 10 and 6); d) HCD (252, 441 and 1,080).

12. Indicate the shaded part of the figure by a fraction:

a)  b)  c) 

13. Simplify a) $\frac{18}{30}$; b) $\frac{108}{144}$.

14. Find

a) LCM (12 and 15); b) LCM (6 and 8);

c) LCM (252, 441 and 1,080).

15. Compare the following numbers

a) $\frac{3}{8}$ and $\frac{7}{12}$;   b) $\frac{3}{8}$, $\frac{5}{6}$ and $\frac{2}{4}$.

16. Calculate

a) $7\frac{3}{4}+4\frac{5}{6}$;   b) $11-10\frac{5}{7}$;

c) $4\frac{1}{2}\cdot\frac{4}{7}\cdot4\frac{2}{3}$;   d) $2\frac{1}{12}\cdot1\frac{7}{20}$.

17. Hasan's step is $63cm$ long whereas his twin brother Husan's step is $56cm$. If they start walking from the same place in the same direction, at what distance will their steps first overlap?

18. Place the following numbers in ascending order

$\frac{1}{39}$, $\frac{1}{13}$ and $\frac{4}{169}$.

19. Compute $\left(1\frac{1}{7}:\frac{5}{14}+3\frac{1}{4}\cdot2\frac{8}{13}-5\frac{1}{2}\right)\cdot5$.

20. Write the number 1,000,000 as the product of two positive integers neither of which has any zeros in it.

21. If $1\cdot2\cdot3\cdots199\cdot200$ is calculated, then how many zeros will be at the end of the product?

22. To pin up a rectangular picture Davron needs 4 pins, one at each corner. For two pictures he needs only 6 pins since he can overlap pictures. What is the smallest number of pins he needs to pin up 12 pictures?

23. If $A=\dfrac{4\frac{1}{3}-1\frac{1}{12}}{6\frac{2}{3}+4\frac{7}{12}}-\dfrac{2\frac{1}{10}-1\frac{47}{50}}{\frac{93}{100}+\frac{87}{100}}$ and $B=\dfrac{10}{1+\dfrac{8}{3+\frac{7}{3}}}$, calculate the product AB.

Homework

1. Find the following sums

a) 367 + 528; b) 1,402 – 973;

c) 37 + 27.4 + (–39); d) 13,528 + 349,711.

2. Do the multiplications

a) $34\cdot5$; b) $132\cdot97$; c) $37\cdot110$.

3. Do the divisions

a) 96:4; b) 294:14; c) 15,291:3.

4. What numbers must fit into the blank places to make these correct?

a) 342 + _ = 831; b) 459 – _ = 199;

c) 35 x _ = 385; d) 2,244 : _ = 17.

5. Qodir thinks of a number, doubles it and adds 7. The answer was 19. What is the number he thought of?

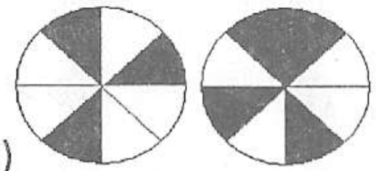
6. A teacher asked how much would be two plus three multiplied by eight. Anvar answered 40, but Dildora replied 26. Figure out who is correct.

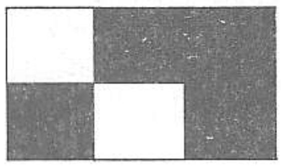
7. In the army there are 3,450 men who are taller than 180*cm*. Of these 882 are taller than 190*cm*. How many men are there who are between 180*cm* and 190*cm* tall?

8. Perform a prime factorization of the numbers: 6,720 and 3,564.

9. Find HCD and LCM of 918 and 2,448.

10. Indicate the shaded part of the figure with a fraction:

a) 

b) 

11. Which of the following fractions lies between $\frac{1}{2}$ and $\frac{2}{3}$?

a) 17/24; b) 1/3; c) 7/12; d) 3/4.

12. Calculate

a) $2\frac{5}{6}+5\frac{3}{4}$; b) $2\frac{1}{2}-1\frac{2}{3}$;

c) $5\frac{2}{3}\cdot\frac{21}{34}$; d) $\frac{15}{21}:\frac{4}{3}:\frac{5}{7}$.

13. Compute $\left(\frac{3\frac{1}{3}+\frac{5}{2}}{2.5-1\frac{1}{3}}\cdot\frac{4\frac{3}{5}-2\frac{1}{3}}{4\frac{3}{5}+2\frac{1}{3}}\right)\cdot 5\frac{1}{5}:\left(\frac{\frac{1}{20}}{\frac{1}{7}-\frac{1}{8}}+5\frac{7}{10}\right)$.

14. Alisher's class is going on a trip to the local art museum. Twenty five students, one teacher, and three parents are going on the trip. The museum charges 400 soums admission for each student and 600 soums for each adult. How much will it cost for all the students and the adults to enter the museum?

According to A. N. Kolmogorov, the periods of development of mathematics are

1. The period of emergence of mathematics

   (from the time of primitive society to 6^{th}-5^{th} centuries BC);

2. The period of elementary mathematics

   (from 6^{th}-5^{th} centuries BC to 16^{th} century AD);

3. The period of mathematics of variables

   (from 17^{th} century to the first half of 19^{th} century AD);

4. The period of contemporary mathematics

   (from the second half of 19^{th} century AD to present time).

Arithmetic is being able to count up to twenty
without taking off your shoes.
Mickey Mouse, famous cartoon character created in 1928 by Walt Disney

I.2. Rational and irrational numbers

Terms

1. **periodic (repeating) decimal fraction** – o'nli davriy kasr|периодическая десятичная дробь;
2. **decimal point (place)** – o'nli nuqta|десятичная точка.
3. **to evaluate, to estimate** – baho bermoq|оценивать;
4. **to convert** – almashtirmoq|переводить;
5. **to solve** – yechmoq|решать;
6. **a solution** – yechim|решение;
7. **to express** – ifodalamoq|представлять;
8. **nonzero** – noldan farqli|отлично от нуля;
9. **to round** – yaxlitlamoq|округлять;
10. **significant figure** – muhim raqam|значащая цифра;
11. **scientific notation** – ilmiy yozuv (standart shakl)|научная запись (стандартная форма).

Learning Objectives

- to recognize a decimal (periodic decimal) fraction and an irrational number;
- to know how to compute expressions containing these numbers;
- to be able to round numbers to certain decimal places and significant figures;
- to express numbers in scientific notation.

Decimal numbers

7.305; – 14.2009; 0.615; etc

Periodic (repeating, recurring) decimal numbers

$2.343434... = 2.\dot{3}\dot{4}$; $5.233333... = 5.2\dot{3}$; $7.103103103... = 7.\dot{1}0\dot{3}$; etc

Rational numbers

$\frac{2}{3}$; 5; – 4; 0; 0.25; etc (They can be expressed as a fraction a/b, where a and b are integers with $b \neq 0$).

Irrational numbers

$\sqrt{2} = 1.414213...$ and $\pi = 3.141592...$ (They can not be expressed as a fraction).

$1.\dot{3}$ and $1.\dot{1}4\dot{3}$ (in American and British system) are written as 1,(3) and 1,(143) (in Uzbek and Russian system).

Also, $1.\dot{3}$ is pronounced as "one point three recurring".

Evaluation of the expressions involving decimals (fractions)

1) To add (subtract) two decimals, we add (subtract) these numbers in a table aligning the decimal points.

   For example,

$$\begin{array}{r} 3.72 \\ -\;1.31 \\ \hline 2.41 \end{array} \qquad\qquad \begin{array}{r} 4.524 \\ +\;2.78 \\ \hline 7.304 \end{array}$$

2) To divide two decimals, we multiply each number by a suitable decimal number to get rid of the decimal points and then divide. For example,

$$\frac{3.56}{0.4}=\frac{3.56\cdot 100}{0.4\cdot 100}=\frac{356}{4}=89 \qquad \text{or}$$

$$\frac{824.61}{0.003}=\frac{824.61\cdot 1000}{0.003\cdot 1000}=\frac{824{,}610}{3}=274{,}870$$

3) To multiply two decimals, we multiply the numbers and put the decimal point as far from the last digit as the total number of digits after the two numbers being multiplied.

For example,

$$\begin{array}{r} 3.72 \\ \times \quad 1.3 \\ \hline 1116 \\ + \quad 372 \\ \hline 4.836 \end{array} \qquad \begin{array}{r} 4.5 \\ \times \quad 2.7 \\ \hline 315 \\ + \quad 90 \\ \hline 12.15 \end{array}$$

Conversion of a periodic decimal fraction to an ordinary fraction

1) The number is written as a mixed number with whole and fractional parts.

2) At the numerator of the fraction write the whole decimal part without brackets minus the number in between the decimal point and the brackets.

3) At the denominator of the fraction write as many digits 9 as the number of digits in the period followed by as many digits 0 as the number of digits between the decimal point and the brackets.

For example,

$$3.2\dot{7}=3\frac{27-2}{90}=3\frac{\cancel{25}^{\backslash 5}}{\cancel{90}^{\backslash 18}}=3\frac{5}{18}.$$

Rounding of a number to a particular number of decimal places

1) Round the number down if the first digit to be excluded is between 0 and 4;

2) Round the number up if the first digit to be excluded is between 5 and 9.

For example,

1) 3.625 expressed to 2 decimal places rounds up to 3.63.

2) 3.62499 expressed to 2 decimal places rounds down to 3.62.

Rounding a number to a particular number of significant figures.

1) Round the number down if the first digit to be excluded is between 0 and 4;

2) Round the number up if the first digit to be excluded is between 5 and 9.

For example,

1) 6,248.500052 rounded to 2, 3, 7 and 9 significant figures will be 6,200, 6,250, 6,248.5 and 6,248.50005, respectively.

2) 0.13546 rounded to 2, 3 and 4 significant figures will be 0.14, 0.135 and 0.1355, respectively.

Scientific notation

A number written in scientific notation has the form $a\cdot10^n$, where $1 \le a < 10$ and n is a whole number.

For example,

1) $952{,}345{,}677{,}626 = 9.52\cdot10^{11}$.

2) $0.000{,}000{,}000{,}000{,}000{,}000{,}014 = 1.4\cdot10^{-20}$.

Notation of number sets

N is a set of natural numbers;

Z is a set of integer numbers;

Q is a set of rational numbers;

I is a set of irrational numbers;

R is a set of real numbers.

$N \in Z \in Q$ *and* $Q \cup I \in R$.

Examples

1. Evaluate the expression $(57.6:0.24+2.3)\cdot2.8-3.6$.

▶ a) $57.6:0.24=(57.6\cdot100):(0.24\cdot100)=5760:24=240$;

b) $240 + 2.3 = 242.3$;

c) $242.3 \cdot 2.8 = 678.44$;

d) $678.44 - 3.6 = 674.84$. ■

2. Convert the following periodic decimal fraction $5.24\dot{3}$ to an ordinary fraction.

► $5.24\dot{3} = 5\frac{243-24}{900} = 5\frac{219}{900} = 5\frac{73}{300}$. ■

3. Show two rational numbers and two irrational numbers between 4 and 5.

► 4.5 and 4.9 are the rational numbers, because they can be expressed as m/n (e.g. 9/2 and 49/10), where m and n are whole numbers and n is nonzero.

4.21251... and $\sqrt{20}$ are the irrational numbers, because they are the non-periodic decimal numbers. ■

4. In 2005 the world population was 6,446,131,400 people and Uzbekistan's population 26,851,195 people. Express these numbers in scientific notation.

► $6{,}446{,}131{,}400 = 6.45 \cdot 10^9$ and $26{,}851{,}195 = 2.69 \cdot 10^7$. ■

5. Round the following number 398,764 to the nearest 100, 1,000 and 100,000.

► 100s digit is 7, therefore the digits after 7 should be eliminated. However, we can not drop the two digits 6 and 4 out, but we have to make them 0s. Thus, 398,764 ≈ 398,800. Similarly,

398,764 ≈ 399,000 and 398,764 ≈ 400,000. ■

6. Write the number 67.469 to 2 and 3 significant figures.

▶ The 2[nd] and 3[rd] significant numbers are 7 and 4, respectively. Thus, 67.469 ≈ 67 and 67.469 ≈ 67.5. ■

Exercises

1. Compute

   a) $2.3 + 0.4 + 14.98$; b) $2.081 \cdot 0.03$;

   c) 13.28:32; d) 13.8:0.25;

   e) $2.3902 \cdot 1000$; f) $(1.4 - 0.29) - 7.3$.

2. A ball was dropped from a 6m height. After each bounce it goes up two thirds of the previous height. How high will the ball rise after the third bounce?

3. If a and b are two numbers with $-3 \le a \le 4$, $\frac{1}{2} \le b \le 3$, find the largest and the least values of $\frac{a}{b}$.

4. Write the following as a periodic decimal fraction:

   $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{2}{15}$.

5. Write the following numbers as ordinary fractions

$0.\dot{3}$; $3.0\dot{3}$; $4.\dot{1}\dot{1}$; $2.8\dot{3}$; $13.81\dot{6}4\dot{5}$.

6. Show that 4.2626... is a rational number.

7. Compute $\dfrac{0.\dot{4}+0.\dot{4}\dot{1}+0.\dot{4}\dot{2}+0.\dot{4}\dot{3}}{0.\dot{5}+0.\dot{5}\dot{1}+0.\dot{5}\dot{2}+0.\dot{5}\dot{3}}$.

8. Find the value of the expression $\dfrac{0.8\dot{3}-0.4\dot{6}}{0.\dot{3}}$.

9. Evaluate $1-\dfrac{1-\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}}}{1+\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}}$.

10. Which numbers are rational (irrational)

$\left(\sqrt{3}\right)^1$, $\left(\sqrt{3}\right)^2$, $\left(\sqrt{3}\right)^3$, $\left(\sqrt{3}\right)^4$?

11. Point out five rational and five irrational numbers among

$\sqrt{2}\cdot\sqrt{8}$, $\left(\sqrt{5}\right)^6$, $\dfrac{\sqrt{3}}{\sqrt{2}}$, $\left(\sqrt{7}\right)^3$, $40-2^{-1}-4^{-2}$,

$\sqrt{5}-2.1$, 0.4, 6π, $\sqrt{49}$, $\sqrt{6}+6$.

12. Identify the rational (irrational) numbers

$$m = \sqrt[4]{256}\,; \qquad n = \pi\,; \qquad p = \sqrt{\sqrt{\sqrt{81}} + 13}\,; \qquad q = \frac{1}{\sqrt{2}}.$$

13. Are these expressions rational or irrational

a) $(1+\sqrt{5})(1+\sqrt{5})$; b) $\dfrac{1+\sqrt{5}}{1-\sqrt{5}}$?

14. The fraction $\dfrac{53}{17}$ can be expressed as $3+\dfrac{1}{x+\dfrac{1}{y}}$. If x and y are integers, what is the value of $x+y$?

15. Round the number 13.2436735 to

a) 2; b) 3; c) 4; d) 5 decimal places.

16. Write the numbers 125.381 and 16,527 to

a) 2; b) 3 significant figures.

17. Express the following numbers in scientific notation:

a) the mass of the Earth is

5,980,000,000,000,000,000,000,000kg.

b) the diameter of water molecule is

0.000,000,03cm.

Homework

1. Compute a) $8.05 \cdot 111.11$; b) 4.29:0.066.

2. Write $\frac{13}{225}$ as a periodic decimal fraction.

3. Calculate a) $\frac{0.0\dot{6}+0.0\dot{3}}{0.1}$; b) $\frac{\frac{2}{9}+3.6\dot{1}}{1.91\dot{6}-1\frac{5}{6}}$.

4. Show one rational and one irrational number between $\sqrt{2}$ and π.

5. Round the number 45.1949645 to

a) 2; b) 3; c) 4; d) 5 decimal places.

6. Write the numbers 25.543 and 8,327 to

a) 2; b) 3 significant figures.

7. Express the following numbers in scientific notation:

a) the diameter of the Sun is 1,390,600,000m;

b) the light speed is 29,979,300,000,000m/s.

8. If $A = 1000 - 999 + 998 - 997 + \ldots + 4 - 3 + 2 - 1$ and

$B = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{1000}\right)$, calculate the value of the product AB.

Historic Number systems (Scales of notation)

1. **Non-positional** The "main numbers" are chosen and denoted by special symbols. For instance, in Roman number system, 1, 5, 10, 50, 100, 500 and 1,000 are chosen as main numbers and denoted as I, V, X, L, C, D and M, respectively. Then, III is 3, VII is 7, IX is 9 and XL is 40.

2. **Alphabetic** The numbers are denoted by alphabet letters that follow that alphabet's sequence. For instance, in ancient Greece (Ionic scale of notation), the numbers are labeled as follows: α=1, β=2, γ=3, ...

3. **Positional** In Babylon the sexagesimal (base-60) scale of notation was used. The Maya used a base-20 number system (it probably descended from the early times when people counted on both fingers and toes). In Roman notation the numbers I and V have indicated 1 and 5 regardless of their positions in the number, whereas in Babylon and Maya scales of notation the significance of the digits also depended on their positions in the writing. Such numerical notations, in particular the decimal notation created in the 9th century in India, are called positional scales of notation.

	1	2	3	4	5	6	7	8	9	10	50	100	500	1000
Roman	I	II	III	IV	V	VI	VII	VIII	IX	X	L	C	D	M
Arabic	١	٢	٣	٤	٥	٦	٧	٨	٩	١٠	٥٠	١٠٠	٥٠٠	١٠٠٠
Hindi	१	२	३	४	५	६	७	८	९	१०	५०	१००	५००	१०००
Egyptian	|	||	|||	||||	|||||	|||| ||	|||| |||	|||| ||||	||||| ||||	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

When I was a boy of fourteen my father was so ignorant
I could hardly stand to have the old man around.
But when I got to be twenty-one,
I was astonished at how much he had learned in seven years.
Mark Twain (1835-1910), American writer and humorist

I.3. Natural and whole exponents

Terms

1. **a natural and a whole exponent (power, degree) of number** – sonning natural va butun darajasi | натуральная и целая степень числа;
2. **an even (odd) number** – juft (toq) son | четное (нечетное) число;
3. **positive (negative) number** – musbat (manfiy) son | положительное (отрицательное) число;
4. **digit** – raqam | цифра;
5. **sum** – yig'indi | сумма;
6. **product** – ko'paytma | произведение;
7. **square** – kvadrat | квадрат;
8. **to expand brackets** – qavslarni ochmoq | раскрывать скобки;
9. **unit** – birlik | единица;
10. **addend (subtrahend)** – qo'shiluvchi (ayiruvchi) | слагаемое (вычитаемое).

Learning Objectives

- to recognize the natural and whole exponents of numbers;
- to evaluate and work out the expressions containing the natural and whole exponents of numbers.

1) $\underbrace{a\cdot a\cdots a}_{n}=a^n$	2) $a^0=1$	3) $a^1=a$
4) $a^{-n}=\dfrac{1}{a^n}$	5) $a^m\cdot a^n=a^{m+n}$	6) $a^m:a^n=a^{m-n}$
7) $(ab)^n=a^nb^n$	8) $\left(\dfrac{a}{b}\right)^n=\dfrac{a^n}{b^n}$	9) $(a^m)^n=a^{mn}$

Examples

1. Compute $\left(4\dfrac{1}{2}\cdot3\right)^2\cdot6^{-2}-\left(\dfrac{2}{3}\right)^{-3}-\left(\dfrac{1}{7}\right)^0.$

▶ We use the rules of exponents.

a) $\left(4\dfrac{1}{2}\cdot3\right)^2\cdot6^{-2}=\left(\dfrac{9}{2}\cdot3\right)^2\cdot\left(\dfrac{1}{6}\right)^2=\left(\dfrac{9}{2}\cdot\dfrac{\overset{1}{\cancel{3}}}{1}\cdot\dfrac{1}{\underset{2}{\cancel{6}}}\right)^2=\left(\dfrac{9}{4}\right)^2=\dfrac{81}{16}.$

b) $\left(\dfrac{2}{3}\right)^{-3}=\left(\dfrac{3}{2}\right)^3=\dfrac{3^3}{2^3}=\dfrac{27}{8}.$

c) $\left(\frac{1}{7}\right)^0=1$.

Now we put these values back to the original expression.

d) $\frac{81}{16}-\frac{^{2/}27}{8}-{}^{16/}1=\frac{81-54-16}{16}=\frac{11}{16}$.

2. What is the unit digit of 2^{1991}?

▶ To find the unit digit, we consider its small exponents and notice that after every four exponents the unit digit repeats itself.

$2^1=2$; $2^2=4$; $2^3=8$; $2^4=16$; $2^5=32$;

$2^6=64$; $2^7=128$; $2^8=256$; $2^9=512$; $2^{10}=1024$; ...

Consequently, 1991:4 = 497 and 3 remainder, which means the third digit after 497 periods of 4 is 8. ■

3. Compare the following two numbers: 12^{50} and 48^{25}.

▶ We should prime factorize the bases as follows:

$(2^2\cdot 3)^{50}=2^{100}\cdot 3^{50}$ and $(2^4\cdot 3)^{25}=2^{100}\cdot 3^{25}$.

Obviously, the first number is greater as $3^{50}>3^{25}$. ■

Exercises

1. Compute

a) 6^3; b) $(-2)^5$; c) $3^2 \cdot 3^3 : 3^4$;

d) $2^8 \cdot 4^2 : 16^2$; e) $5^{-3} \cdot (0.6)^{-2}$; f) $(0.4)^3 \cdot (2.5)^5$;

g) $(-3)^3 \cdot \left(-\frac{1}{3}\right)$.

2. Find a half of the sum $4^{12} + 4^{12} + 4^{12} + 4^{12}$.

3. Simplify

a) $(3 \cdot 2^5)^2$; b) $((-2) \cdot 3^2 \cdot 5^2)^2$;

c) $((-0.125) \cdot 4^3 \cdot 9^2 \cdot 11^4)^3$.

4. What is the sum of the digits of $25^3 \cdot 2^6 - 2$ when expressed as a single number?

5. If the side of a square is extended two (three) times, how many times will its area increase?

6. Which number is greater

a) 24^{50} or 12^{100}; b) 100^{20} or 8100^{10};

c) 3^{-4} or $(-4)^3$; d) 2^{30} or 3^{20}?

7. Which one of the following has the largest value?

$5 \cdot 5^5$; 5^{5^5}; $(5^5)^5$; 5^{55}; $(5 \cdot 5)^5$.

8. The mass of the Earth is $6 \cdot 10^{24}\,kg$, the mass of the Sun is $2 \cdot 10^{30}\,kg$. How many times is the mass of the Sun greater than the mass of the Earth?

9. What digit does the number end with:

1) 2^{2003};   2) 3^{2000};   3) 19^{19}?

10. Calculate

a) $\dfrac{2\cdot 5^{22}-9\cdot 5^{21}}{25^{10}}$;   b) $\dfrac{\left(4\cdot 3^{22}+7\cdot 3^{21}\right)\cdot 57}{\left(19\cdot 27^{4}\right)^{2}}$.

11. The table shows the population and area, in square kilometers, of five Central Asian countries.

Country	Population	Area
Kazakhstan	$1.5\text{x}10^{7}$	$2.7\text{x}10^{6}$
Kyrgyzstan	$5.1\text{x}10^{6}$	$0.2\text{x}10^{6}$
Tajikistan	$7.2\text{x}10^{6}$	$0.1\text{x}10^{6}$
Turkmenistan	$5.0\text{x}10^{6}$	$0.5\text{x}10^{6}$
Uzbekistan	$2.7\text{x}10^{7}$	$0.4\text{x}10^{6}$

Find the combined population of Kazakhstan, Kyrgyzstan, Turkmenistan and Uzbekistan and give your answer in the standard form (or scientific notation), i.e. $a\cdot 10^{k}$, where $1\le a<10$ and $k\in Z$.

12. If $a = 1$, $b = 2$, $c = 3$ and $d = 4$, find the value of

$$\frac{a^{b}+b^{c}+c^{d}}{b^{a}+c^{b}+d^{c}+(a+b)(b+c)}+3\left(a^{a}+b^{b}+c^{c}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Homework

1. Compute

a) $\left(\frac{1}{9}\right)^3 \cdot (4.5)^3$;

b) $\frac{2^9 \cdot \left(2^6\right)^2}{2^5 \cdot 2^8}$;

c) $\frac{2^6 \cdot 3^5}{6^4}$;

d) $\frac{2^3 \cdot 8^3 \cdot 3^6}{12^5}$;

e) $\frac{5\left(5 \cdot 7^{15} - 19 \cdot 7^{14}\right)}{7^{16} + 3 \cdot 7^{15}}$;

f) $\frac{2^{19} \cdot 27^3 + 15 \cdot 4^9 \cdot 9^4}{6^9 \cdot 2^{10} + 12^{10}}$.

2. Find the last digit of the number

a) $6 \cdot 16^{10}$;

b) 311^5.

3. Let $x = 5.2 \cdot 10^3$ and $y = 2.9 \cdot 10^{-5}$. Find

a) y^2 / x;

b) $3x - 2y$.

Give your answers in scientific notation.

Time	Event
30000 BC	Paleolithic people in central Europe record numbers on bones.
25000 BC	Early geometric designs used.
4000 BC	Babylonian and Egyptian calendars in use.
3000 BC	First number systems are in use in Egypt and Babylon.
1950 BC	Babylonians solve quadratic equations.
575 BC	Thales brings Babylonian mathematical knowledge to Greece.
530 BC	Pythagoras studies the relationship between the sides of right-angle triangle.
500 BC	The Babylonian base-60 number system is used to record the positions of the Sun, etc.
450 BC	Greeks begin to use written numerals. Zeno of Elea presents his paradoxes.
300 BC	Euclid studies geometry as an axiomatic system in his "Elements".
250 BC	Archimedes gave the formulae for calculating the volume of a sphere and a cylinder.
225 BC	Apollonius studies the ellipse, parabola and hyperbola in his "Conic sections".
90	Nichomachus' book treats arithmetic as a separate topic from geometry.
250	Use of base-20 number system in the Maya civilization of Central America. Diophantus uses symbols for unknown numbers.
534	Chinese mathematics is introduced to Japan.
594	Decimal number system is used in India.
628	Brahmagupta uses zero and negative numbers, gives methods to solve quadratic equations, sum series, and compute square roots.
700	In the Mayan civilization a symbol for zero is invented.
810	The Greek and Indian mathematical and astronomy works are translated into Arabic. Khwarizmi writes important works on

Time	Event
	arithmetic, algebra, geography and astronomy.
1000	Alhazen writes works on optics, astronomy and mathematics.
1144	Gherard de Cremona begins translating Arabic works into Latin.
1150	Arabic numerals are introduced into Europe.
1200	Chinese start to use a symbol for zero.
1321	Gersonide's "Book of numbers" deals with permutations and combinations.
1437	Ulugbek publishes his star catalogue indicating accurate positions of 1018 stars.
1591	Viete creates symbolic algebra using letters as symbols for quantities.
1614-17	Napier and Burgi independently discover logarithms.
1635-37	Fermat and Descartes found analytic geometry by applying algebra to geometry.
1654	Pascal and Fermat create the theory of probability.
1684-87	Leibniz and Newton create the differential and integral calculus.
1691	Jacob Bernoulli invents polar coordinates.
1712	Brook Taylor develops Taylor series.
1748	Euler's "Analysis of the Infinite" systematically studies mathematical analysis.
1780	Lagrange develops variational calculus.

Time	Event
1799	Gauss proves the fundamental theorem of algebra.
1806	Legendre develops the method of least squares.
1812	Fourier discovers his method of representing functions by a trigonometric series.
1828	Gauss introduced differential geometry.
1829	Lobachevski develops non-Euclidean geometry.
1854	Boole formalizes symbolic logic and defines Boolean algebras. Bernhard Riemann introduces Riemannian geometry.
1887	Levi-Civita publishes a paper developing the calculus of tensors.
1893-94	Pearson introduced the study of statistics and Poincare introduced topology.
1900	Hilbert poses 23 problems as a challenge for the 20th century.
1901	Plank proposes quantum theory. Lebesgue formulates the theory of measure.
1905	Einstein publishes the spectral theory of relativity.
1908	Zermelo axiomizes set theory, thus avoiding Cantor's contradictions.
1913	Ramanujan sends a long list of theorems without proofs to G. H. Hardy.
1928	John von Neumann begins devising the principles of game theory.

Time	Event
1933	Andrei Kolmogorov publishes his book on axiomatization of probability.
2000	The Clay Mathematics Institute establishes the seven Millennium Prize problems.

Learning starts with wondering.
Aristotle (384-322 BC), Greek philosopher and scientist

The beginning of learning is silence, then listening, then studying, then following, then spreading it.
Imam Gazzoli (1058-1111), Persian philosopher

I.4. Rational exponents

Terms

1. **rational exponent (power) of a number** – sonning ratsional darajasi | рациональная степень числа;
2. **to raise to the fractional (negative) power** – kasr (manfiy) darajaga oshirmoq | возведение в дробную (отрицательную) степень;
3. **to extract a (square) root** – (kvadrat) ildiz chiqarmoq | извлекать (квадратный) корень;
4. **index (degree) of a root** – ildiz ko'rsatkichi | показатель корня;
5. **to bring to the same base** – bir asosga keltirmoq | приводить к общему основанию;
6. **formula** – formula | формула;
7. **value** – qiymat | значение;
8. **step by step** – qadam ba qadam | шаг за шагом;
9. **to compare** – taqqoslamoq | сравнивать;
10. **to use, to apply** – foydalanmoq, qo'llamoq | использовать, применять.

Learning Objectives

- to recognize numbers expressed as a rational exponent;
- to be able to estimate and manipulate the expressions containing rational exponents.

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1) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

2) $\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$

3) $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

4) $\left(\sqrt[m]{a^n}\right)^p=\sqrt[m]{\left(a^n\right)^p}$

5) $\sqrt[m]{a^{mn}}=a^n$

6) $\sqrt[mn]{a^m}=a^n$

7) $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$

8) $a\sqrt[n]{b}=\sqrt[n]{a^n b}$

Examples

1. Find the value of the expression

$$\left(\frac{2^{-1}\cdot 3^3}{\sqrt[3]{2}}\right)^{\frac{5}{4}}:\sqrt[6]{\frac{2^2\cdot\sqrt{3}}{3^2}}.$$

▶ We will calculate the value of the expression step by step, starting from the first,(then the second and finally the whole expression itself:

$$a)\ \left(\frac{2^{-1}\cdot 3^3}{\sqrt[3]{2}}\right)^{\frac{5}{4}} = \left(\frac{2^{-1}\cdot 3^3}{2^{\frac{1}{3}}}\right)^{\frac{5}{4}} = \left(2^{-1-\frac{1}{3}}\cdot 3^3\right)^{\frac{5}{4}} = \left(2^{-\frac{4}{3}}\cdot 3^3\right)^{\frac{5}{4}} =$$

$$\left(2^{-\frac{4}{3}}\right)^{\frac{5}{4}}\left(3^3\right)^{\frac{5}{4}} = 2^{\left(-\frac{4}{3}\right)\frac{5}{4}}\left(3^3\right)^{\frac{5}{4}} = 2^{-\frac{5}{3}}\cdot 3^{\frac{15}{4}}.$$

$$b)\ \sqrt[6]{\frac{2^2\cdot\sqrt{3}}{3^2}} = \left(\frac{2^2\cdot 3^{\frac{1}{2}}}{3^2}\right)^{\frac{1}{6}} = \left(2^2\cdot 3^{\frac{1}{2}-2}\right)^{\frac{1}{6}} =$$

$$\left(2^2\cdot 3^{-\frac{3}{2}}\right)^{\frac{1}{6}} = \left(2^2\right)^{\frac{1}{6}}\cdot\left(3^{-\frac{3}{2}}\right)^{\frac{1}{6}} = 2^{2\frac{1}{6}}\cdot 3^{\left(-\frac{3}{2}\right)\frac{1}{6}} = 2^{\frac{1}{3}}\cdot 3^{-\frac{1}{4}}.$$

$$c)\ \left(2^{-\frac{5}{3}}\cdot 3^{\frac{15}{4}}\right):\left(2^{\frac{1}{3}}\cdot 3^{-\frac{1}{4}}\right) = 2^{-\frac{5}{3}-\frac{1}{3}}\cdot 3^{\frac{15}{4}-\left(-\frac{1}{4}\right)} = 2^{-2}\cdot 3^4 =$$

$$\frac{3^4}{2^2} = \frac{81}{4} = 20\frac{1}{4} = 20.25.\ \blacksquare$$

2. Evaluate the expression $7^3\cdot 7^{\frac{4}{3}} : 49^{\frac{1}{6}} - \left(343\cdot 7^{-2}\right)^2$.

▶ We convert all values to the same bases, here 7, and use the relevant formulas.

$$a)\ 7^3\cdot 7^{\frac{4}{3}} = 7^{3+\frac{4}{3}} = 7^{\frac{13}{3}};$$

$$b)\ 7^{\frac{13}{3}} : 49^{\frac{1}{6}} = 7^{\frac{13}{3}} : \left(7^2\right)^{\frac{1}{6}} = 7^{\frac{13}{3}} : 7^{2\frac{1}{6}} = 7^{\frac{13}{3}-\frac{1}{3}} = 7^4 = 2{,}401;$$

c) $\left(343\cdot 7^{-2}\right)^2=\left(7^3\cdot 7^{-2}\right)^2=\left(7^{3-2}\right)^2=\left(7^1\right)^2=7^2=49$;

d) $2{,}401-49=2{,}352$. ■

3. Evaluate the expression $\dfrac{3\cdot\left(\sqrt[3]{3}\right)^4\cdot\sqrt[3]{9^2}\cdot\left(\dfrac{1}{3}\right)^6}{\left(\sqrt[3]{3}\right)^{-1}\cdot 27^{-\frac{2}{3}}}$.

▶ To evaluate the expression above we will convert all radicals to rational powers. Then, we will bring them to the same base and use relevant formulas.

a) $\left(\sqrt[3]{3}\right)^4=\left(3^{\frac{1}{3}}\right)^4=3^{\frac{1}{3}4}=3^{\frac{4}{3}}$.

b) $\sqrt[3]{9^2}=\sqrt[3]{\left(3^2\right)^2}=\sqrt[3]{3^4}=3^{\frac{4}{3}}$.

c) $\left(\dfrac{1}{3}\right)^6=\left(3^{-1}\right)^6=3^{(-1)6}=3^{-6}$.

d) $3\cdot 3^{\frac{4}{3}}\cdot 3^{\frac{4}{3}}\cdot 3^{-6}=3^{1+\frac{4}{3}+\frac{4}{3}-6}=3^{-\frac{7}{3}}$.

e) $\left(\sqrt[3]{3}\right)^{-1}=\left(3^{\frac{1}{3}}\right)^{-1}=3^{\frac{1}{3}(-1)}=3^{-\frac{1}{3}}$.

f) $27^{-\frac{2}{3}}=\left(3^3\right)^{-\frac{2}{3}}=3^{3\left(-\frac{2}{3}\right)}=3^{-2}$.

g) $3^{-\frac{1}{3}}\cdot 3^{-2}=3^{-\frac{1}{3}-2}=3^{-\frac{7}{3}}$. h) $3^{-\frac{7}{3}}:3^{-\frac{7}{3}}=1$. ■

Exercises

1. Compute

a) $\sqrt{16 \cdot 25}$; b) $\sqrt[3]{\frac{8}{27}}$; c) $8^{\frac{4}{3}} \cdot 9^{\frac{1}{2}}$;

d) $\sqrt{50}$; e) $\sqrt[3]{32}$; f) $\left(\sqrt[4]{3^3}\right)^2$;

g) $\sqrt{\sqrt[3]{64}}$.

2. Evaluate

a) $\left((0{,}28)^0 - \frac{5}{6}\right)^{-\frac{1}{2}}$; b) $(2.25)^{-\frac{1}{2}} \cdot 81^{\frac{3}{4}}$;

c) $\left(\frac{1}{16}\right)^{-0.75} + 810000^{0.25} - \left(7\frac{19}{32}\right)^{\frac{1}{5}} + (0.63)^0$.

3. Estimate the expression

a) $\left(4^{-0.25} - 2^{0.5}\right)\left(4^{-0.25} + \left(2\sqrt{2}\right)^{\frac{1}{3}}\right)$;

b) $\left((-17)^{-4}\right)^{-6} : \left((-17)^{-13}\right)^{-2} - \left(\frac{1}{17}\right)^2$.

4. Compare the numbers

$\sqrt{1001} + \sqrt{999}$ and $\sqrt{1000} + \sqrt{1000}$.

5. Express $\sqrt{560}$ in terms of m and n, if $m = \sqrt{7}$ and $n = \sqrt{5}$.

6. Calculate

a) $\dfrac{8^{\frac{2}{3}} \cdot 2^3 \cdot (0.5)^{-2}}{2^{-\frac{4}{5}} \cdot 4^{1\frac{2}{5}}}$; b) $\dfrac{\sqrt{196} \cdot \sqrt{19.6}}{\sqrt{0.196} \cdot \sqrt{1.96}}$.

7. Evaluate

a) $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}+\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$; b) $\dfrac{3^{0.2}}{27^{-0.6}}-\dfrac{2^{-1}}{4^{-1.5}}$.

Homework

1. Compute

a) $1^{-0.43}-(0.008)^{-\frac{1}{3}}+(15.1)^0$;

b) $(1.5)^3 \cdot (2.25)^{-1.5} \cdot (0.75)^{-1} \cdot \left[\left(-\frac{1}{3}\right)^{-2}+\left(-\frac{1}{8}\right)^{-\frac{1}{3}}-(0.0625)^{-\frac{1}{2}}\right]$.

2. Find the value of

a) $\dfrac{\left(\frac{1}{2}\right)^{\frac{3}{4}} \cdot \frac{\sqrt[3]{64}}{\sqrt[4]{32}}}{4^{-\frac{1}{3}} \cdot 16^{\frac{2}{3}}}$; b) $\dfrac{\sqrt[4]{7\sqrt[3]{54}+15\sqrt[3]{128}}}{\sqrt[3]{4\sqrt[4]{32}+\sqrt[3]{9\sqrt[4]{162}}}}$.

3. Evaluate $R=\dfrac{\sqrt{P}}{Q}$ when $P = 1.44\text{x}10^{-6}$ and $Q = 4.8\text{x}10^{-6}$.

Developments of mathematics before the 15th century AD

Ancient Egypt

The Egyptians have been using symbols for numbers, simple straight lines and the abacus since 3000 BC. By about 2000 BC they knew how to add and multiply natural numbers. The main sources of Egyptian mathematics are two Papyruses (Rhind and Moscow) dating to about 1800 BC. The Rhind Papyrus (volume 5,25x0,33*m*) contains 84 problems and the Moscow Papyrus (volume 5,44x0,08*m*) 25 problems dealing with amounts of bread and foodstuffs, animal breeding, grain storage, determining the number of bricks needed for particular constructions, operations with decimals and other practical problems. Some problems deal with geometrical principles. For instance, the area of a triangle is found as a half of the product of its height and base and a circle's area with the help of the formula $(d-d/9)^2$, where d is the circle's diameter (which results roughly in $\pi \approx 256/81 \approx 3{,}1605$). The most beautiful result of Egyptian mathematics is the invention of the formula $V=h(a^2+ab+b^2)/3$ for the pyramid with square bases, where a and b are the sides of the squares and h is the pyramid's height. In 1000 BC common (vulgar) fractions were used by Egyptians. **Anaxagoras** (500-428 BC) was the first to introduce the notions of infinitely numerous and infinitesimally small. In about 200 BC **Eratosphenes**, the head of the library of Alexandria, Egypt, calculated the Earth's circumference only about 15 percent too large. In about 60 AD, **Heron** of Alexandria wrote "Metrica" (Measurements), in which he gave formulas for calculation of areas and volumes, and contributed to geometry and geodesy (a branch of mathematics that studies the shape and size of the Earth, and the location of objects or areas on the Earth). In about 900 **Abu Kamil** (850-930 AD) stated and proved basic laws and rules of algebra and solved complicated equations such as $x+y+z=10$, $x^2+y^2=z^2$ and $xy=z^2$. His book was the main reference for the Italian mathematician Leonardo Fibonacci in the 12th century.

Developments of mathematics before the 15th century AD

Babylon

According to over 400 clay tablets found in the land between the Rivers Tiger and Euphrates, arithmetic, simple algebra and geometry and calendars have been developed from about 3000 BC. The Babylonians knew how to express commercial problems mathematically such as exchanging money and merchandise, computing simple and compound interest, taxes, second and third degree equations with two unknowns, finding the area and volume of ordinary geometrical figures, and others. They used the sexagesimal number system to record and predict the positions of the Sun, Moon and planets. The division of the circle into 360 parts and the division of the degree and the minute each into 60 parts originated in Babylonian astronomy. The Babylonians also divided the day into 24 hours, the hour into 60 minutes, and the minute into 60 seconds. Their number system was based on the number 60. Multiplication tables existed since 1800 BC. The Babylonians knew how to solve second degree equations.

I.1. **1.** a) 134; b) 653; c) 1,727; d) 122; e) 851; f) 11; g) 27; h) 3,923. **2.** a) 48; b) 84; c) 351; d) 21,252; e) 35,535. **3.** a) 5; b) 31; c) 471; d) 23; e) 231. **4.** 3,200 and 6,800. **5.** a) 30; b) 18; c) 11; d) 47. **6.** a) Yes; b) Yes; c) Yes; d) Yes; e) Yes; f) Yes; g) Yes; h) All but 9. **7.** 2, 13, 19, 41, 37. **8.** $2^2 \cdot 5^2$; $2^2 \cdot 3^2 \cdot 5$. **9.** 28: 1, 2, 4, 7, 14, 28; 36: 1, 2, 3, 4, 6, 9, 12, 18, 36; 4. **10.** 12: 12, 24, 36, 48, 60, 72, 84, 96; 15: 15, 30, 45, 60, 75, 90; 60. **11.** a) 4; b) 6; c) 1; d) 9. **12.** a) 2/4, 1/4; b) 2/5; c) 1/4. **13.** a) 3/5; b) 3/4. **14.** a) 60; b) 24; c) 52,920. **15.** a) $\frac{3}{8}<\frac{7}{12}$; b) $\frac{3}{8}<\frac{2}{4}<\frac{5}{6}$. **16.** a) $12\frac{7}{12}$; b) $\frac{2}{7}$; c) 12; d) $2\frac{13}{16}$. **17.** 504. **18.** 4/169<1/39<1/13. **19.** 1. **20.** $64 \cdot 15{,}625$. **21.** 49. **22.** 26. **23.** 4/5.

Homework: **1.** a) 895; b) 429; c) $89\frac{4}{5}$; d) 363,239. **2.** a) 170; b) 12,804; c) 4,070. **3.** a) 24; b) 21; c) 5,097. **4.** a) 489; b) 260; c) 11; d) 132. **5.** 6. **6.** Dildora, because $2+3\cdot 8=26$. **7.** 2,568. **8.** $2^6 \cdot 3 \cdot 5 \cdot 7$ and $2^2 \cdot 3^4 \cdot 11$. **9.** 306 and 7,344. **10.** a) 3/8, 4/8; b) 4/6. **11.** 7/12. **12.** a) $8\frac{7}{12}$; b) $\frac{5}{6}$; c) 7/2; d) 3/4. **13.** 1. **14.** 12,400 soums.

I.2. **1.** a) 17.68; b) 0.06243; c) 0.415; d) 55.2; e) 2,390.2; f) – 6.19. **2.** 16/9m. **3.** Largest = 4/0.5 = 8; Smallest = (– 3)/0.5 = – 6. **4.** $0.\dot{6}$; $0.\dot{7}1428\dot{5}$ and $0.1\dot{3}$. **5.** 1/3; 91/30; 37/9; 17/6; 115,022/8,325. **6.** Denote 4.2626...=A. **7.** 170/211. **8.** 1.1. **9.** 5/6. **10.** 1st, 3rd are irrational; 2nd, 4th are rational. **11.** Rational: 1st, 2nd, 6th, 8th, 9th; Irrational: the rest. **12.** Rational: m, p; Irrational: n, q. **13.** 1st is rational; 2nd is irrational. **14.** 8+2=10. **15.** a) 13.24; b) 13.244; c) 13.2437; d) 13.24367. **16.** a) 130; 17,000; b) 125; 16,500. **17.** a) $5.98 \cdot 10^{24}$ kg; b) $3 \cdot 10^{-8}$ cm.

Homework: 1. a) 894.4355; b) 65. **2.** $0.05\dot{7}$. **3.** a) 1; b) 46. **4.** Rational: 2; Irrational: $\sqrt{3}$. **5.** a) 45.19; b) 45.195; c) 45.195; d) 45.19496. **6.** a) 26; 8,300; b) 25.5; 8,330. **7.** a) $1.39 \cdot 10^9\, m$; b) $3 \cdot 10^{13}\, m/s$. **8.** 0.5.

I.3. 1. a) 216; b) – 32; c) 3; d) 16; e) 1/45; f) 25/4; g) 9. **2.** 2^{25}. **3.** a) 9,216; b) 202,500; c) $(-512)\cdot 33^{20}$. **4.** 53. **5.** 4; 9 times. **6.** a) 12^{100}; b) 100^{20}; c) 3^{-4}; d) 3^{20}. **7.** 5^{55}. **8.** 1,000,000/3 times. **9.** a) 8; b) 1; c) 9. **10.** a) 5; b) 1/9. **11.** 5.21×10^7. **12.** 177.

Homework: 1. a) 1/8; b) 256; c) 12; d) 12; e) 8/7; f) 1/2. **2.** a) 6; b) 1. **3.** a) 1.62×10^{-13}; b) 1.56×10^4.

I.4. 1. a) 20; b) 2/3; c) 48; d) $5\sqrt{2}$; e) $2\sqrt[3]{4}$; f) $3\sqrt{3}$; g) 2. **2.** a) $\sqrt{6}$; b) 18; c) 37.5. **3.** a) – 1.5; b) 1/289. **4.** $1^{st} < 2^{nd}$. **5.** $4mn$. **6.** a) 32; b) 100. **7.** a) 10; b) 5.

Homework: 1. a) – 3; b) 4. **2.** a) 1/4; b) 0.6. **3.** 250.

Millions saw the apple fall, but Newton asked why.
Bernard Baruch (1870-1965), American financier and economist

CHAPTER II. ALGEBRAIC EXPRESSIONS

II.1. Polynomials

Terms

1. **polynomial (monomial, binomial, trinomial)** – ko'phad (birhad, ikkihad, uchhad)|полином, многочлен (одночлен, двучлен, трехчлен);
2. **to collect like terms** – o'xshash hadlarni ixchamlamoq|собирать подобные члены;
3. **the brackets (parentheses)** – qavslar|скобки;
4. **algebraic expression** – algebraik ifoda|алгебраическое выражение;
5. **common factor** – umumiy ko'paytuvchi|общий множитель;
6. **identity** – ayniyat|тождество;
7. **coefficient** – koeffitsient|коэффициент;
8. **arithmetic operation** – arifmetik amal|арифметическое действие;
9. **radical sign** – radikal belgi, ildiz|радикальный знак, корень;
10. **unknown** – noma'lum|неизвестное.

Learning Objectives

– to be able to identify various polynomials and their degrees;

– to simplify polynomials by collecting like terms.

$4x$;	-1	are monomials
$2x+4$;	$2x^2y-1$	are binomials
x^2+x+2;	$xy+x+11$	are trinomials.

Arithmetic operations

1) $-(x+y-z)=-x-y+z$ **2)** $a(x+y-z)=ax+ay-az$

3) $(x+y-z):n=x:n+y:n-z:n$ **4)** $abx=(ab)x=(ax)b=a(bx)$

5) $a+(-b)=a-b$ **6)** $(-a)(-b)=ab$

7) $(-a):(-b)=a:b$ **8)** $(-a)b=a(-b)=-(ab)$

Examples

1. Simplify the algebraic expression

▶ $12a(3a-1)+(7b-a)(5a+b-4)-23a^2-2b^2-17a+13b.$

We expand the brackets and then collect like terms.

$12a(3a-1)+(7b-a)(5a+b-4)-23a^2-2b^2-17a+13b=$

$36a^2-12a+35ab+7b^2-28b-5a^2-ab+$

$+4a-23a^2-2b^2-17a+13b=$

$36a^2-5a^2-23a^2+35ab-ab+7b^2-2b^2-12a+$

$+4a-17a-28b+13b=8a^2+34ab+5b^2-25a-15b.$ ■

2. Simplify $\left(\dfrac{\sqrt[6]{a^{-2}b^{-3}}}{\sqrt[3]{a^2}\sqrt{b^5}}\right)^{-1}$.

► It is convenient to convert all radical signs to rational exponents. Then we will work with the same bases.

$$\left(\frac{\sqrt[6]{a^{-2}b^{-3}}}{\sqrt[3]{a^2}\sqrt{b^5}}\right)^{-1}=\left(\frac{\left(a^{-2}b^{-3}\right)^{\frac{1}{6}}}{a^{\frac{2}{3}}b^{\frac{5}{2}}}\right)^{-1}=\left(\frac{a^{(-2)\cdot\frac{1}{6}}b^{(-3)\cdot\frac{1}{6}}}{a^{\frac{2}{3}}b^{\frac{5}{2}}}\right)^{-1}=$$

$$\left(\frac{a^{-\frac{1}{3}}b^{-\frac{1}{2}}}{a^{\frac{2}{3}}b^{\frac{5}{2}}}\right)^{-1}=\left(a^{-\frac{1}{3}-\frac{2}{3}}b^{-\frac{1}{2}-\frac{5}{2}}\right)=\left(a^{-1}b^{-3}\right)^{-1}=ab^3.$$ ■

Exercises

1. Evaluate $a^2+2b-ab$, when $a=4$ and $b=-3$.

2. Simplify

a) $12a+23a$; b) $12a+3b-5a+8b$;

c) $17(2x + 3y) - 3(9x + 13y)$.

3. Simplify

a) $(ab^3)^4$;

b) $(-2a^2b^3)^2$;

c) $(-0.5a^3b^2c^4)^3$;

d) $3xy^2z \cdot \left(-4yz^4\right)$;

e) $10a^4b^2c^5 : 5ab^2c^6$.

4. Simplify by collecting like terms

$(4a^2 + 2b - 2x^2y^2) - (12a^2 - c) + (7b - 2x^2y^2)$.

5. Multiply out the brackets, then simplify

a) $(3x^2 + 2y)(x - 4y)$;

b) $(2x - 1)(2x - 1)$.

6. Find the square of the binomials

a) $(2x - 1)$;

b) $(3x + 2)$.

7. Divide the polynomials

a) $\dfrac{4a^{-2}}{(2a)^{-3}}$;

b) $\dfrac{3x^2 + 15x}{3x}$;

c) $\dfrac{12x^5 - 6x^3 + 4x^2}{2x^2}$;

d) $(5x^2 - 9ax - 2a^2) : (x - 2a)$.

8. Simplify the expressions

a) $\dfrac{3^{4n+3} \cdot 3^{3n-2}}{3^{2n-1}}$;

b) $\dfrac{4^9 \cdot \left(b^3\right)^4}{2^{15} \cdot b^6}$.

9. Which of the following four expressions is an identity

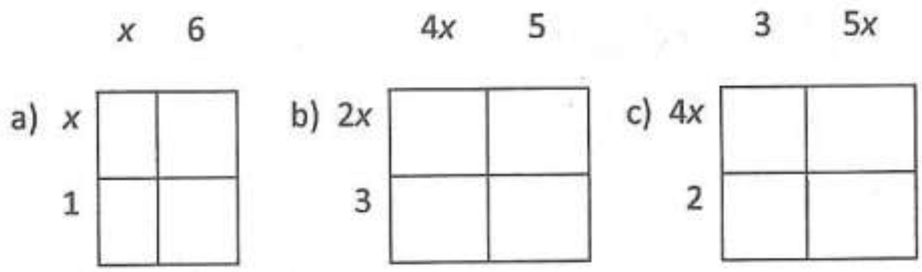
$$\frac{p^2-q^2}{p^2+q^2}=-\frac{p^2-q^2}{q^2-p^2};\qquad \frac{p^2-q^2}{p^2+q^2}=-\frac{p^2-q^2}{p^2+q^2};$$

$$-\frac{p^2+q^2}{p^2-q^2}=\frac{p^2+q^2}{q^2-p^2};\qquad -\frac{p^2-q^2}{q^2-p^2}=-\frac{p^2-q^2}{q^2+p^2}?$$

10. Find the unknown coefficient α if

$$(\alpha x-2y)(x+3y)=\alpha x^2+5xy-6y^2.$$

11. Evaluate the areas of the large and the small rectangles given below (Figures not to scale)



12. Factorize

a) $8x^3+16x^2-4x$; b) $8xyz+16x^2yz$;

c) $20x^2y^2z^2+16xyz^2$; d) $8b^3+16y^2+32t+8$;

e) $2ab^2-6a^2b^2+18abc$; f) $3x+3y+ax+ay$.

13. Simplify

a) $(x-1)(x+1)$; b) $(x-1)^2(x+1)$;

c) $2(2x+1)(x-0.5)$; d) $(x-y)(x^2+xy+y^2)$;

e) $\frac{a-5}{a^2-5a}$; f) $\frac{ax-bx}{2}:(a-b)$;

g) $\frac{p-1}{p}:\frac{2p-2}{p}$;

h) $\left(\sqrt{x}-\sqrt{y}\right):\left(\frac{1}{\sqrt{x}+\sqrt{y}}\right)$.

14. Evaluate

a) $\frac{3^{n+4}-6\cdot 3^{n+1}}{7\cdot 3^{n+3}}$;

b) $\frac{\left(\frac{a^{-1}b^{3}}{\sqrt[3]{a}}\right)^{\frac{5}{4}}}{\sqrt[6]{\frac{a^{2}\sqrt{b}}{b^{2}}}}$;

c) $\frac{\left(\sqrt[5]{a^{\frac{4}{3}}}\right)^{\frac{3}{2}}}{\left(\sqrt[5]{a^{4}}\right)^{3}}\cdot\frac{\left(\sqrt{a\sqrt[3]{a^{2}b}}\right)^{4}}{\left(\sqrt[4]{a\sqrt{b}}\right)^{6}}$.

15. Find the value of $x-y+3\left[\left\{x-y+xy\left(\frac{1}{x}+\frac{1}{y}\right)\right\}^{2}-4x^{2}+y\right]$

when $x = 2/3$; $y = -1/3$.

Homework

1. Evaluate $3x^2-2xy+y^2$ for $x=-3$ and $y=2$.
2. Simplify

a) $4x-7y-8x-15y$;

b) $16x-3(2x-5)$;

c) $(4x^3-7y^2+5x-15)-(x^4+2x^3-4y^2-11x+5)$.

3. Multiply out the brackets and simplify where possible

a) $2\frac{1}{3}\left(\frac{6}{7}m+3\right)-1\frac{2}{3}\left(\frac{3}{5}m-3\right)$;

b) $14(2m-n)+2(3n-6m)$; c) $4x^2\left(x+2+\frac{1}{x}\right)$;

d) $4pq(2+r)+5qr(2p+7)$.

4. Simplify the following as much as possible

a) $(3y^4)^2(4y^2)$; b) $\frac{27x^4y^2z}{9x^3yz^2}$;

c) $\frac{4p^3q^3}{(2pr)^3}$; d) $\frac{10z^3}{xy}\cdot\frac{4x^3}{5z}$;

e) $\frac{16xyz}{3}:\frac{4x^2}{9}$; f) $\frac{3^{4n+3}\cdot 3^{3n-2}}{3^{2n-1}}$;

g) $\frac{4^9\cdot\left(b^3\right)^4}{2^{15}\cdot b^6}$; h) $(8x^2-2x-15):(4x+5)$.

5. A rectangular pool has a length of $(3x-2)$ meters and a width of $(5-x)$ meters. Write down simplified expressions for the perimeter and the area of this pool.

6. Evaluate $\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\left(a^2+ab+b^2\right)}{\left(\frac{a}{b^2}-\frac{b}{a^2}\right)}$.

Developments of mathematics before the 15th century AD

China

In 2000-1000 BC in China each letter (hieroglyph) represented a certain number, which has been used up to days. Chinese people have used a computing board with sticks made out of bamboo or elephant bones to perform arithmetic calculations. Decimal numbers have emerged simultaneously with natural numbers and operations on decimal numbers were applied to such problems as finding areas and distributing inheritance.

The first mathematics book "Nine volume mathematics" by **Chjan Tsan** (who lived in about 150 BC) contained 246 practical problems and was meant for land measurers, engineers, officials and traders. In the 5th century the algorithm for the approximate calculation of roots of cubic equation $x^3+ax^2=b$ was used in China. In the 8th century the algorithm for solving general cubic equations was produced. In about 534 Chinese mathematics was introduced to Japan. China started to use a symbol for zero in about 1200.

A problem well stated is half-solved.
John Dewey (1859-1952), American philosopher, psychologist and educator

He who cannot describe the problem
will never find the solution to that problem.
Confucius (551-479 BC), Chinese philosopher

II.2. Short multiplication formulae

Terms

1. **short multiplication formula (SMF)** – qisqa ko'paytirish formulasi (QKF) | формула сокращенного умножения (ФСУ);
2. **incomplete square of a difference (sum)** – chala kvadrat ayirma (yig'indi) | неполный квадрат разности (суммы);
3. **double (triple) product** – ikkilangan (uchlangan) ko'paytma | удвоенное (утроенное) произведение;
4. **lowest common denominator** – eng kichik umumiy mahraj | наименьший общий знаменатель;
5. **in terms of** – orqali | через;
6. **to be equal to** – teng bo'lmoq | быть равным;
7. **as much as possible** – imkon qadar | настолько, насколько возможно;
8. **term** – had | член;
9. **difference** – ayirma, farq | разность, разница;

10. **to find** – topmoq | находить.

Learning Objectives

- to get acquainted with the short multiplication formulae;
- to be able to apply the short multiplication formulae efficiently.

Short multiplication formulae

1) $(a+b)^2 = a^2 + 2ab + b^2$

2) $(a-b)^2 = a^2 - 2ab + b^2$

3) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

4) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

5) $a^2 - b^2 = (a-b)(a+b)$

6) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

7) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Examples

1. Calculate

$$\left(35.5^2-27.5^2\right):\left(\frac{57^3+33^3}{90}-57\cdot 33\right)\cdot\frac{78^2-2\cdot 78\cdot 18+18^2}{33^2-27^2}.$$

▶ a) $35.5^2-27.5^2=(35.5-27.5)(35.5+27.5)=8\cdot 63=2^3\cdot 3^2\cdot 7$;

b) $\frac{57^3+33^3}{90}-57\cdot 33=\frac{(57+33)\left(57^2-57\cdot 33+33^2\right)}{90}-57\cdot 33=$

$57^2-2\cdot 57\cdot 33+33^2=(57-33)^2=24^2=\left(2^3\cdot 3\right)^2=2^6\cdot 3^2$;

c) $\frac{78^2-2\cdot 78\cdot 18+18^2}{33^2-27^2}=\frac{(78-18)^2}{(33-27)(33+27)}=\frac{60^2}{6\cdot 60}=10=2\cdot 5$;

d) $\left(2^3\cdot 3^2\cdot 7\right):\left(2^6\cdot 3^2\right)\cdot(2\cdot 5)=\frac{2^3\cdot 3^2\cdot 7\cdot 2\cdot 5}{2^6\cdot 3^2}=$

$\frac{2^4\cdot 3^2\cdot 5\cdot 7}{2^6\cdot 3^2}=\frac{35}{4}=8.75$. ■

2. Simplify the expression

$$\frac{x^3+y^3}{x+y}:\left(x^2-y^2\right)+\frac{2y}{x+y}-\frac{xy}{x^2-y^2}.$$

▶ $\frac{(x+y)\left(x^2-xy+y^2\right)}{x+y}\cdot\frac{1}{x^2-y^2}+\frac{2y}{x+y}-\frac{xy}{x^2-y^2}=$

$\frac{x^2-xy+y^2}{(x+y)(x-y)}+\frac{^{(x-y)/}2y}{x+y}-\frac{xy}{(x+y)(x-y)}=$

$\frac{x^2-xy+y^2+2xy-2y^2-xy}{x^2-y^2}=\frac{x^2-y^2}{x^2-y^2}=1$. ■

Exercises

1. Use SMF to find

a) $(10+5)^2$; b) $(10-6)^2$; c) 19^2;

d) $(10-2)(10+2)$; e) $101\cdot 99$; f) 13^2-7^2;

g) $(10+2)^3$; h) 11^3; i) $(10-3)^2$;

j) 19^3; k) 3^3-2^3; l) 11^3-9^3;

m) 6^3+4^3; n) 11^3+9^3.

2. Find

a) $(2ab^2+0.5cd^2)^2$; b) $(3x-2y)^3$;

c) $\left(2x^{\frac{1}{2}}-y^{-\frac{1}{4}}\right)\left(2x^{\frac{1}{2}}+y^{-\frac{1}{4}}\right)$.

3. If $a^2+b^2=4$ and $(a-b)^2=2$, what is the value of ab?

4. What is the value of $x^2+12x+45$, when $x=64$?

5. If $x^2+y^2=\mathrm{A}$ and $xy=\mathrm{B}$, express $(x+y)^4$ in terms of A and B.

6. First simplify, then compute

$$\frac{a^2-1}{n^2+an}\cdot\left(\frac{1}{1-\frac{1}{n}}-1\right)\cdot\frac{a-an^3-n^4+n}{1-a^2},$$

when $a = 5, n = 1.5$.

7. What is the following equal to

$$\left(\frac{1}{a}+a\right)^2-\left(\frac{1}{a}-a\right)^2?$$

8. Calculate

a) $\dfrac{4\left(0.8^2-0.8\cdot1.7+1.7^2\right)}{1.6^3+3.4^3};$

b) $\left(\left\{\sqrt[4]{2}-\sqrt[4]{8}\right\}^2+5\right)\left(\left\{\sqrt[4]{2}+\sqrt[4]{8}\right\}^2-5\right).$

9. Simplify as much as possible

a) $\dfrac{\sqrt{x^2+6x+9}}{x^3+27}\cdot\left(\dfrac{(x-3)^2}{3x}+1\right)\cdot2x;$

b) $\left(\dfrac{1+\frac{a}{b}}{\frac{b}{a}-1}\right):\left(\dfrac{a^2-b^2}{1-\frac{2b}{a}+\frac{b^2}{a^2}}\right);$

c) $\dfrac{x^2-2x}{x^2-x-2}\cdot\dfrac{(x+1)\cdot y^2}{\left(xy^3\right)^2}.$

10. Calculate

a) $\frac{\left(\sqrt{2}-\sqrt{3}\right)^2+2\sqrt{6}}{\left(\sqrt{6}+1\right)\left(\sqrt{6}-1\right)}$;

b) $\left(\frac{15}{\sqrt{6}+1}+\frac{4}{\sqrt{6}-2}-\frac{12}{3-\sqrt{6}}\right)\cdot\left(\sqrt{6}+11\right)$.

11. Find $A = 11^3 + 12^3 + 13^3 + 14^3 + 15^3 + 16^3 + 17^3 + 18^3 + 19^3 + 20^3$

given that $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.

Homework

1. Use short multiplication formulae to evaluate

a) $(15 + 11)^2$; b) $(25 - 9)^2$; c) $17^2 - 7^2$;

d) $37 \cdot 43$; e) $(30 + 6)^3$; f) $(40 - 4)^3$;

g) $13^3 + 17^3$; h) $25^3 - 15^3$.

2. Calculate

a) $\frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}}-\frac{\sqrt{2}}{\sqrt{6}+\sqrt{2}}$;

b) $\sqrt[3]{1-\sqrt{3}}\cdot\sqrt[6]{4+2\sqrt{3}}$.

3. Simplify

a) $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}$;

b) $\left(\dfrac{m-2}{m+2}-\dfrac{m+2}{m-2}\right):\dfrac{8m}{m^2-4}$;

c) $\left(\dfrac{a+x}{a}-\dfrac{x-y}{x}\right)\cdot\dfrac{a^2}{x^2+ay}:\dfrac{a}{8x}$;

e) $\dfrac{\sqrt{x}+1}{x\sqrt{x}+x+\sqrt{x}}:\dfrac{1}{x^2-\sqrt{x}}$;

d) $\dfrac{x}{x^2+y^2}-\dfrac{y(x-y)^2}{x^4-y^4}$;

f) $\left(\dfrac{\sqrt{y}-\sqrt{x}}{y-\sqrt{xy}+x}+\dfrac{x}{x\sqrt{x}+y\sqrt{y}}\right)\cdot\dfrac{x\sqrt{x}+y\sqrt{y}}{y^3}$.

4. Simplify as much as possible the expression

$$\left(\frac{a^4+b^4}{b^6}\right)\left(\frac{1}{a-b}-\frac{1}{a+b}-\frac{2b}{a^2+b^2}-\frac{4b^3}{a^4+b^4}\right).$$

Developments of mathematics before the 15th century AD

India

The first concepts of mathematics emerged in 2000-1000 BC. For example, in the work "Shulbasutra" methods of mathematical calculation of various buildings were given. In the 6th century special symbols referred to as brahma digits 1 through 9 were the basis for creating decimal notation. The Indians referred to zero as "shunya" (empty), Arab scientists called zero "sifr". The science of arithmetic was firstly founded and developed systematically in India, that is, the rules of arithmetic operations based on decimal notation were created by Indian mathematicians who also introduced negative numbers. The work "Aryabhatiya" of **Aryabhata** (476-550 AD), who is known as the "Eastern Copernicus", was a turning point for the development of natural sciences in India. He proposed that Earth orbited the Sun and properly explained ecclipses of the Sun and the Moon. Aryabhata knew how to extract square and cube roots and solved the equation $ax+by=c$ for whole numbers, whereas **Brahmagupta** (598-660 AD) investigated the solvability of equations $ax^2+b=y^2$. Brahmagupta used zero and negative numbers and computed square roots. He considered $\pi=\sqrt{10}$. For instance, Brahmagupta's writing of the equation $3x^2+10x-8=x^2+1$ would be

$$\textit{ya va } 3 \textit{ ya } 10 \textit{ ru } \dot{8}$$
$$\textit{ya va } 1 \textit{ ya } 0 \textit{ ru } 1$$

where ya from yavat (tavat), is unknown, va, from varga, is a square number, ru, from rupa, is a free term and the point above the number means the number being subtracted.

In about 850 **Mahavira** wrote a nine-chapter book that included all mathematical knowledge up to that time. In about 900 **Sridhara** solved quadratic equations, summed series and proposed the methods for calculating the areas of polygons. **Bhaskara** (12th AD) knew the formula for the volume of a sphere and used the trigonometric relations $\sin^2 x+\cos^2 x=1$ and $\sin x=\cos(90^\circ-x)$ for solving astronomical problems. Bhaskara wrote six books on mathematics, including "Lilavati" ("The Beautiful"), which summarized mathematical knowledge in India up to his time.

II.1. **1.** 22. **2.** a) $35a$; b) $7a+11b$; c) $7x+12y$. **3.** a) a^4b^{12}; b) $4a^4b^6$; c) $-0.125a^9b^6c^{12}$; d) $-12xy^3z^5$; e) $2a^3c^{-1}$. **4.** $-8a^2+9b-4x^2y^2+c$. **5.** a) $3x^3-12x^2y+2xy-8y^2$; b) $4x^2-4x+1$. **6.** a) $4x^2-4x+1$; b) $9x^2+12x+4$. **7.** a) $32a$; b) $x+5$; c) $6x^3-3x+2$; d) $5x+a$. **8.** a) 3^{5n+2}; b) $8b^2$. **9.** 3^{rd}. **10.** $\alpha=7/3$. **11.** a) x^2+5x+4; x^2; $6x$; x; 6; b) $8x^2+22x+15$; $8x^2$; $10x$; $12x$; 15; c) $12x$; $20x^2$; 6; $10x$. **12.** a) $4x(2x^2+4x-1)$; b) $8xyz(1+2x)$; c) $4xyz^2(5xy+4)$; d) $8(b^3+2y^2+4t+1)$; e) $2ab(b-3ab+9c)$; f) $(x+y)(3+a)$. **13.** a) x^2-1; b) x^3-x^2-x+1; c) $4x^2-1$; d) x^3-y^3; e) $1/a$; f) $x/2$; g) $1/2$; h) $x-y$. **14.** a) $1/3$;

b) $a^{-2}b^{12}$; c) $a^{-1/6}b^{-1/4}$. **15.** 0.

Homework: **1.** 43. **2.** a) $-4x-22y$; b) $10x+15$; c) $-x^4+2x^3-3y^2+16x-20$. **3.** a) $m+12$; b) $16m-7n$; c) $4x^3+8x^2+4x$; d) $8pq+14pqr+35qr$. **4.** a) $36y^{10}$; b) $3xy/z$; c) $q^3/(2r^3)$; d) $8x^2z^2/y$; e) $12yz/x$; f) 3^{5n+2}; g) $8b^6$; h) $2x-3$. **5.** $P=4x+6$; $S=-3x^2+17x-10$. **6.** 1.

II.2. **1.** a) 225; b) 16; c) 361; d) 96; e) 9,999; f) 120; g) 1,728; h) 1,331; i) 49; j) 6,859; k) 19; l) 602; m) 280; n) 2,060. **2.** a) $4a^2b^4+2ab^2cd^2+0.25c^2d^4$; b) $27x^3-54x^2y+36xy^2-8y^3$; c) $4x-y^{-0.5}$. **3.** $ab=1$. **4.** 4,909. **5.** $(A+2B)^2$. **6.** 19/6. **7.** 4. **8.** a) 1/5; b) 17. **9.** a) 2/3; b) $-1/(ab)$; c) $1/(xy^4)$. **10.** a) 1; b) -115. **11.** 41075.

Homework: **1.** a) 676; b) 256; c) 240; d) 1,591; e) 46,656; f) 46,656; g) 7,110; h) 12,250. **2.** a) 1; b) $(-2)^{1/3}$. **3.** a) $a-b$; b) -1; c) 8; d) $1/(x+y)$; e) $x-1$; f) $1/y^2$. **4.** $8b/(a^4-b^4)$.

God wrote the universe in the language of mathematics.
Galileo Galilei (1564-1642), Italian physicist and astronomer

Go down deep enough into anything and you will find mathematics.
Dean Schlicter

CHAPTER III. EQUATIONS AND SIMULTANEOUS EQUATIONS

III.1. Linear equations

Terms

1. **equality** – tenglik | равенство;
2. **equation** – tenglama | уравнение;
3. **an equivalent equation** – teng kuchli tenglama | равносильное уравнение;
4. **root (degree) of an equation** – tenglamaning ildizi (darajasi) | корень (степень) уравнения;
5. **linear equation** – chiziqli tenglama | линейное уравнение;
6. **to determine** – aniqlamoq | определять;
7. **constant** – o'zgarmas, konstanta | постоянное, константа;
8. **to transfer** – o'tkazmoq | переносить;
9. **to denote** – belgilamoq | обозначать;
10. **to satisfy** – qanoatlantirmoq | удовлетворять.

Learning Objectives

- to determine the degree of an equation;
- to be able to reduce a problem into a linear equation and solve it.

Determining the degree of an equation

All terms of an equation must be transferred to one side and simplified as much as possible. If the equation contains one unknown, then the highest degree of the unknown in the equation will be the degree of the equation. If the equation contains several unknowns, then the highest sum of exponents of the unknowns for all terms is considered as the degree of the equation.

$3x-6=9$;	$4x=2y-18$	are the 1st degree equations
$7x^2+3x=1$;	$xy+x+y=4$	are the 2nd degree equations
$2x+4=x^3$;	$xyz+2x^2=xy$	are the 3rd degree equations.

Linear equation

$ax+b=0$,

where a and b are constant real numbers.

Solution of linear equation ($ax+b=0$)

$$x=-\frac{b}{a}.$$

Method of solution of a linear equation

Transfer all unknowns to one side and all constants to other side of equal sign, simplify and then divide both sides of the equation by the coefficient of the unknown.

Examples

1. Determine the degree of the equations
   a) $(x+2)^3+x^2=12x+13$;
   b) $(x+y)^2+2x^2y-9=x^2$.

▶ a) We expand the brackets, transfer all terms to the left side and simplify to end up with $x^3+7x^2-5=0$. The highest degree of the unknown x is 3, therefore the expression is of the 3rd degree equation.

b) We expand the brackets, transfer all terms to the left side and simplify to end up with $2xy+y^2+2x^2y-9=0$. The highest sum of exponents of the unknowns for all the terms is 3, so it is a 3rd degree equation. ■

2. Solve the linear equation $\frac{3x-5}{4}+\frac{12-11x}{6}=3$.

▶ First we multiply both sides of the equation by the least common denominator 12. Then, the unknowns and constant numbers are sorted out to opposite sides of the equation and simplified. Finally, both sides of the equation are divided by the coefficient of the unknown to find the root of the equation.

$$12\left(\frac{3x-5}{4}+\frac{12-11x}{6}\right)=3\cdot 12$$

$$3(3x-5)+2(12-11x)=36$$

$$9x-15+24-22x=36 \Rightarrow -13x=27 \Rightarrow$$

$$\frac{-13x}{-13}=\frac{27}{-13} \Rightarrow x=-2\frac{1}{13}. \blacksquare$$

3. A number of people boarded a bus at the terminal. At the first stop, half of the passengers got off and one man got on. At the second stop, one third of the passengers on the bus got off and one man got on. If the bus finally had 15 passengers, what was the initial number of passengers?

▶ There are two methods to solve this problem.

1^{st} method. Let us denote the original number of passengers as x. Then we can register the number of passengers at each stop as follows

After the first stop we have $x-\frac{x}{2}+1$ passengers

After the second stop we have $\left(x-\frac{x}{2}+1\right)-\frac{1}{3}\left(x-\frac{x}{2}+1\right)+1$ passengers. This figure we know is 15. So,

$$\frac{2}{3}\left(x-\frac{x}{2}+1\right)+1=15 \Rightarrow$$

$$x-\frac{x}{2}+1=21 \Rightarrow \frac{x}{2}=20 \Rightarrow x=40.$$

2nd method. We can start from the last stop and work backward. At the last stop one person boarded. Before he boarded we must thus have had 14 passengers on board. But these 14 are a result of one third of the passengers leaving the bus. Hence we had 21 passengers before stop 2, since

$$14\cdot\frac{3}{2}=21 \quad \left(\textit{check } 21-\frac{1}{3}\cdot 21=\frac{2}{3}\cdot 21=14\right).$$

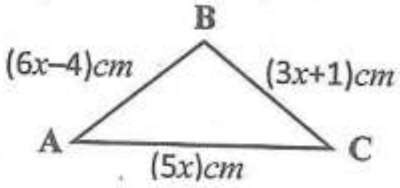
If we do the same for the first stop we have that before the one passenger got on, we had 20 passengers. These were there after half the passengers got out, so the bus must have started with 40 passengers.

Question: Which method is better? Which method would be better if we had data for 5 or 10 stops? ■

Exercises

1. Determine the degree of the following equations
   a) $2x-4=3$; b) $x^2+x=1$;
   c) $x^3=x^2+3$; d) $3x^4+2x^3-x=3x^4-1$;
   e) $5x^5+4x^4y^2=3xy^4$; f) $2xyz+1=x^2y^2$.
2. Solve the equations
   a) $5x=10$; b) $-4x=2$;
   c) $4x+5=2x+11$; d) $2(x+3)-3(4-5x)=7x-2$;
   e) $\frac{3+25x}{3x+7}=5$; f) $\frac{3x-2}{4}+\frac{2x+3}{2}-2.5x+2=0$.
3. Find x in
   a) $12\left(1\frac{3}{4}x+\frac{5}{8}\right)=-6\frac{1}{2}$; b) $5-3(x-2(x-2(x-2)))=2$;
   c) $\left(1.7:\left(1\frac{2}{3}x-3.75\right)\right):\frac{8}{25}=1\frac{5}{12}$.

4. A triangle has lengths as shown below. Find the length of each side, if the length of AC exceeds that of AB by half a centimeter.



5. If the average of 35, 38, 41, 43 and x is 37, what is the value of x?
6. If the sum of 5 consecutive numbers is 115, find the numbers.
7. If a and b are real numbers with $ab(a-2b) \neq 0$, solve the equation

$$\frac{1}{b}\left(x+\frac{1}{a}\right)=\frac{1}{a}(1+2x).$$

8. Shirin's father was 24 years old when Shirin was born. Now he is four times as old as Shirin. How old is Shirin now?
9. Solve the equations

a) $\dfrac{0.1\dot{6}+0.\dot{6}}{0.\dot{3}+1.1\dot{6}}(x+1)=0.3\dot{8}x$; b) $\dfrac{\left(\sqrt[6]{4}\right)^5\cdot 16^{\frac{1}{2}}}{\sqrt[3]{64}\cdot x}=16^{\frac{1}{6}}\cdot 4^{-\frac{1}{2}}$;

c) $\dfrac{x-1}{x}+\dfrac{x-2}{x}+\dfrac{x-3}{x}+\cdots+\dfrac{1}{x}=4$;

d) $\dfrac{x}{2}+\dfrac{x}{6}+\dfrac{x}{12}+\dfrac{x}{20}+\dfrac{x}{30}+\dfrac{x}{42}=-6$;

e) $\dfrac{x}{4}\left(3-\dfrac{8}{x}\right)-\dfrac{7}{8}\left(7-\dfrac{3x}{4}\right)=15\left(\dfrac{1}{3}-\dfrac{x}{64}\right)$.

10. A basket of melons is emptied by one person taking half of them plus one more, a second person taking half of the remainder and one more, and a third person taking half the remainder and six more. How many melons did the basket contain to start with?
11. A slow train traveling from Toshkent to Angren arrives 9 minutes late when traveling at $36km/h$. If it travels at $27km/h$ it arrives 39

minutes late. What is the distance between Toshkent and Angren?

12. A taxi service charges 300 soums for the first kilometer and 50 soums for each additional kilometer. If Botir paid a 3,200 soums fare, how far did he go?

13. An egg dealer bought a number of eggs at 60 pennies for 6, and five times that number for 900 pennies for 100. He sold them all at 72 pennies per 6 eggs and made a profit of 1020 pennies. How many eggs did he buy?

Homework

1. What are the degrees of the following equations

   a) $(x+1)^3 = 4(x-1)^2 + x^3$; b) $x(y-z^2) = x^2y(y^2 - z^2)$;

   c) $\dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + 2x + 1} = 0$; d) $2(x-1) = x-1$.

2. Solve the equations

   a) $1.2(0.5 - 5x) + 4.2 = 3(4 - 1.2x)$;

   b) $2(x - 3) = 1.2 - x$;

   c) $2 - \dfrac{2x - \dfrac{4-3x}{5}}{5} = \dfrac{7x - \dfrac{x-3}{2}}{5}$;

   d) $1 - \dfrac{x - \dfrac{1+x}{3}}{3} = \dfrac{x}{2} - \dfrac{2x - \dfrac{10-7x}{3}}{2}$.

3. Sevara found she had 5,200 soums in her purse, but she forgot how much money she had taken with her in the morning. However, she knew that she had spent 200 soums for the metro, 300 soums for two somsas and a cup of tea, 1,800 soums for two blank DVD disks and 1,200 soums for a notebook. How much money did Sevara have in the morning?

4. Dilfuza is sawing embroidered scarves at an art fair. Each scarf costs 35,312.5 soums. If she sells the scarves for 40,000 each,

how many will she have to sell to make a profit of exactly 70,000?

5. Jamshid receives a fixed weekly payment of \$200, plus a commission on sales over \$2,000 at a rate of \$50 for every \$1,000. How much sales should he achieve to earn \$550 in a week?

6. Bobur has constructed a number game. He asks his friends to choose a number, add 3 to it, multiply by 2, add 8, divide by 2, and then tell the answer. Nozima picked a number, followed Bobur's instructions and ended up with the number 32. What number did Nozima think of?

Developments of mathematics before the 15th century AD
Ancient Greece

The Greeks adopted elements of mathematics from the Babylonians and the Egyptians. In ancient Greece mathematics developed in various philosophical schools: The Ionic school (7th-6th centuries BC), the school of Pythagoras (6th-5th centuries BC) and Plato's academy (6th-5th centuries BC). The Greeks developed a new branch of mathematics – logistics, which included such problems as operations with whole and decimal numbers, calculation of square roots, solution of first and some second degree equations, architectural computations and land measurements. In addition, the ancient Greeks proved the existence of irrational numbers. In 575 BC, Thales uses geometry to solve such problems as calculating the height of pyramids, finding the distance of ships from the shore and stipulates the famous theorem about parallel lines that cut equal parts on one side of an angle also cut equal parts on another.

In ancient Greece, the solution of the following three classic problems was attempted using only a ruler and a compass (6th-5th centuries BC):

1. Doubling the cube. To make a cube that is twice the size of a given cube. The difficulty of the problem is that it is impossible to make a line of length $\sqrt{2}$ with only a ruler and a compass, as was proved later in the 19th century AD.
2. Trisecting the angle (Greek: tria = three; Latin: sectus = cut, divide). To divide an angle into three equal angles. The problem can only be solved in some particular cases.
3. Squaring the circle. To make a square of the same size as any given circle with the help of ruler and compass. The insolvability of the problem was proved in the 19th century AD.

Developments of mathematics before the 15th century AD

The Greeks' insistence on deductive proof was an extraordinary step. No other civilization had conceived the idea of establishing conclusions exclusively by deductive reasoning based on explicitly stated axioms. The Greek mathematician **Pythagoras** of Samos (582-500 BC) used pebbles or dots to represent whole numbers. (The English word "calculation" derives from the Greek word for stone or pebble). He established the rule for any right-angle triangle, known today as the Pythagoras theorem, which states that the square of the hypotenuse equals the sum of the squares of the other two sides. **Hippocrates** of Chios (460-377 BC) wrote his book "Elements", in which he first compiles the elements of geometry. **Democritus** (460-370 BC) showed that a pyramid and a cone are equal to a third of a prism and a cylinder with the same bases and heights respectively. **Eudoxus** of Cnidus (408-355 BC) was one the founders of axiomatic method in mathematics. He founded the study of spherical geometry. It is believed that Eudoxus discovered much of the geometry later included in "Elements", the comprehensive treatise on mathematics written by the Greek mathematician Euclid. In about 340 BC, **Aristaeus** wrote his "Five books" about conic sections. **Aristarchus** of Samos (310-250 BC) applied geometry to the calculation of the distance of the Sun and the Moon from the Earth. He also proposed that the Earth orbits the Sun. In about 300 BC **Euclid** systematized the work of many Greek mathematicians in his popular work "Elements" which forms the basis of Geometry to this day. **Archimedes** (287-212 BC) gave the formulae for calculating the volume of a sphere and a cylinder in his book "On the sphere and the cylinder". In about 225 BC, **Apollonius** of Perga introduced the terms "parabola", "ellipse" and "hyperbola" in his book "Conics". **Hipparchus** (190-120 BC) estimated the duration of a year to within 6.5 minutes of the correct value.

The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs.
Thomas Hill (1836-1882), British philosopher and educator

III.2. Quadratic equations

Terms

1. **unknown quantity** – noma'lum qiymat | неизвестная величина;
2. **quadratic equation** – kvadrat tenglama | квадратное уравнение;
3. **variable** – o'zgaruvchi | переменная величина;
4. **method of trial and error** – taxmin qilish usuli | метод пробок и ошибок;
5. **incomplete quadratic equation** – chala kvadrat tenglama | неполное квадратное уравнение;
6. **discriminant** – diskriminant | дискриминант;
7. **method** – usul, metod | способ, метод;
8. **biquadratic** – bikvadrat | биквадратный;
9. **to substitute** – o'rniga qo'ymoq | подставлять;
10. **Viete's theorem** – Viyet teoremasi | теорема Виета.

Learning Objectives

- to identify quadratic and biquadrate equations;
- to solve quadratic equations by factorization, the standard formula and completing the square.

Quadratic equation

$ax^2 + bx + c = 0$,

where a, b and c are constant numbers

Incomplete quadratic equations

$ax^2 + bx = 0$; $ax^2 + c = 0$; $ax^2 = 0$.

Solution of $ax^2 + bx + c = 0$

Quadratic formula

Discriminant $D = b^2 - 4ac$

I. If $D > 0$, then $x_1 \neq x_2$;

II. If $D = 0$, then $x_1 = x_2$;

III. If $D < 0$, then there is no solution among real numbers.

$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If b is an even number, then

$$x_{1/2} = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a}.$$

Viete's theorem

If x_1 and x_2 are the solutions of $ax^2 + bx + c = 0$, then

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 \cdot x_2 = \frac{c}{a} \end{cases}$$

Factorization of quadratic expression

$ax^2 + bx + c = a(x - x_1)(x - x_2)$,

where x_1, x_2 are the roots of the equation.

Examples

1. Solve $2x^2 - 15x - 8 = 0$ using the quadratic formula.

▶ Coefficients are a = 2; b = − 15 and c = − 8.

$$D = b^2 - 4ac = (-15)^2 - 4 \cdot 2 \cdot (-8) = 289 = 17^2.$$

$$x_1 = \frac{-b+\sqrt{D}}{2a} = \frac{-(-15)+\sqrt{17^2}}{2\cdot 2} = 8 \text{ and}$$

$$x_2 = \frac{-b-\sqrt{D}}{2a} = \frac{-(-15)-\sqrt{17^2}}{2\cdot 2} = -\frac{1}{2}. \blacksquare$$

2. Find the roots of the quadratic equation $x^2 - 39x + 74 = 0$ by Viete's theorem.

▶ According to Viete's theorem, the roots of the quadratic equation must satisfy the conditions

$$\begin{cases} x_1 + x_2 = -\dfrac{b}{a} \\ x_1 \cdot x_2 = \dfrac{c}{a} \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = \dfrac{-39}{1} \\ x_1 \cdot x_2 = \dfrac{74}{1} \end{cases} \Rightarrow$$

$$\begin{cases} x_1 + x_2 = 39 \\ x_1 \cdot x_2 = 74 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 37 \end{cases},$$

which we can find by trying factors of 74 until we find two that add up to 39. ■

3. Factorize the quadratic $3x^2 - 57x + 210$.

▶ We will find the roots of the corresponding quadratic equation $3x^2 - 57x + 210 = 0$ and factorize using the formula

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Our equation has coefficients a = 3, b = − 57 and c = 210. So,

$$D = b^2 - 4ac = (-57)^2 - 4\cdot 3\cdot 210 = 729 = 27^2.$$

$$x_1 = \frac{-b+\sqrt{D}}{2a} = \frac{-(-57)+\sqrt{27^2}}{2\cdot 3} = 14 \text{ and}$$

$$x_2 = \frac{-b-\sqrt{D}}{2a} = \frac{-(-57)-\sqrt{27^2}}{2\cdot 3} = 5.$$

Thus, $3x^2 - 57x + 210 = 3(x-5)(x-14)$. ■

4. Solve $\quad (2x^2-5x)^2 - 28(2x^2-5x) + 75 = 0$.

▶ Let $2x^2 - 5x = y$.

Then $y^2 - 28y + 75 = 0$, which is a quadratic equation for y with a = 1, b = − 28 and c = 75. So,

$D = b^2 - 4ac = (-28)^2 - 4\cdot 1\cdot 75 = 484 = 22^2$, yielding

$$y_1 = \frac{-b+\sqrt{D}}{2a} = \frac{-(-28)+\sqrt{22^2}}{2\cdot 1} = 25 \text{ and}$$

$$y_2 = \frac{-b-\sqrt{D}}{2a} = \frac{-(-28)-\sqrt{22^2}}{2\cdot 1} = 3.$$

Now, we substitute these values of y back into the expression we had above:

1) $2x^2 - 5x = 3 \Rightarrow 2x^2 - 5x - 3 = 0$

$D = b^2 - 4ac = (-5)^2 - 4\cdot 2\cdot(-3) = 49 = 7^2$

$x_1 = \frac{5+\sqrt{49}}{4} = 3$ and $x_2 = \frac{5-\sqrt{49}}{4} = -\frac{1}{2}$.

2) $2x^2 - 5x = 25 \Rightarrow 2x^2 - 5x - 25 = 0$

$D = b^2 - 4ac = (-5)^2 - 4\cdot 2\cdot 25 = 225 = 15^2$

$x_3 = \frac{5+\sqrt{225}}{4} = 5$ and $x_4 = \frac{5-\sqrt{225}}{4} = -\frac{5}{2}$. ■

Exercises

1. Classify and solve the quadratic equations

   a) $16x^2 = 0$; b) $x^2 - 4 = 0$;

   c) $3x^2 + 3 = 0$; d) $x^2 - 3x = 0$;

   e) $3x^2 + 4x = 0$; f) $16x^2 + 25x = 34x$.

2. Solve the equations using the quadratic formula .

   a) $x^2 - 8x + 15 = 0$; b) $3x^2 - 7x + 2 = 0$;

   c) $x^2 - 3x + 12 = 0$; d) $x^2 - 10x + 25 = 0$.

3. Solve the following by completing the square

   a) $x^2 - 12x - 28 = 0$; b) $x^2 - 3x + 2 = 0$;

   c) $x^2 + 5x + 6 = 0$.

4. Classify and solve by factorizing

   a) $p^2 - 3p = 0$; b) $x^2 - 4x + 4 = 0$;

   c) $x^2 - x - 20 = 0$; d) $x^2 - 5x + 6 = 0$;

   e) $3x^2 - 7x + 4 = 0$.

5. One root of $x^2 + 5x + a = 0$ is 2. Find the value of a using Viete's theorem.

6. Factorize the quadratics

   a) $x^2 - 8x + 15$; b) $3x^2 - 7x + 2$;

   c) $7x^2 + 17x - 12$.

7. If $(t - 8)$ is a factor of $t^2 - kt - 48$, what is the value of k?

8. Find the values of $(1+a)(1+b)$ and $\frac{1}{a}+\frac{1}{b}$, where a and b are the roots of the quadratic equation $2x^2 + 3x - 4 = 0$, without solving the quadratic equation.

9. Find the product of all the real (rational or irrational) roots of the equation:

$$(4x^2-7x-5)(5x^2+13x+3)(3x-x^2-8)=0.$$

10. Solve the biquadrate equations

a) $x^4-3x^2+2=0$; b) $2x^4-19x^2+9=0$.

11. If 7 is a solution to $x^3-6x^2+px+14=0$, find p and the other two roots of the equation.

12. If $ax^2 + bx - 6 = 0$ has roots (-2) and 3, solve the equation $bx^2 + 2x + 3a = 0$.

13. Reduce to a quadratic equation and solve

a) $x^6+7x^3-8=0$; b) $(x^2-5x)-8(x^2-5x)-84=0$;

c) $x^2+\frac{1}{x^2}+x+\frac{1}{x}-4=0$; d) $x-3\sqrt{x}+2=0$.

14. Solve the equation

$(3x^2 + 13x + 12)(x + 1) = (x + 4)(3x^2 - 2x - 8)$.

15. Solve $(x^2 + 4x - 4)^2 - (x^2 - 4x - 4)^2 = 0$.

Homework

1. Solve by factorizing

a) $9x^2 - 49 = 0$; b) $x^2 + x - 20 = 0$;

c) $x + 3 = 2x^2$; d) $1 - 5x + 6x^2 = 0$.

2. Solve by completing the square

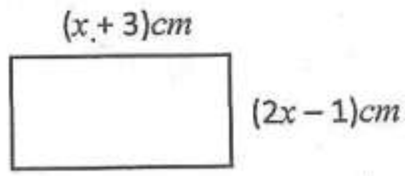
a) $x^2 - 6x + 5 = 0$; b) $4x^2 - 24x + 11 = 0$.

3. Solve using the quadratic formula

a) $4x^2 - 15x + 9 = 0$; b) $5 - 3x - 2x^2 = 0$;

c) $3x^2 - 11x - 20 = 0$.

4. Find the sides of the rectangle below if its area is $9cm^2$.



5. On what value of m the equation $x^2 + (m+2)x + m + 5 = 0$ has two equal roots?

6. If one root of $x^2 - ax - 32 = 0$ is 4, determine a.

7. Solve $(x^2 + \frac{x}{2} - 10)^2 - (x^2 - \frac{x}{2} - 8)^2 = 0$.

8. Find the difference between the largest and smallest roots of

   a) $x^4 - 13x^2 + 36 = 0$;

   b) $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$.

9. Find the sum of all roots of

   $(x^2 - 7x + 2)^2 - 13(x^2 - 7x) - 26 = 0$.

The real danger is not that computers will begin to think like men, but that men will begin to think like computers.
Harris, Sydney J. (1917–1986), American journalist

III.3. Simultaneous linear equations

Terms

1. **simultaneous linear equations** – chiziqli tenglamalar sistemasi|система линейных уравнений;
2. **to substitute** – o'rniga qo'ymoq|подставлять;
3. **infinite solutions** – cheksiz ko'p yechim|бесконечно много решений;
4. **method of elimination** – noma'lumlarni yo'qotish usuli|метод исключения неизвестных;
5. **method of successive substitutions** – noma'lumlarni o'rniga qo'yish usuli|метод последовательных подстановок;
6. **double equation** – ikkitalik (qo'sh) tenglama|двойное уравнение;
7. **total** – yig'indi|сумма;
8. **two (three) times as much (many) as** – ikki (uch) barobar ko'p|два (три) раза больше;
9. **half (twice) as much as** – ikki barobar kam (ko'p)|два раза меньше (больше);
10. **two thirds (three fourth)** – uchdan ikki (to'rtdan uch)|две трети (три четвертых).

Learning Objectives

- to formulate problems as simultaneous equations;
- to solve simultaneous equations using the methods of elimination and of successive substitutions.

Two linear simultaneous equations

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases},$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constant numbers.

Examples

1. Solve by the method of substitution

$$\begin{cases} 12x+y=4 \\ 23x-y=66 \end{cases}.$$

▶ Use one equation to express one unknown in terms of the other and substitute the result into the other equation.

$$\begin{cases} 12x+y=4 \\ 23x-y=66 \end{cases} \Rightarrow \begin{cases} y=4-12x \\ 23x-y=66 \end{cases} \Rightarrow$$

$$\begin{cases} y=4-12x \\ 23x-(4-12x)=66 \end{cases} \Rightarrow \begin{cases} y=4-12x \\ x=2 \end{cases} \Rightarrow$$

$$\begin{cases} y=4-12\cdot 2=-20 \\ x=2 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=-20 \end{cases}. \blacksquare$$

2. Use the method of elimination to solve

$$\begin{cases} 14x+y=30 \\ 15x-2y=26 \end{cases}.$$

▶ Eliminate y by multiplying the first equation by 2 and then adding the two equations.

$$\begin{cases} 14x+y=30 \quad |\cdot 2 \\ 15x-2y=26 \end{cases} \Rightarrow \begin{cases} 28x+2y=60 \\ 15x-2y=26 \end{cases} (+) \Rightarrow$$

$$\begin{cases}43x=86\\15x-2y=26\end{cases}\Rightarrow\begin{cases}x=2\\15\cdot2-2y=26\end{cases}\Rightarrow\begin{cases}x=2\\y=2\end{cases}. ■$$

3. Solve the following three simultaneous equations

$$\begin{cases}2x+y+3z=14\\3x-y-z=2\\x+y+z=6\end{cases}.$$

▶ We will use both substitution and elimination. First express y in terms of x and z in the first equation and substitute into the 2nd and 3rd equations. Then, solve the system of two equations on x and z by elimination.

$$\begin{cases}y=14-2x-3z\\3x-(14-2x-3z)-z=2\\x+(14-2x-3z)+z=6\end{cases}\Rightarrow\begin{cases}y=14-2x-3z\\5x+2z=16\\-x-2z=-8\end{cases}\Bigg|(+)\Rightarrow$$

$$\begin{cases}y=14-2x-3z\\4x=8\\-x-2z=-8\end{cases}\Rightarrow\begin{cases}y=14-2x-3z\\x=2\\-2-2z=-8\end{cases}\Rightarrow\begin{cases}y=1\\x=2\\z=3\end{cases}. ■$$

4. A meal made with four eggs and $60g$ cheese contains 560 calories. Another meal made with six eggs and $20g$ cheese also contains 560 calories. How many calories does one egg contain?

▶ Let the calories of 1 egg and $1g$ of cheese be x and y, respectively. Then,

$$\begin{cases}4x+60y=560\\6x+20y=560\,|\cdot(-3)\end{cases}\Rightarrow\begin{cases}4x+60y=560\\-18x-60y=-1680\end{cases}\Bigg|(+)\Rightarrow$$

$$\begin{cases}-14x=-1120\\4x+60y=560\end{cases}\Rightarrow\begin{cases}x=80\\y=4\end{cases}.$$

Thus, one egg contains 80 calories. ■

Exercises

1. Which of the following will be the solution of $\begin{cases} 2x+y=13 \\ 4x-3y=11 \end{cases}$

   a) (4, 5); b) (1, 11); c) (5, 3); d) (8, 7)?

2. Which of the following will be the solution of $\begin{cases} 3x-4y+10z=14 \\ 10x-y-2z=36 \\ x+y+z=7 \end{cases}$

   1) (12, 8, 1); 2) (4/3, 0, 1); 3) (0, −1, 1); 4) (4, 2, 1)?

3. Solve by substitution

   a) $\begin{cases} x+y=4 \\ x-y=2 \end{cases}$; b) $\begin{cases} x+y=5 \\ x-y=-1 \end{cases}$;

   c) $\begin{cases} x+y=7 \\ x-y=1 \end{cases}$; d) $\begin{cases} 4x-3y=-4 \\ 4y-10x=3 \end{cases}$.

4. Solve by eliminating either the x term or the y term

   a) $\begin{cases} 4x-y=13 \\ 2x-y=5 \end{cases}$; b) $\begin{cases} x+3y=10 \\ 2x-3y=2 \end{cases}$;

   c) $\begin{cases} 3x+2y=12 \\ 2x+y=7 \end{cases}$; d) $\begin{cases} 7y-3x=2 \\ 5y-2x=2 \end{cases}$.

5. Find x, y and z

   a) $\begin{cases} x+y+z=9 \\ x-2y-z=1 \\ 2x+y+z=14 \end{cases}$; b) $\begin{cases} x-y+3z=7 \\ 2x+y-2z=7 \\ 3x+y-z=13 \end{cases}$.

   c) $\begin{cases} x-y-z=5 \\ x+2y-3z=4 \\ 2x+5y+2z=25 \end{cases}$; d) $\begin{cases} 3x-4y+2z=-1 \\ 4x+2y-7z=52 \\ 5x-3y+3z=7 \end{cases}$.

6. Six apples and four oranges cost £1.90, whereas eight apples and two oranges cost £1.80. Find the cost of an apple and an orange.

7. Solve the double equation $\frac{3(x-y)}{5}=x-3y=x-6$.

8. If $\begin{cases} x+2y=2 \\ 2x+y=k \end{cases}$, then for what value of k will $x+y=2$ be true?

9. If $a+b=5$ and $ab=2$, then what is a^4+b^4 equal to?

10. The combined total of the annual salaries of Otabek and Barchinoy was 2,400 thousand soums in 2005. In 2006 Otabek and Barchinoy earned 25% and 12.5% more than in the previous year, respectively, which totaled 2,880 thousand soums. Find the monthly salary of each in 2006.

11. The ninth graders at a high school are raising money by selling T-shirts and baseball caps. The number of T-shirts sold was three times the number of caps. The profit they received for each T-shirt sold was 1,000 soums, and the profit on each cap was 500 soums. If the students made a total profit of 42,000 soums, how many T-shirts and how many caps were sold?

12. The Taxi Luxe service in Tashkent charges x soums for the first 6 kilometers and y soums for every next kilometer. Anvar used the taxi service to travel 10 kilometers and paid 2,300 soums. Shoira used the taxi service to travel 15 kilometers and paid 3,300 soums. Find out how much the taxi services charges for the first 6 kilometers and for every additional kilometer afterwards.

13. A box contained 31 sweets. On Monday Malika ate three fourths of the number Sarvar ate on Monday. On Tuesday Malika ate two thirds of the number Sarvar ate on Tuesday. Then all the sweets had been eaten. Find the number of sweets Malika ate.

14. A store sold 213 bicycles during a year. For the first few months they sold 20 bicycles per month, then for some months they sold 16 bicycles per month and in the remaining month(s) they sold 25 bicycles per month. For how many months did they sell 25 bicycles per month?

15. Altogether Olim, Alan and Farrukh earned \$104. Alan earned twice as much as Olim and Farrukh earned \$4 more than Alan. How much did each person earn?

16. A group of 35 friends went on an excursion and each spent on average 1,434 soums for his or her expenses and souvenirs. If each girl spent on average 1,470 soums and each boy on average 1,400 soums, how many girls and how many boys took part in the excursion?

Homework

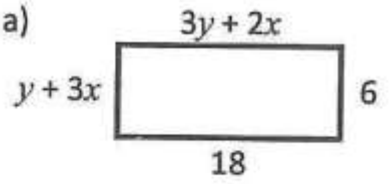
1. Solve for x and y

a) $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$; b) $\begin{cases} 3x-4y=3 \\ x+2y=1 \end{cases}$;

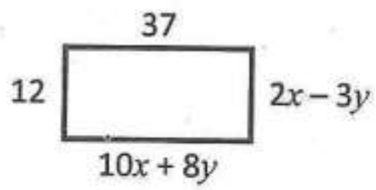
c) $\begin{cases} x-12y=16 \\ 5x+12y=8 \end{cases}$; d) $\begin{cases} 5x+3y=24 \\ x+5y=-4 \end{cases}$.

2. Solve the simultaneous equations $y+2x=7$ and $y=x-2$.
3. The ages of a boy and a girl add up to 20. In 4 years time the age of the girl will be three times as much as the age of the boy. How old will each be in 2 years time?
4. Find the value of x and y for each of the following rectangles, by first writing down a pair of simultaneous equations and then solving them

a)



b)



5. Solve

a) $\begin{cases} 4r+2s=56 \\ r+5s=50 \end{cases}$; b) $\begin{cases} 6p+2q=38 \\ 6q-5p=22 \end{cases}$;

c) $\begin{cases} 3x+8=2y \\ 5y=1-2x \end{cases}$.

6. Shirin is three times as old as Malika will be next year. Nargiza is now eight times as old as Malika, and in twelve years she will be three times as old as Shirin is now. How old is Nargiza now?

7. If $x + y = 4$, $y + z = 7$ and $x + z = 5$, what is the value of $(x + y + z)^2$?

8. At Oloy market 2 kilograms of nuts and 3 kilograms of raisins cost 23,000 soums, while 4 kilograms of nuts and 2 kilograms of raisins cost 26,000 soums. Find the prices of one kilogram of nuts and one kilogram of raisins.

9. A sum of money was divided between A, B and C. C got twice as much as A, and A and B together got £50. When A and C gave a fifth of their money to charity and B gave a tenth, £10 were collected. What was the original sum of money equal too?

Developments of mathematics before the 15th century AD

Middle East and Central Asia.

Ibn Sina (in Europe known as Avicenna, 9th-10th centuries) contributed to philosophy, medicine, psychology, geology, mathematics, astronomy and logic. His book "Kitab al-Shifa" (The Book of Healing) divides mathematics into four major groups: geometry, astronomy, arithmetic and music. **Al-Battani** (late 9th-early 10th centuries AD) introduced trigonometrical functions of sine, cosine and tangent in his book "Kitab al-Zij" consisting of 57 chapters. Abu Rayhan **Al-Biruni** (973-1048 AD), a well-known scholar from Khwarazm, an author of over 150 works (of which only 30 survive) on various topics such as arithmetic, summation of series, combinatorial analysis, irrational numbers, ratio theory, algebraic definitions, methods of solving algebraic equations, geometry, trisection of the angle and other problems which can not be solved with ruler and compass alone, conic sections, stereometry, stereographic projection, trigonometry, the sine theorem in the plane and solving spherical triangles. Biruni expressed the problem of finding the side of a regular nonagon as a cubic equation $x^3+1=3x$ and obtained an approximate solution in sexagesimal fraction. Also, the problem of finding the side of regular septangle (heptagon) was reduced to the equation $x^3+1=2x+x^2$. **Abu al-Wafa** (940-998 AD) translated the works of Diophantus into Arabic, explained all six trigonometrical functions and produced the sine table. The Persian mathematician **Omar Khayyam** (11th-12th centuries AD) systematically analyzed the third degree equations, presented their classification and determined the conditions of their solvability (in the sense of existence of positive roots). He finds the duration of the year very precisely to be 365.24219858. He wrote "Discussions of the difficulties in Euclid" (a book of flaws in Euclid's "Elements"). He explained the method of extracting the root of any natural degree in his work, but the work has not reached us; later, **Nasireddin Tusi** (1201-1274 AD) also described them, and formulated orally the binomial theorem of Newton and the rule of finding binomial coefficients: ($C_n^m=C_{n-1}^{m-1}+C_{n-1}^m$.) He was the first to treat trigonometry separately from astronomy. The beautiful end of the epoch was the work of **Mirzo Ulugbek** (1394-1449 AD), who gathered over hundred scholars at the observatory at court in Samarqand and organized the precise astronomical observations, calculations of mathematical tables, etc for which he is famous (Ashraf A 1994). In 1444 he finished his work "Ziji Jadidi Kuragoni" consisting of four books, which talks about different calendars, mathematics, spherical astronomy, mathematical geography, astrology and trigonometry. **Al-Kashi** worked in Ulugbek's observatory and wrote the book "The Key of Arithmetics", in which for the first time he explained the theory of decimal numbers, gave the rule of extracting roots of any index from whole numbers and the table of binomial numbers. **Al-Kushchi**, a student of Ulugbek, wrote commentaries to the tables of Ulugbek (which did not survive) and chaired the observatory after his death.

When people thought the Earth was flat, they were wrong. When people thought the Earth was spherical they were wrong. But if you think that thinking the Earth is spherical is just as wrong as thinking the Earth is flat, then your view is wronger than both of them put together.
Isaac Asimov (1920-1992), Russian-born American writer

III.4. Simultaneous nonlinear equations

Terms

1. **nonlinear equation** – nochiziqli (chiziqli bo'lmagan) tenglama | нелинейное уравнение;
2. **naught (0.5 reads 'naught point five' or 'zero point five')** – yo'q, nol (0.5 nol butun o'ndan besh deb o'qiladi) | ничто, ноль (0.5 читается как ноль целых пять десятых);
3. **arithmetic operation** – arifmetik amal | арифметическое действие (операция);
4. **sign** – ishora, belgi | знак; обозначение, символ;
5. **negative sign** – manfiy ishora | отрицательный знак числа, знак минус;
6. **to make up an equation** – tenglama tuzmoq | составлять уравнение;
7. **condition** – shart | условие;
8. **to replace** – o'rniga qo'ymoq | подставлять;
9. **technique** – usul, texnika | приём, способ, техника;
10. **unique solution** – yagona yechim | единственное решение.

Learning Objectives

– to make up simultaneous nonlinear equations based on given conditions of a problem;
– to learn the miscellaneous techniques to solve simultaneous nonlinear equations.

Methods of solving simultaneous nonlinear equations

Method of substitution.

Method of elimination.

Examples

1. Solve for x and y $\begin{cases} x+y=10 \\ xy=24 \end{cases}$.

▶ It is appropriate to use the method of substitution.

$$\begin{cases} x+y=10 \\ xy=24 \end{cases} \Rightarrow \begin{cases} x=10-y \\ (10-y)y=24 \end{cases} \Rightarrow \begin{cases} x=10-y \\ y^2-10y+24=0 \end{cases}$$

$$\begin{cases} x=10-y \\ y_1=4,\ y_2=6 \end{cases} \Rightarrow \begin{cases} x_1=6,\ x_2=4 \\ y_1=4,\ y_2=6 \end{cases}. \blacksquare$$

2. Solve the following simultaneous nonlinear equations

$$\begin{cases} \dfrac{12}{\sqrt{x-1}}+\dfrac{5}{\sqrt{y+\dfrac{1}{4}}}=5 \\ \dfrac{8}{\sqrt{x-1}}+\dfrac{10}{\sqrt{y+\dfrac{1}{4}}}=6 \end{cases}.$$

▶ The radical sign should be denoted for simplicity as follows:

$\dfrac{1}{\sqrt{x-1}}=a,\ \dfrac{1}{\sqrt{y+\dfrac{1}{4}}}=b$. Then,

$$\begin{cases} 12a+5b=5 \\ 8a+10b=6 \,|:2 \end{cases} \Rightarrow \begin{cases} 12a+5b=5 \\ 4a+5b=3 \end{cases} \Big| (-) \Rightarrow$$

$$\begin{cases} 8a=2 \\ 4a+5b=3 \end{cases} \Rightarrow \begin{cases} a=\dfrac{1}{4} \\ 4\cdot\dfrac{1}{4}+5b=3 \end{cases} \Rightarrow \begin{cases} a=\dfrac{1}{4} \\ b=\dfrac{2}{5} \end{cases}.$$

Now we replace these values of a and b back into our designations to find x and y.

$$\begin{cases} \dfrac{1}{\sqrt{x-1}}=\dfrac{1}{4} \\ \dfrac{1}{\sqrt{y+\dfrac{1}{4}}}=\dfrac{2}{5} \end{cases} \Rightarrow \begin{cases} \sqrt{x-1}=4 \\ 2\sqrt{y+\dfrac{1}{4}}=5 \end{cases} \Rightarrow$$

$$\begin{cases} x-1=16 \\ 4\left(y+\dfrac{1}{4}\right)=25 \end{cases} \Rightarrow \begin{cases} x=17 \\ y=6 \end{cases}. \blacksquare$$

Exercises

1. Solve

a) $\begin{cases} x+4=0 \\ xy^2=-12 \end{cases}$; b) $\begin{cases} x+y=4 \\ xy=-5 \end{cases}$;

c) $\begin{cases} x+y=3 \\ x^2-y^2=6 \end{cases}$; d) $\begin{cases} y-x^2=-2 \\ y-3x=8 \end{cases}$.

2. Find all solutions of

a) $\begin{cases} y-x^3=0 \\ y-16x=0 \end{cases}$; b) $\begin{cases} x^3+y^3=7 \\ x^3y^3=-8 \end{cases}$;

c) $\begin{cases} x^2-y=23 \\ x^2y=50 \end{cases}$.

3. Solve for x and y

a) $\begin{cases} x^2+xy+y^2=13 \\ x+y=4 \end{cases}$; b) $\begin{cases} x^2-xy+y^2=7 \\ x-y=1 \end{cases}$;

c) $\begin{cases} x+y+xy=7 \\ x^2+xy+y^2=13 \end{cases}$.

4. On what value of a does $\begin{cases} x+y=a \\ xy=9 \end{cases}$ have a unique solution?

5. If $x+y=A\neq 0$ and $x^3+y^3=B$ express

a) x^2y+xy^2 and b) xy

in terms of A and B.

6. Solve

a) $\begin{cases} x^2y+xy^2=6 \\ xy+x+y=5 \end{cases}$; b) $\begin{cases} (x+1)(y+1)=10 \\ (x+y)(xy+1)=25 \end{cases}$;

c) $\begin{cases} x^2+y^2=25 \\ (x-3)(y-5)=0 \end{cases}$; d) $\begin{cases} x^2+xy=15 \\ y^2+xy=10 \end{cases}$.

7. Find the product xyz, if

a) $\begin{cases} x^2y^3=8 \\ x^3y^2=4 \\ zx=6 \end{cases}$; b) $\begin{cases} xy=6 \\ yz=12 \\ zx=8 \end{cases}$; c) $\begin{cases} \dfrac{xy}{x+y}=\dfrac{2}{3} \\ \dfrac{yz}{y+z}=\dfrac{4}{5} \\ \dfrac{xz}{x+z}=\dfrac{4}{7} \end{cases}$.

Homework

1. Solve

a) $\begin{cases} x+2=1 \\ x^2y^3=8 \end{cases}$; b) $\begin{cases} x+y=7 \\ xy=12 \end{cases}$;

c) $\begin{cases} x-y=4 \\ xy=5 \end{cases}$;

d) $\begin{cases} 3x+y=2 \\ x^2-xy+6y=-4 \end{cases}$.

2. Solve for x and y

a) $\begin{cases} x^2+y^2=10 \\ x+y=4 \end{cases}$;

b) $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=13 \\ \dfrac{1}{x}+\dfrac{1}{y}=1 \end{cases}$;

c) $\begin{cases} (x+y)(x-y)=0 \\ 2x-y=1 \end{cases}$.

3. Solve the simultaneous equations

$$\begin{cases} \dfrac{7}{\sqrt{x-7}}-\dfrac{4}{\sqrt{y+6}}=\dfrac{5}{3} \\ \dfrac{5}{\sqrt{x-7}}+\dfrac{3}{\sqrt{y+6}}=\dfrac{13}{6} \end{cases}.$$

4. Find a two digit number if the difference between its digits is equal to 2 and the sum of squares of its digits is equal to 52.

Developments of mathematics before the 15th century AD

Europe

For Europe the 12th-15th centuries were the period of adoption of the heritage of ancient time and the East. The translation of Greek and Arabic works into Latin led to a rise in mathematical study in Europe. The English philosopher **Adelard** translated Al-Khwarizmi's astronomical tables and an Arabic version of Euclid's "Elements" into Latin in the 12th century. Italian mathematicians such as **Leonardo Fibonacci** and **Luca Pacioli** depended heavily on Arabic sources in improving business mathematics used for accounting and trade. Fibonacci's "Liber Abaci" (Book of the Abacus, 1202) introduced Arabic numbers, the Hindu-Arabic place-value decimal system, and Arabic algebra to Europe. It also introduces the famous sequence of numbers now called the "Fibonacci sequence". Fibonacci also wrote "Liber Quadratorum" (The Book of the Square), his most impressive work. It is the first major European advance in number theory since the work of Diophantus a thousand years earlier. In about 1336 mathematics became a compulsory subject for a degree at the University of Paris. In the mid 15th century **Nicholas of Cusa** studied the concepts infinitely large (and small). He explained a circle as a limit of regular polygons. **Widman** wrote an arithmetic book in German and introduced the symbols "+" and "–". **Del Ferro** and **Tartaglia** independently found the formula to solve cubic equations. In 1536 **Hudalrichus Regius** found the fifth perfect number ($2^{12}(2^{13}-1)=33{,}550{,}336$), which is the first perfect number to be discovered since ancient times. In 1543 **Copernicus** published his book "On the Revolutions of Heavenly Spheres", in which he explains Copernican theory, in particular proposes that the Sun (not the Earth) is at rest in the center of the Universe. In 1555 **Scheybl** gave the sixth perfect number ($2^{16}(2^{17}-1)=8{,}589{,}869{,}056$), which remains unknown until 1977.

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.
Albert Einstein (1879-1955), German-born American physicist

III.5. Arithmetic and geometric means

Terms

1. **arithmetic mean** – arifmetik o'rta qiymat|арифметическое среднее значение;
2. **geometric mean** – geometrik o'rta qiymat|геометрическое среднее значение;
3. **average** – o'rta qiymat|среднее значение;
4. **concept** – tushuncha|понятие, концепция;
5. **less than** – kichik|меньше;
6. **greater than** – katta|больше;
7. **to figure out** – hisoblamoq, topmoq|вычислять, находить;
8. **fifth root** – beshinchi darajali ildiz|корень пятой степени;
9. **to extract a root** – ildiz chiqarish|извлекать корень;
10. **quantity** – son, miqdor|число, количество, величина.

Learning Objectives

- to introduce the concepts of arithmetic and geometric mean;
- to be able to find the arithmetic and geometric means of numbers.

Arithmetic mean of the numbers $a_1, a_2, \ldots, a_n$

$$\frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Geometric mean of the numbers $g_1, g_2, \ldots, g_n$

$$\sqrt[n]{g_1 \cdot g_2 \cdots g_n}.$$

Examples

1. Find the average (arithmetic mean) of the numbers:

   13, 25, 28, 37, 81, and 92.

▶ The arithmetic mean of six numbers is the sum of the six numbers divided by the quantity of the numbers. So,

$$\text{Arithmetic mean} = \frac{13+25+28+37+81+92}{6} = \frac{276}{6} = 46 . \blacksquare$$

2. Find the geometric mean of the numbers:

   24, 28, 49, 147, and 27.

▶ The geometric mean of five numbers is the product of the five numbers extracted from the fifth root. So,

$$\text{Geometric mean} = \sqrt[5]{24 \cdot 28 \cdot 49 \cdot 27 \cdot 147} =$$

$$\sqrt[5]{\left(2^3 \cdot 3\right) \cdot \left(2^2 \cdot 7\right) \cdot \left(7^2\right) \cdot \left(3^3\right) \cdot \left(3 \cdot 7^2\right)} = \sqrt[5]{\left(2 \cdot 3 \cdot 7\right)^5} = 42 . \blacksquare$$

Exercises

1. Find the arithmetic mean of the following numbers

   1, 2, −2, 0, 1, 8, 3, −3, 2, 4, −2 and 2.
2. One number is less than the other by 8, their arithmetic mean is 21. Find these numbers.
3. Malika gained on average for three tests 83 points. Her average grade for the first two tests is 76 points. Figure out her grade for the last test.
4. Find the geometric mean of 40, 50 and 32.
5. How many times is the arithmetic mean of numbers 4 and 64 greater than their geometric mean?
6. The arithmetic mean of 5 numbers is 13. What number should be added to them so that their arithmetic mean becomes 19?

7. The average height of six children is 120cm. The height of the shortest child is 105cm. What is the average height of the other children?

8. If $2^a \cdot 2^b \cdot 2^c = 256$, what is the average of a, b and c?

9. The cost of item A is $y/2$ each and the cost of item B is $t/3$ each. What is the average (arithmetic mean) cost per unit of a collection consisting of x units of A and k units of B?

10. The average weight of 6 travel bags is 6kg. If two of them weigh 18kg altogether, then what is the average weight of the remaining 4 travel bags?

11. A class of 20 students in a school has average grade 64% in Mathematics. Another class of 30 students has average grade 69% in Mathematics.

    a) What is the average grade of all 50 students together?

    b) When one student left the school, the average Mathematics grade of the remaining students became 67.5%. What was the grade of the student who left the school?

Homework

1. Find the arithmetic mean of 83, 87, 81 and 90.
2. Find the geometric mean of numbers 2, 4, 8, 16 and 32.
3. A tourist walked 10km at the speed of 5km/h and rode a bike 60km at the speed of 20km/h. Find the average speed of the tourist.
4. John works a variety of different jobs. On Monday he earned \$50. Tuesday he earned \$40. On Wednesday and Thursday he earned \$30 each day, and on Friday he earned \$100. What was John's average daily pay for the 5 days?

Development of mathematics in the 16th century AD

In the 16th century mathematicians used mixed writing – some words and mathematical symbols. For example, x^3+5x=18 would be written by **G. Cardano** (1545) as: cubus p*5 positionibus aequantur 18 (cubus – "cube", positio – "unknown", aequantur – "equal"); by **R. Bombelli** (1572) as: ^3p*5^1 equale a 18 (^{3} – "unknown's cube", ^{1} – "unknown", equale a – "equal"); by **F. Viete** (1591) as: C.++5N aequantur 18 (C – subus – "cube", N – numerus "number"); and by **Th. Harriot** as: aaa+5*a = 12.

Science and art belong to the whole world, and before them vanish the barriers of nationality.
Goethe (1749-1832), German poet, dramatist, novelist and scientist

Mathematics knows no races or geographic boundaries.
David Hilbert (1862-1943), German mathematician and mathematical philosopher

III.6. Ratios and proportions

Terms

1. **ratio** – nisbat|отношение;
2. **constant** – konstanta|константа;
3. **direct and inverse proportion** – to'g'ri va noto'g'ri nisbat|прямая и обратная пропорциональность;
4. **a value of** x – x ning qiymati|значение x;
5. **to cross-multiply** – diagonal bo'yicha ko'paytirmoq|умножать по диагонали;
6. **variable** – o'zgaruvchi|переменная;
7. **to represent** – ifodalamoq, ko'rsatmoq|представлять;
8. **approximate** – taqribiy|приблизительный;
9. **to complete a table** – jadvalni to'ldirmoq|заполнять таблицу;
10. **to imply** – anglatmoq, kelib chiqmoq|означать, следовать;
11. **scaling factor** – masshtab ko'paytuvchisi|масштабный множитель.

Learning Objectives

- to understand ratio and proportion;
- to be able to use ratio and proportion to solve problems.

Ratio

$a:b$ or $\frac{a}{b}$.

Proportion

$a:b=c:d$ or $\frac{a}{b}=\frac{c}{d}$.

Direct proportion

$\frac{y}{x}=k\ (y\propto x)$.

Inverse proportion

$yx=k\left(y\propto \frac{1}{x}\right)$.

Division of an amount in a certain ratio

Multiply the amount by corresponding scaling factor.

For example,

60 divided in the ratio 2:3 is

$60\cdot\frac{2}{2+3}=24$ and $60\cdot\frac{3}{2+3}=36$.

Distance of a vehicle

$S=vt$,

where S is a distance, v is a velocity (speed) and t is a time.

Examples

1. A fish is cut into three pieces, which are in ratio 3:4:2 by weight. If the second piece weighs 400 grams, find the weight of the whole fish.

▶ 1st method: Let's denote the total weight of the fish by x.

Then, we can make up the equation for the second piece's weight as

$$x \cdot \frac{4}{3+4+2} = 400,$$

which solved results in $x = 1{,}000$.

2nd method: Let's denote the weights of the three pieces by A, B and C. Then, we can consider A:B:C = $3x$:$4x$:$2x$. In other words, A = $3x$, B = $4x$ and C = $2x$. We are given that the weight of the second piece is 400 grams, that is B = $4x$ = 400. This implies that $x = 100$ is a scaling factor.

Hence, A = $3x$ = 300 and C = $2x$ = 200.

Finally, the weight of the whole fish is A + B + C = 900 grams. ■

2. A trip takes 5 hours in a car moving 40 miles per hour. How long would the trip take in a train moving at 100 miles per hour?

▶ There are two variables: the speed of the car (v) and the time (t). These variables are inversely related, because the higher the speed of the car, the less time will be needed to travel a distance. This means, their product remains constant

$$v \cdot t = const = 40\frac{m}{h} \cdot 5h = 100\frac{m}{h} \cdot t \Rightarrow$$

$$t = \frac{40\frac{m}{h} \cdot 5h}{100\frac{m}{h}} = \frac{200h}{100} = 2 \ hours.$$ ■

Exercises

1. If 24 of the 40 students in a class are girls, what is the ratio of boys to girls in the class?
2. Two books cost \$90. The ratio of the first book's price to the price of the second book is as 1:2. Find the prices of these books.

3. An alloy consists of silver and 0.45kg of gold. Find the weight of the alloy, if the ratio of silver's weight to the weight of gold is equal to 3:5.

4. Represent the number 459 as a sum of three numbers, which are in ratio 1:2:6. Find the difference between the largest and smallest of these numbers.

5. £640 is shared between Anne, Bill and Carl in the ratio 4:5:7. Calculate how much each person receives.

6. The ratio of sheep to chickens to goats on a farm can be expressed as the triple ratio 6:15:4. If there are 120 chickens on the farm, find the number of goats.

7. The scale (ratio) of a topological map is 1:1,000,000. Find the distance of a geographical place if it has the following length on the map:

   a) 1.5mm; b) 2.8cm.

8. Find x from the proportion $12 : x = 6 : 5$.

9. In one minute a tortoise crawls 50cm. How many hours will it take for it to crawl 0.1km?

10. If three apples cost 50¢, how many apples can you buy for $20?

11. On Earth there are about 10^{16} ants and $6 \cdot 10^{9}$ humans. What is the approximate ratio of humans to ants?

12. In a mahalla (local governing community) election, Dilshod and Olim were running for chairman. There were 30,500 people eligible to vote, and three fourths of them actually voted. Olim received one third of the votes cast. How many people voted for Olim?

13. Numbers $a - 2b$, 4, $a + 3b$, 24 are consecutive terms of a proportion. Calculate the value of the expression $\dfrac{a^2 - b^2}{2ab}$.

14. Establish a direct or an inverse proportional relationship between variables:

    a) the time spent for study for a test and a mark for the test;

b) the number of workers at a factory and the production;

c) the difficulty of a math problem and the chance to solve the problem;

d) the price of a TV-set and the demand for the TV-set.

15. y is directly proportional to x. If $y = 5$ when x is 25, find y when x is 100.

16. If $y \propto x$ and $y = 132$ when $x = 10$, find the value of y when $x =$ 14.

17. Numbers a and $b^2 - 3$ are directly related. $a = 88$, when $b = 5$. Find a, when $b = 3$.

18. Complete the following tables, where $y \propto x$

a)

x	3	6	9
y		9	

b)

x	27		
y	5	10	15

c)

x	2	4	6
y		10	

19. If $y = 3$ *when* $x = 8$ and y and x vary inversely, find the value of y when $x = 12$.

20. A hospital has enough pills on hand to treat 10 patients for 14 days. How long will the pills last if there are 35 patients?

21. Complete the following table, where $y \propto \frac{1}{x}$

x	1	2	3	4	5	6
y					9.6	

22. If 15 workers can paint a certain number of houses in 24 days, how many days will 40 workers take, working at the same rate, to do the same job?

23. One-fifth of the light switches produced by a certain factory are defective. Four-fifths of the defective switches are rejected and 1/20 of the non-defective switches are rejected by mistake. If all the switches not rejected are sold, what part of the switches sold by the factory is defective?

Homework

1. If the ratio of boys to girls in a class is 5:3 and the total number of students in the class is 32, figure out the number of boys and girls.
2. If a apples cost c cents, how many apples can be bought for d dollars?
3. The price of solution A is 1,000 soums for $1kg$. The price of solution B is 2,000 soums for $1kg$. Find the price of $1kg$ of the solution made of B and A in ratio 3:1.
4. Find the smallest part of the number 25.5, when it is broken into three parts proportional to numbers 7, 8, 2.
5. If $y \propto \frac{1}{x}$ and $x = 4$ when $y = 5$, find the value of x when $y = 10$.
6. Find unknowns from

   a) $2x:7 = 18:5$;　　　　b) $12.5:2.5 = 16.\dot{6}:y$.
7. If a person earns \$5.15 per hour, how much does he earn in 7 hours?
8. The current ratio of men to women in a committee is 2 to 5. If 4 men were added to the committee, the ratio of men to women would be 2 to 3. How many men are currently in the committee?
9. An expenditure budget in a company is split between three departments (Marketing, Production and Quality) in the ratio 12:3:5. If the total budget is \$40,000, find how much money each department receives.

Development of mathematics in the 17th century AD

The 16th century was the first century in which West Europe excelled over the ancient world and the East with the inventions of **Copernicus** in astronomy and the discoveries of **Galileo** in mechanics, the usage of symbols, such as +, −, x, =, >, < that make the algebraic thinking and writing simpler, the systematic use of letters for variables in equations introduced by the French mathematician **François Viète's**, and the solutions of third and fourth degree equations. In 1603 the Accademia dei Lincei was founded in Rome. The biggest invention of the century was calculus (differential and integral), which was made by **Isaac Newton** in England and **Wilhelm Leibniz** in Germany (there has been a dispute between them on who first invented calculus, but it was later proved they did it independently). In 1612 **Bachet** published a work on mathematical puzzles and tricks, which became the main reference for later works. He finds the method of constructing magic squares. The Scottish mathematician **John Napier** and the Swiss mathematician **Justus Byrgius** independently introduced the concept of logarithms. In 1621 **Bachet** translated Diophantus's Greek text "Arithmetica" into Latin. In 1623 **Schickard** built a "mechanical clock", a wooden calculating machine that adds and subtracts and aids with multiplication and division. French mathematicians **Rene Decartes** and **Pierre de Fermat** discovered the analytic geometry (analytic geometry enables the study of geometric figures using algebraic equations). Rene Descartes introduced x and y coordinates (Cartesian system of coordinates). Fermat greatly contributed to mathematics in differential calculus (working on maxima and minima) and number theory. He made his famous conjecture known as Fermat's Great theorem. In 1639 **Desargues** began the study of the projective geometry (a branch of mathematics, which considers what happens to shapes when they are projected on to a non-parallel plane). In 1649 **De Beaune** gave the equations of hyperbolas, parabolas and ellipses. **Blaise Pascal** and Fermat set the groundwork for the investigations of probability theory and the corresponding rules of combinatorics in their discussions over a game of gambling. In 1656-1657 **Huygens** invented the pendulum clock and introduced the term of "mathematical expectation" in probability theory. In 1660 **Viviani** measured the velocity of sound. In 1662 the Royal Society of London was founded, which promotes natural sciences including mathematics. In 1663 **Barrow** became the first Professor of Mathematics at the University of Cambridge in England. In 1666 the Academie des Sciences in Paris was founded. In 1669 **Wallis** published his "Mechanica" (Mechanics), in which he analyzes the study of mechanics in detail. In 1675-1677 Leibniz introduced the modern notation for integral (later the term "coordinate"), discovered differentials of elementary functions independently of Newton and found the rules of differentiating products, quotients and the function of a function. Later he found the method of integrating rational functions by expanding to simple fractions. Moreover, Leibniz formulated the principle of using the binary number system, which is used by computers nowadays. In 1691 **Jacob Bernoulli** discovered the polar coordinates (a method for identifying a point in space using angles and distances). In 1693 **Johann Bernoulli** found "L'Hopital rule".

It isn't enough just to learn – one must learn how to learn, how to learn without classrooms, without teachers, without textbooks. Learn, in short, how to think and analyze and decide and discover and create.

Michael Bassis

Try to learn something about everything and everything about something.

Thomas Henry Huxley (1825-1895), British biologist

III.7. Percentages

Terms

1. **original amount (value)** – boshlang'ich miqdor (qiymat)|первоначальная величина (значение);
2. **percent** – foiz|процент;
3. **percentage increase (decrease)** – foiz o'sishi (kamayishi)|процентное увеличение (понижение);
4. **to label** – belgilamoq|обозначать;
5. **to cross multiply** – diagonal bo'ylab ko'paytirmoq|умножать по диагонали;
6. **diagram** – rasm, diagramma|рисунок, диаграмма;
7. **graph** – rasm, grafik|рисунок, график;
8. **to result in** – kelib chiqmoq|означать, вытекать;
9. **figure** – qiymat, figura|значение, фигура;
10. **respectively** – mos ravishda|соответственно.

Learning Objectives

- to know how to convert a percent to a number and vice versa;
- to learn to work with percentages, in particular, to find a certain percent of a given amount.

---

1 percent (1%) = 100^{th} part of a number.

For example, 1% of 350 = $\frac{350}{100} = 3.5$.

To convert a percentage to a fraction

Divide by 100%.

For example, 145% = 145%:100% = 1.45.

To convert a fraction to a percentage

Multiply by 100%.

For example, $0.45 = 0.45 \cdot 100\% = 45\%$.

To find a percent of a number

$a\%\ of\ b = b \cdot \frac{a\%}{100\%}$.

For example, 10% of 420 = $420 \cdot \frac{10\%}{100\%} = 420 \cdot 0.1 = 42$.

To find a number as a percent of another

a as a percent of $b = \frac{a}{b} \cdot 100\%$.

For example, 5 as a percent of 50 is $\frac{5}{50} \cdot 100\% = 10\%$.

To increase a quantity by a percent

Change the percent to a number, add 1 and multiply by the quantity. For example, to increase 20 by 5%, we change 5% to the number 0.05, add 1 to it and multiply the result by 20 to get 21.

To decrease a quantity by a percent

Change the percent to a number, subtract from 1 and multiply by the quantity.

For example, to decrease 50 by 10%, we change 10% to the number 0.1, subtract it from 1 and multiply the result by 50 to obtain 45.

Examples

1. Karim said that 12% of the boxes of apples were not sold. Kamola said that is the same as 360 boxes of apples. How many boxes of apples were sold?

▶ Let x be the total number of boxes of apples, which is the original amount. Then we have the following:

x is 100%
360 is 12%

To find the unknown term of the proportion, one needs to cross multiply the known terms and divide be the third known term.

$$x = \frac{360 \cdot 100\%}{12\%} = 3{,}000\ . ■$$

2. In selling stock an investor made a profit of $140 plus 20% of the amount originally paid for the stock. If the cost of the stock was originally $800 what percent of the cost was the total profit?

▶ 20% of the original price is

$$\$800 \cdot \frac{20\%}{100\%} = \$160\ .$$

Total profit = $160 + $140 =$300. Total cost = $800.
We therefore have the following:

$800 is 100%
$300 is X %

The unknown term X will be found by cross multiplying $300 and 100% and dividing by $800, which will result in 37.5%. ■

3. An antiques dealer tries to sell a vase at 45% above the 180,000 soums which the dealer paid at auction. What is the new sale price?

▶ The problem is to increase the price of 180,000 soums by 45%. So, we change 45% to the number 0.45. Add 1 to it to get 1.45.

Finally, we multiply the price 180,000 by this number to find the new price of 261,000 soums. ■

Exercises

1. Calculate

   a) 10% of 1,000; b) 40% of 200;

   c) 150% of 50; d) 50% of 500.

2. What number is 15% of 420?

3. What is $\frac{3}{4}$ of $\frac{1}{10}$ of 1 percent of 100,000?

4. Find the number value of the percent:

   13.5%, 2.3%, 145%, $\frac{2}{5}\%$.

5. Express the percents as ordinary (decimal) fractions:

   50%, 12%, 40%, 25%.

6. Express the following numbers as percents

   $\frac{1}{8}$, 0.23, $\frac{12}{40}$, 0.34.

7. Dildora paid 75% of her $150 contract. How much money did she pay?

8. 90 out of 120 students have personal computers. What percent does it make?

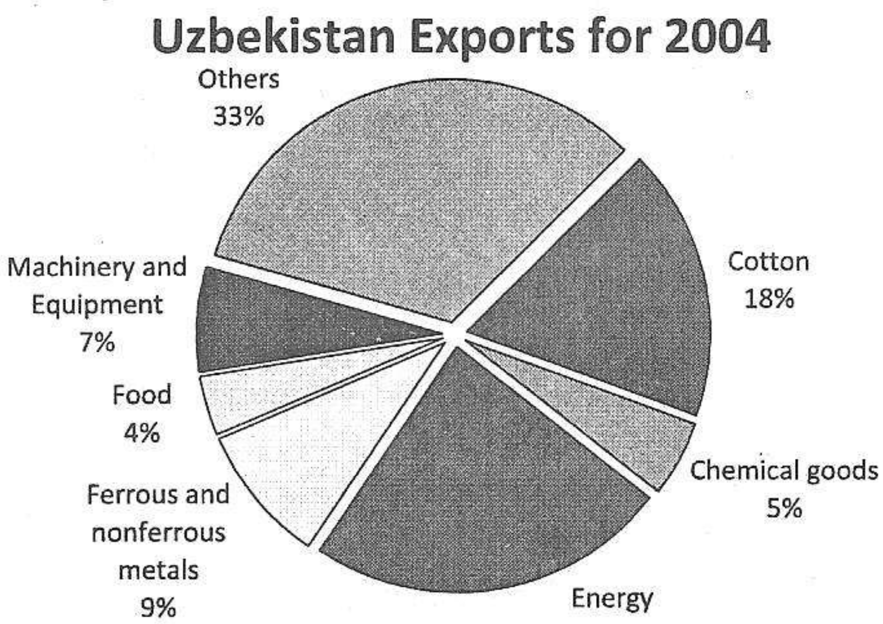
9. A teacher's wage was 20,000 soums. The government decided to increase it by 25%. How much did it become?

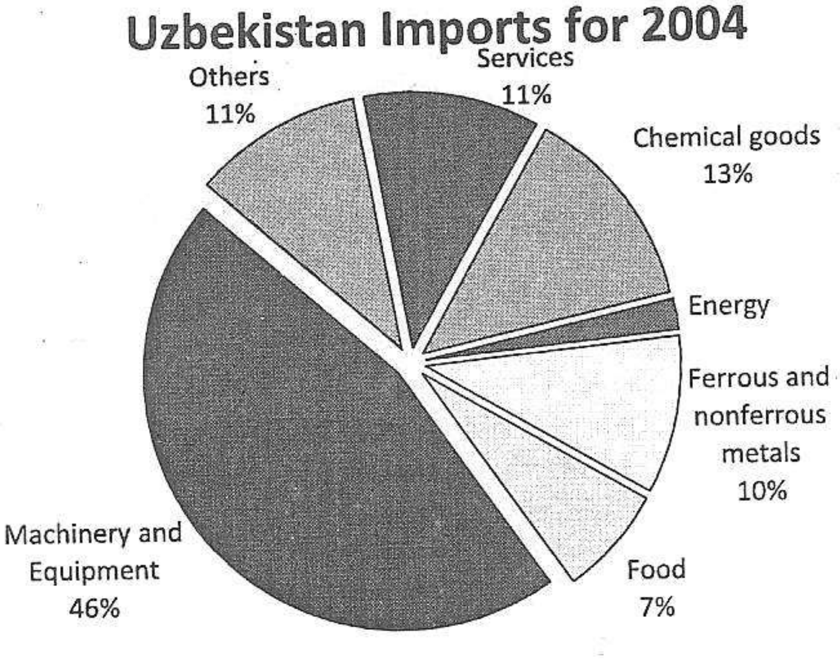
10. The salary of a worker is 20,000 soums. His salary was first raised by 20%, then after a certain period, was lowered by 20%. How much does he receive now?

11. The price of a tennis racket was reduced by 15% to $68. How much did it cost before the discount?

12. If 45% of the students at a college are male, what is the ratio of male students to female students?

13. The price of a notebook was first decreased by 15%, then by 150 soums. Now it costs 190 soums. Determine the notebook's price before the two decreases.
14. After a double increase, each time by 10%, the price of an item became 484 soums. How much did the item cost after the first increase and how much did it cost originally?
15. Three friends visited a restaurant, where there was a service charge of 5% added to the bill. If they paid 10,500 soums altogether, how much was the service charge?
16. In 2004, the total exports of Uzbekistan were 4,853 million US dollars and the total imports were 3,816 million US dollars. The diagrams below show the Uzbekistan export and import of goods (in %) for 2004.



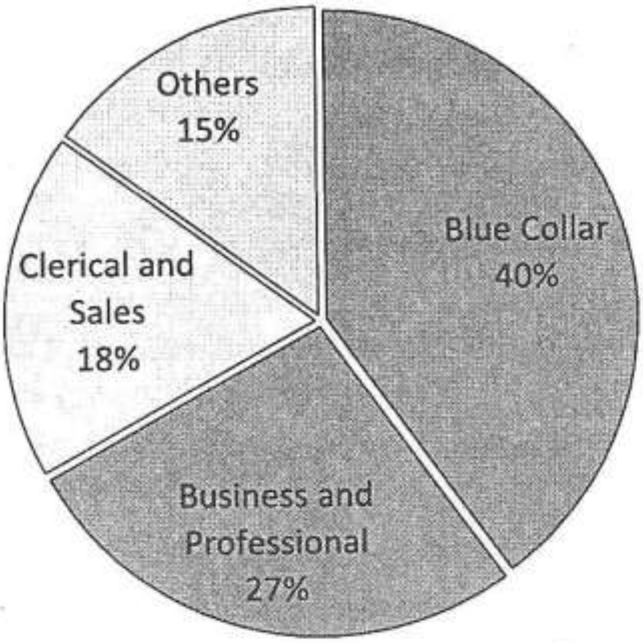


Source: The State Committee of Statistics of RUz.

a) In 2004, what percentage of total exports was due to Energy? Similarly, what percentage of total imports was due to Energy?

b) Find the difference between the Uzbek export and import of Energy in 2004.

17. The chart below shows the percent of the people in a certain survey who belonged to each of four occupational categories. Of those surveyed, if 160 were in blue-collar occupations, how many were in occupations classified as "Other"?



18. A person purchased 200 shares of stock X priced at \$25.4 per share and 100 shares of stock Y priced at \$51.7 per share. The next day the prices per share were \$24.7 and \$49.8, respectively. What is the one day percentage decrease in the total price of the 300 shares?
19. Statistics show that 10% of smokers get lung cancer, and 90% of lung cancer patients are smokers. If 20% of the population smokes, what percent of the population have lung cancer?
20. A shop has 50 computer games to sell. If they sell them at £31.50 (thirty one and a half pounds) each they make a profit of 26%. How much will their total profit from the games be if they sell half the games at £31.50, 24 of the games at £26.50 and one game is never sold?
21. A travelling salesman spends 125,000 soums to buy two kinds of cloth at 2,500 soums and 2,000 soums per meter. He manages to sell it all at 2,150 soums per meter and make a profit of 3.2%. How many meters of each cloth did he buy?

Homework

1. Compute

   a) 15% of 1,500;　　　　b) 115% of 500;

   c) 5% of 10% of 200.

2. Find the original price of a good, if its price was 1100 after a 12% decrease.

3. If $L = MN$, by what percent will L increase if M increases by 15% and N increases by 20%?

4. By what percentage is the number comprising 50% of 720 greater than the number comprising 500% of 24?

5. If x is 80 percent of y, then what percent is y of x?

6. In a small town there were 3,400 registered voters, 40 percent of whom voted in an election. If Alisher received 408 votes, what percent of the vote did he receive?

7. The Burguts and the Alps are high school basketball teams. The table below shows the final scores of their playoff games.

Teams/Games	1	2	3	4	5
Burguts	50	28	63	48	39
Alps	49	35	64	40	45

   Refer to the table above to find the percent of all games that the Alps won.

8. Of those surveyed in a poll, 60 percent were members of the "Kamolot" youth organization. If 75 percent of the Kamolot members polled were males and 55 percent of the non-Kamolot people polled were males, what percent of those polled were males?

9. The price of a model M camera is \$209 and the price of a special lens is \$69. When the camera and lens are purchased together, the price is \$239. The amount saved by purchasing the camera

and lens together is approximately what percent of the total price of the camera and lens when purchased separately?

10. At a shareholders' meeting it was decided to divide the company's profit in 2006 as follows: 15% for the company's reserve fund, 35% for the company's technical equipment, and the rest for dividends paid to shareholders. If the company's profit was 1,245,350 soums, find the amount of money allocated to the company's technical equipment and as dividend payments altogether.
11. A dealer bought a horse, expecting to sell it at a 10% profit, but he was forced to sell it for 5,000 soums less than he expected. When he calculated what had happened, he found he had lost 15% on what the horse cost him. What did he pay for the horse?

Development of mathematics in the 18th century AD

In 1715 **Taylor** published his book, in which he explains the rule of expanding a function to a polynomial. In 1724 the Academy of Sciences was founded in St. Petersburg. **Leonard Euler** studied limits and introduced the constant number e ($e \approx 2.72$). He named the square root of minus one with the symbol i. He obtained the most remarkable identity in all of mathematics: $e^{i\pi} + 1 = 0$, which links five fundamental mathematical constants: number 0, number 1, number π, number e, number i, and contains three basic arithmetic functions: addition, multiplication and exponentiation. It is said Gauss has commented that if a student did not recognize the equation, he would not be considered a first-class mathematician. **Euler's** "Introductio in analysin infinitorum" (Introduction to the Analysis of Infinites) published in 1748 interpreted calculus in terms of functions and introduced the notation $f(x)$ for the first time. **Cramer** investigated curves and proved the famous "Cramer's rule". **Lambert** proved that π was irrational. **Lagrange** proved Wilson's theorem that n is prime if and only if $(n-1)!$ is divisible by n. **Bufon** mathamtically calculated the age of the Earth to be 75,000 years. **Bezout** published his book on the theory of equations, in which he proves the theorem that bears his name. **William Herschel** discovered the planet Uranus. Lagrange published "Mecanique Analytique" (Analytical Mechanics), which summarises all the findings in the field of mechanics since Newton's time. Lagrange also published "Theory of Analytical Functions" on the theory of functions of a real variable and used the notation dy/dx for derivatives. **Karl Gauss** proved the fundamental theorem of algebra in 1799, which states that the number of roots of an equation is equal to the degree of the equation.

Chapter III Answers. Equations and simultaneous equations

III.1. 1. a) 1^{st}; b) 2^{nd}; c) 3^{rd}; d) 3^{rd}; e) 6^{th}; f) 4^{th}. **2.** a) 2; b) – 0.5; c) 3; d) 0.4 e) 3.2; f) 4. **3.** a) –2/3; b) 3; c) 4.5. **4.** AB=17; AC=17.5; BC=11.5. **5.** 28. **6.** 21; 22; 23; 24; 25. **7.** $(b-1)/(a-2b)$. **8.** 8 years old. **9.** a) –10/3; b) 4; c) 9; d) 7. e) 8. **10.** 54. **11.** 54*km*. **12.** 59*km*. **13.** 360.

Homework: 1. a) 2^{nd}; b) 5^{th}; c) 3^{rd}; d) 1^{st}. **2.** a) – 3. b) 2.4; c) 93/91; d) 5/13. **3.** Sevara had 8,700 soums. **4.** 16. **5.** \$9,000. **6.** 18.

III.2. 1. a) incomplete quadratic equation; x=0; b) incomplete quadratic equation; x_1=2; x_2=2; c) Ø; d) incomplete quadratic equation; x_1=0; x_2=3; e) incomplete quadratic equation; x_1=0; x_2= –4/3; f) incomplete quadratic equation; x_1=0; x_2=3/4. **2.** a) x_1=5; x_2=3; b) x_1=2; x_2=1/3; c) Ø; d) $x_{1,2}$=5. **3.** a) x_1=14; x_2= –2; b) x_1=2; x_2=1; c) x_1= –2; x_2= –3. **4.** a) Incomplete; p_1=0; p_2=3; b) complete; $x_{1,2}$=2; c) complete; x_1=5; x_2= –4; d) complete; x_1=3; x_2=2; e) complete; x_1=4/3; x_2=1. **5.** a= –14. **6.** a) $(x-5)(x-3)$; b) $(x-2)(x-1/3)$; c) $7(x+3)(x-4/7)$. **7.** k=2. **8.** –2.5 and 0.75. **9.** –0.75. **10.** $x_{1,2}=\pm\sqrt{2}$; $x_{3,4}=\pm 1$. **11.** p= –9; x_2= –2; x_3=1. **12.** x_1= –1; x_2=3. **13.** a) x_1= –2; x_2=1; b) $x_{1,2,3,4}$=7; –2; 3; 2; c) $x_{1,2}=\left(-3\pm\sqrt{5}\right)/2$; $x_{3,4}=1$; d) x_1=2; x_2=1. **14.** – 11/4; – 2/3. **15.** – 2; 0; 2.

Homework: 1. a) $x_{1,2}$=±7/3; b) x_1= –5; x_2=4; c) x_1=1.5; x_2= –1; d) x_1=0.5; x_2=1/3. **2.** a) x_1=5; x_2=1; b) x_1=5.5; x_2=0.5. **3.** a) x_1=3; x_2=3/4; b) x_1=2.5; x_2=1; c) x_1=5; x_2= –4/3. **4.** 6*cm* and 5*cm*. **5.** $m_{1,2}$=±4. **6.** a= –4. **7.** 2; –3; 3. **8.** a) 6; b) 13. **9.** 14.

III.3. 1. (x, y) = (5, 3). **2.** (x, y, z) = (4, 2, 1) **3.** a) (x, y)=(3, 1); b) (x, y)=(2, 3); c) (x, y)=(3, 4); d) (x, y)=(0.5, 2). **4.** a) (x, y)=(4, 3); b) (x, y)=(4, 2); c) (x, y)=(2, 3); d) (x, y)=(2, 4). **5.** (x, y, z) = (5, 0, 4); b) (x, y, z) = (4, 3, 2); c) (x, y, z) = (8, 1, 2); d) (x, y, z) = (5, 2, -4). **6.** An apple costs £0.17 and an orange costs £0.22. **7.** (x, y)=(–5/14, –15/14). **8.** k=4. **9.** 433. **10.** 150,000 and 90,000 soums. **11.** 36 T-shirts and 12 caps. **12.** 1,500 soums and 200 soums. **13.** 13. **14.** 1. **15.** Olim – \$20, Alan – \$40, Farrukh – \$44. **16.** 18 boys and 17 girls.

Homework: 1. a) (x, y)=(3, 2); b) (x, y)=(1, 0); c) (x, y)=(4, –1); d) (x, y)=(6, –2). **2.** (3, 1). **3.** Boy = 5, Girl = 19. **4.** a) (x, y)=(0, 6); b) (x, y)=(4.5, –1). **5.** a) (r, s)=(10, 8); b) (p, q)=(4, 7); c) (x, y)=(–2, 1). **6.** 24. **7.** 64. **8.** 4,000 soums and 5,000 soums. **9.** 10+40+20=70.

III.4. 1. a) $(x_1, y_1)=(-4, \sqrt{3})$; $(x_2, y_2)=(-4, -\sqrt{3})$; b) $(x_1,y_1)=(5, -1)$; $(x_2, y_2)=(-1, 5)$; c) $(x, y)=(2.5, 0.5)$; d) $(x_1, y_1)=(5, 23)$; $(x_2, y_2)=(-2, 2)$. **2.** a) $(x_1, y_1)=(0, 0)$; $(x_2, y_2)=(4, -64)$; $(x_3, y_3)=(-4, -64)$; b) $(x_1, y_1)=(2, -1)$; $(x_2, y_2)=(-1, 2)$; c)$(x_1, y_1)=(5, 2)$; $(x_2, y_2)=(-5, 2)$. **3.** a) $(x_1, y_1)=(3, 1)$; $(x_2, y_2)=(1, 3)$; b) $(x_1, y_1)=(3, 2)$; $(x_2, y_2)= (-2, -3)$; c) $(x_1, y_1)=(1, 3)$; $(x_2, y_2)=(3, 1)$. **4.** $a_1=6$; $a_2= -6$. **5.** a) $(A^3-B)/3$; b) $(A^3-B)/(3A)$. **6.** a) $(x_1, y_1)=(2, 1)$; $(x_2, y_2)=(1, 2)$; b) $(x_1, y_1)=(4, 1)$; $(x_2, y_2)=(1, 4)$; c) $(x_1, y_1)=(3, 4)$; $(x_2, y_2)=(3, -4)$; $(x_3, y_3)=(0, 5)$; d) $(x_1, y_1)=(3, 2)$; $(x_2, y_2)=(-3, -2)$. **7.** a) 12; b) 24; c) 8/3.

Homework: 1. a) $(x, y)=(-1, 2)$; b) $(x_1, y_1)=(3, 4)$; $(x_2, y_2)=(4, 3)$; c) $(x_1, y_1)=(5, 1)$; $(x_2, y_2) = (-1, -5)$; d) $(x_1, y_1)=(4, -10)$; $(x_2, y_2)=(1, -1)$. **2.** a) $(x_1, y_1)=(1, 3)$; $(x_2, y_2)=(3, 1)$; b) $(x_1, y_1)=(1/3, -1/2)$; $(x_2, y_2)=(-1/2, 1/3)$; c) $(x_1, y_1)=(1/3, -1/3)$; $(x_2, y_2)=(1, 1)$. **3.** $(x, y) = (16, -2)$. **4.** 46; 64.

III.5. 1. 4/3. **2.** 17; 25. **3.** 97. **4.** 40. **5.** 17/8. **6.** 49. **7.** 123. **8.** 8/3. **9.** $\dfrac{x\cdot\frac{y}{2}+k\cdot\frac{t}{3}}{x+k}=\dfrac{3xy+2kt}{6(x+k)}$. **10.** 4.5$kg$. **11.** a) 67; b) 42.5.

Homework: 1. 85.25. **2.** 8. **3.** 14km/h. **4.** \$50.

III.6. 1. 2/3. **2.** 30; 60. **3.** 0.27. **4.** 51+102+306; 255. **5.** Anne – £160, Bill – £200, Carl – £280. **6.** 32. **7.** a) 1.5km; b) 28km. **8.** 10. **9.** 10/3 hours. **10.** 120. **11.** 1 to 1,666,667. **12.** $30{,}500\cdot\frac{3}{4}\cdot\frac{1}{3}=7{,}625$. **13.** 4/3. **14.** a) direct; b) direct; c) inverse; d) inverse. **15.** 20. **16.** 148.8. **17.** 24. **18.** a) 4.5; 13.5; b) 54; 81; c) 5; 15. **19.** 2. **20.** 4 days. **21.** 48; 24; 16; 12; 8. **22.** 9. **23.** 1/20.

Homework: 1. 20; 12. **2.** $100ad/c$. **3.** 1750. **4.** 3. **5.** 2. **6.** a) 12.6; b) 10/3. **7.** 36.05. **8.** 6. **9.** 24,000; 6,000; 10,000.

III.7. 1. a) 100; b) 80; c) 75; d) 250. **2.** 63. **3.** 75. **4.** 0.135; 0.023; 1.45; 0.004. **5.** 1/2; 3/25; 2/5; 1/4. **6.** 12.5%; 23%; 30%; 34%. **7.** \$112.5. **8.** 75%. **9.** 25,000. **10.** 19,200. **11.** 80. **12.** 9/11. **13.** 400. **14.** 440 soums and 400 soums. **15.** 500 soums. **16.** a) 12%; 2%; b) 506.04 million. **17.** 60. **18.** 3.22%. **19.** 20/9%. **20.** 173.5. **21.** 10 and 50 meters.

Homework: 1. a) 225; b) 575; c) 1. **2.** 1250. **3.** 38%. **4.** 200% **5.** 125%. **6.** 30%. **7.** 60%. **8.** 67%. **9.** 14.03%. **10.** 1,058,547.5 soums. **11.** 20,000 soums.

Mathematics is well and good but
nature keeps dragging us around by the nose.
Albert Einstein (1879-1955), German-born American physicist

CHAPTER IV. INEQUALITIES AND SIMULTANEOUS INEQUALITIES

IV.1. Linear inequalities

Terms

1. **linear inequality** – chiziqli tengsizlik|линейное неравенство;
2. **greater than or equal to** – katta yoki teng|больше или равно;
3. **less than or equal to** – kichik yoki teng|меньше или равно;
4. **variable range** – o'zgaruvchining qiymatlar sohasi|область значений переменной;
5. **union of intervals** – intervallar birlashmasi|объеденение интервалов;
6. **property** – xossa|свойство;
7. **relationship** – munosabat|соотношение;
8. **to prove** – isbotlamoq|доказывать;
9. **ascending (descending) order** – o'sib (kamayib) borish tartibi|возрастающий (убывающий) порядок;
10. **both (opposite) sides of an equation** – tenglamaning har ikki (qarama-qarshi) tomoni|обе (противоположные) стороны уравнения.

Learning Objectives

- to be able to identify a linear inequality;
- to know how to find solutions of a linear inequality and to write down the solution properly.

Properties of inequalities:

1) If $a > b$ and $b > c$, then $a > c$.

2) If $a > b$ and $c > d$, then $a + c > b + d$.

3) If $a > b > 0$ and $c > d > 0$, then $a \cdot c > b \cdot d$.

4) If $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

5) If $a > b, c > 0$ ($a > b, c < 0$), then $a \cdot c > b \cdot c$ ($a \cdot c < b \cdot c$).

6) If $a > b > 0$, $n \in N$, then $a^n > b^n$.

Linear inequality:

$ax > b$,

where a and b are known constant numbers.

Solution of a linear inequality $ax > b$**:**

$$\begin{cases} x > \frac{b}{a}, \text{ if } a > 0 \\ x < \frac{b}{a}, \text{ if } a < 0 \end{cases}$$

Examples

1. Solve the inequality and show the answer as an interval
   $2(x-14)+4(3-7x)<13(x+4)-5(8+3x)$.

▶ Eliminate the brackets, sort out the unknown and constant numbers to opposite sides of the inequality, tidy up and then divide both sides by the coefficient of the unknown.

1) $2x-28+12-28x<13x+42-40-15x$.

2) $2x-28x-13x+15x<42-40+28-12$.

3) $-24x<18$.

4) $\frac{-24x}{-24}>\frac{18}{-24}$.

5) $x > -\frac{3}{4} \Rightarrow x \in \left(-\frac{3}{4}, +\infty\right)$. ■

2. A team of four coworkers work for a tailor. Their productivity is as follows: Madina makes five more items of clothing than Dilfuza, Feruza makes half as many as Madina makes, and Nodira makes one third of the number Feruza makes. If the price of an item is 10,000 soums, what is the least number of items that Dilfuza has to saw in order for the four of them to make at least 240,000 soums between them?

▶ We divide 240,000 by 10,000 to find 24, which is the number of clothes the four workers need to produce.

If we denote the number of clothes made by Dilfuza by x, then the numbers of clothes of her coworkers are as follows:

Madina: $x + 5$;

Feruza: $0.5(x + 5)$;

Nodira: $(x + 5)/6$.

The sum of these numbers must be at least 24, so we make up the following inequality:

$$x + (x+5) + \frac{x+5}{2} + \frac{x+5}{6} \geq 24$$

To solve the inequality, we multiply both sides by the least common denominator, separate unknowns and constant numbers, tidy up and finally divide both sides by the coefficient of the unknown.

1) $6 \cdot \left[x + (x+5) + \frac{x+5}{2} + \frac{x+5}{6}\right] \geq 24 \cdot 6$;

2) $12x + 30 + 3(x+5) + x + 5 \geq 144$;

3) $16x \geq 96$; 4) $x \geq 6$.

Thus, Dilfuza has to produce at least 6 items. ■

Exercises

1. Consider the real numbers A, B, C, and D, and the following relationship: $A < B$, $B < C$, $D > C$. Which of the following is true?

   a) $D - C > D - B$; b) $D - C < B - A$;

   c) $D - B > D - A$; d) $D - B < D - A$.

2. Given $q = 3\sqrt{3}$, $r = 1 + 2\sqrt{3}$, $s = 3 + \sqrt{3}$, which of the following is true?

   a) $q > r > s$; b) $q > s > r$;

   c) $r > q > s$; d) $s > q > r$;

3. Solve for x

   a) $4x > 8$; b) $3x < 15 - 3$;

   c) $-3x < 6$.

4. Solve for x and express your solution as an interval

   a) $-x > 2$; b) $3 - 5x \geq 18$;

   c) $5x - 4 > 6x - 6$.

5. Find the range of x such that

   a) $12x < 5(2x + 4)$; b) $2(x + 3) > 5(x - 3)$;

   c) $1 - \frac{17 - 3x}{2} > 1.5$;

6. If $p > q$ and $r < 0$, which of the following is (are) true?

   I. $pr < qr$ II. $p + r > q + r$

   III. $p - r < q - r$?

7. If $p > 0$, $q < 0$, which is greater $p + q$ or $p - q$?

8. The formula for converting a temperature from degrees Fahrenheit (°F) to degrees Celsius (°C) is

   $$C = \frac{5}{9}(F - 32),$$

   where F is the temperature in degrees Fahrenheit and C is the temperature in degrees Celsius. Aziz needs to store some food in a refrigerator at a temperature between $0°C$ and $4°C$. What range of temperature in degrees Fahrenheit would be suitable?

9. Prove the following inequalities for any real numbers

   a) $a^2+b^2 \geq 2ab$; b) $a^4+b^4 \geq ab(a^2+b^2)$.

10. Solve for x

    a) $(x+7)^2-4 \geq (x+6)^2-3$;

    b) $(4-x)^2-(x+6)^2 \geq (x+4)^2-(2-x)^2$.

Homework

1. Place w, x, y and z in ascending order if

   $w > x$, $y < z$ and $y > w$.

2. Solve for x

   a) $3x+2 > 2x+7$; b) $5x-5 \geq -9+3x$;

   c) $6y+2 < 8y+14$.

3. Find the solution

   a) $\frac{x}{6}-\frac{x}{7}<1$; b) $\frac{x+4}{7}-\frac{x+7}{4} \geq -3$.

4. An electricity supplier uses two different formulae for calculating monthly charges for electricity. For a consumer using an amount E of electricity, the cost C is given by either

   $C = 60 + 0.2E$ or $C = 0.3(E-50)$.

   For what amount of electricity will the first formula result in a cheaper cost?

5. There are 461 students and 20 teachers taking buses on a trip to a museum. Each bus can seat a maximum of 52. What is the least number of buses needed for the trip?

Development of mathematics in the 19th century AD.

In this century a great effort was made to produce strong theoretical foundations in all areas of mathematics. Attention was paid to the critical revision of orignial conditions (axioms), to the establishment of systems of definitions and proofs and to the reevaluation of logical methods used in these proofs. **Nikolai Lobachevsky** investigated non-Euclidean geometry. **Niels Abel** (Norwegian) and **Evariste Galois** (French) proved that there is no general algebraic method for solving polynomial equations of degree greater than four, and other 19th century mathematicians utilized this in their proofs that straightedge and compass alone are not sufficient to trisect an arbitrary angle, to construct the side of a cube twice the volume of a given cube, nor to construct a square equal in area to a given circle. Mathematicians had vainly attempted to solve all of these problems since the time of the ancient Greeks (Ashraf A 1994). **Legendre** found the method of least squares to find the best approximations to a set of observed data. **Fourier** (French) discovered his method of representing continuous functions by the sum of a series of trigonometric functions and his book "Analytic Theory of Heat" widely explained the techniques of Fourier analysis. **Bessel** discovered a class of integral functions, which are now known as Bessel functions. **Bolzano** published a book, in which he tries to avoid using the concept of infinitesimals in defining continuous functions. **Ampere** published his book, in which he derives mathematically the law of electromagnetic force and lays the foundation for electromagnetic theory. **Gauss** introduced differential geometry. **Green** published his book, in which he studied the properties of electric and magnetic fields, introduced the term potential and gave the formula connecting surface and volume integrals, now known as Green's Theorem. **Liouville** developed the "Sturm-Liouville theory" and found the first transcendental numbers (numbers that can not be expressed as the roots of an algebraic equation with rational coefficients). **De Morgan** introduced the term "mathematical induction" and formulated the method. **Lame** proved Fermat's Last Theorem for n=7. **Boole** published "The Mathematical Analysis of Logic", in which he shows that the rules of logic can be treated mathematically rather than metaphysically. Boole's work laid the foundation of computer logic (Prokhorov Yu (ed) 1988). Also, Boole developed the algebra of logic known as Boolean algebra. **Chebyshev** proved Bertrand's conjecture that there is always at least one prime number between n and $2n$ for n>1. The term matrix was first introduced in **Sylvester's** book. The London and later the Moscow, the France and the St. Petersburg Mathematical Societies were founded. **Maxwell** published his book "Electricity and Magnetism", in which the four partial differential equations known as "Maxwell's equations" are introduced. **Venn** introduced his "Venn diagrams" in set theory. **Volterra** studied integral equations. **Levi-Civita** developed the calculus of tensors. **Galton** introduced the concept of correlation. **Poincare** developed algebraic topology. **Hensel** found the p-adic numbers. **Hilbert** suggested the 23 problems at the Congress of Mathematicians in Paris, which paved the way for great work and progress in 20th century mathematics.

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself.
The larger the denominator, the smaller the fraction.
Leo Tolstoy (1828-1910), Russian writer and moral philosopher

IV.2. Simultaneous linear inequalities

Terms

1. **simultaneous linear inequalities** – chiziqli tengsizliklar sistemasi|система линейных неравенств;
2. **double inequality** – ikkitalik tengsizlik|двойное неравенство;
3. **to identify** – aniqlamoq|определять;
4. **particular solution** – xususiy yechim|частное решение;
5. **general solution** – umumiy yechim|общее решение;
6. **the number axis** – sonlar o'qi|числовая ось;
7. **a common intersecting interval** – umumiy kesishuvchi oraliq|общий пересекающийся интервал;
8. **an overlapping region** – kesishuvchi soha|пересекающяяся область;
9. **inequality sign** – tengsizlik ishorasi|знак неравенства.

Learning Objectives

– to be able to identify simultaneous linear equations;
– to know how to find solutions of simultaneous linear inequalities and to write the solution properly.

Simultaneous linear inequalities

$\begin{cases} a_1x > b_1 \\ a_2x > b_2 \end{cases}$ or $\begin{cases} a_1x > b_1 \\ a_2x < b_2 \end{cases}$,

where a_1, a_2, b_1, b_2 are real numbers.

Examples

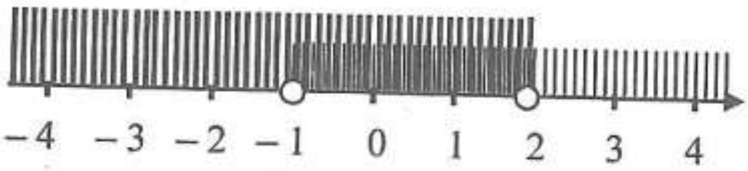
1. Solve the following simultaneous linear inequalities

$$\begin{cases}7(x+3)-11>2x+5\\2(x-3)+18>5x+6\end{cases}.$$

▶ First, we solve each inequality to find particular solutions. Then we show the solutions on the same number axis to determine the common intersecting interval(s) and find the general solution for the simultaneous nonlinear inequalities.

$$\begin{cases}7(x+3)-11>2x+5\\2(x-3)+18>5x+6\end{cases}\Rightarrow\begin{cases}7x-2x>5-10\\2x-5x>6-12\end{cases}\Rightarrow$$

$$\begin{cases}x>-1 \text{ } -a \text{ } particular \text{ } solution\\x<2 \text{ } -a \text{ } particular \text{ } solution\end{cases}.$$



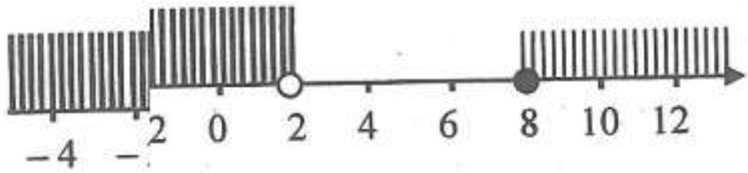
From the diagram above, it is clear that the two intervals have a common part at (– 1, 2), which is the general solution to the simultaneous linear inequalities. ■

2. Solve the simultaneous linear inequalities

$$\begin{cases}6(x-3)+4\geq 2x+18\\5(x-7)+18>3(2x-1)-16\end{cases}.$$

▶ $$\begin{cases}6x-18+4\geq 2x+18\\5x-35+18>6x-3-16\end{cases}\Rightarrow\begin{cases}6x-2x\geq 18+14\\6x-5x<-17+19\end{cases}\Rightarrow$$

$$\begin{cases}x\geq 8-a \text{ } particular \text{ } solution\\x<2-a \text{ } particular \text{ } solution\end{cases}.$$



From the diagram above it is obvious that the two intervals do not have a common overlapping part, so there is no solution. ■

Exercises

1. Solve the following simultaneous linear inequalities and express the solution as an interval

a) $\begin{cases} x>3 \\ x<7 \end{cases}$; b) $\begin{cases} 4x-3>5x-5 \\ 2x+4<8x \end{cases}$;

c) $\begin{cases} 2x-3>3x-5 \\ 2x+4>8x \end{cases}$; d) $\begin{cases} 3x-3>5x-15 \\ 3x+4\le 7x-8 \end{cases}$.

2. Which of the following simultaneous linear inequalities do not have a solution?

a) $\begin{cases} x-3>2x-1 \\ x+3<3x-2 \end{cases}$; b) $\begin{cases} 3x-3>5x-15 \\ 3x+4\le 7x-28 \end{cases}$;

c) $\begin{cases} x-3>4x \\ x+4\le 2x-2 \end{cases}$; d) $\begin{cases} 5x-3\ge 6x \\ 3x+4\le 0.5(4x+2) \end{cases}$.

3. How many prime roots does the double inequality $1\le \frac{x+2}{4}\le 4$ have?

4. Find the arithmetic mean of all whole roots of

$\begin{cases} 2x-1\ge 3x-4 \\ 8x+7>5x+4 \end{cases}$.

Homework

1. Find the least whole root of $\begin{cases} x+8<12 \\ -3x<15 \end{cases}$.

2. Find the sum of all whole roots of $\begin{cases} -x-5<-2x-2 \\ -2x+2\geq 3-3x \end{cases}$.

Development of mathematics in the 20th century AD

Mathematics grows exponentially. **Fredholm** developed the theory of integral equations. **Plank** suggested the quantum theory. **Lebesgue** introduced the theory of measure and defined "Lebesgue integral". **Einstein** published the special theory of relativity. New areas of mathematics such as mathematical logic, the mathematics of computers, statistics and game theory emerged. **Zermelo** suggested the seven axioms in set theory in his book. **Turing** published "On Computable Numbers", which describes theoretical machines, now known as the "Turing machine". **Kolmogorov** published "Analytic Methods in Probability Theory", which puts the foundations of the theory of random Markov processes. **George Dantzig** found the simplex method of optimization. **Norbert Wiener** published a book on the theory of information control and introduced the term "cybernetics". **Schwartz** published a book on the theory of distributions. **Mauchly** and **John Eckert** constructed the Binary Automated Computer (BINAC), which enables data storage on magnetic tape rather than on punched cards. **Hormander** developed the theory of partial differential equations. Kolmogorov published a book on the theory of dynamical systems. **Edward Lorenz** showed a simple mathematical system with chaotic behavior and a new theory of chaos emerged. **Jacobson** published his book "Lie algebras". **Sobolev** published his book "Applications of functional analysis and mathematical physics". **Wiles** proved Fermat's last theorem. At a meeting of the American Mathematical Society in Los Angeles "Mathematical Challenges of the 21st century" were proposed. Unlike "Hilbert's problems" from 100 years earlier, these were given by a team of 30 leading mathematicians of whom eight were Fields Medal winners (Prokhorov Yu (ed) 1988). A prize of seven million dollars is put up for the solution of seven famous mathematical problems. Called the Millennium Prize Problems they are: P versus NP; The "Hodge Conjecture"; The Poincaré Conjecture; The Riemann Hypothesis; "Yang-Mills Existence and Mass Gap"; "Navier-Stokes Existence and Smoothness"; and The "Birch and Swinnerton-Dyer Conjecture" (Prokhorov Yu (ed) 1988).

I think, therefore I am.
René Descartes (1596-1650), French philosopher, scientist and mathematician

IV.3. Nonlinear inequalities

Terms

1. **nonlinear inequality** – nochiqizli tengsizlik|нелинейное неравенство;
2. **segment** – segment|сегмент;
3. **the infinity** – cheksizlik|бесконечность;
4. **the method of intervals** – oraliqlar usuli|метод интервалов;
5. **semi open interval** – yarim ochiq interval|полуоткрытый интервал;
6. **critical value** – kritik qiymat|критическое значение;
7. **to indicate** – ko'rsatmoq, belgilamoq|показывать, указывать, означать;
8. **leftmost (middle, rightmost) interval** – eng chapki (o'rta, eng o'nggi) oraliq|крайний интервал слева (по середине, справа);
9. **left hand (right hand) side** – chap (o'ng) taraf|левая (правая) сторона;
10. **to belong to** – tegishli bo'lmoq|принадлежать;
11. **to yield** – kelib chiqish, hosil bo'lish|давать, приносить, производить, получить(ся).

Learning Objectives

- to be able to identify simultaneous nonlinear inequalities;
- to know how to solve simultaneous nonlinear inequalities by the method of intervals.

Examples

1. Solve for x:

$$2x^2 + 4x - 14 \geq (x + 9)(x - 1).$$

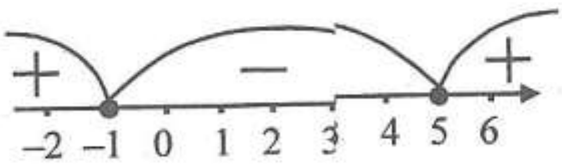
▶ Eliminate the brackets, transfer all terms to the left side and factorize the polynomial:

$$2x^2+4x-14 \geq x^2+9x-x-9 \Rightarrow$$

$$x^2-4x-5 \geq 0 \Rightarrow$$

$$(x+1)(x-5) \geq 0 .$$

Now, the critical values of x for each of the two factors in the last inequality are 5 and –1, which, when placed on the number axis, divide it into three intervals. We can then identify the sign of the left hand side of our inequality by trying different values from these intervals.



For example, consider the number -2 from the leftmost interval, which yields a positive value for $(x + 1)(x - 5)$; or the number 0 from the middle interval, which yields a negative value; and the number 6 from the rightmost interval, which yields positive again. The main inequality requires a sign that is greater than or equal to 0, which implies the solution is contained in the intervals that yield positive values.

Thus, $x \in (-\infty,-1] \cup [5,+\infty)$. ■

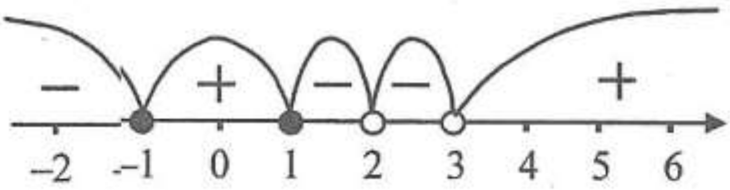
2. Solve the inequality

$$\frac{(x^2-1)(x-2)}{x^2-5x+6} \leq 0 .$$

▶ We factorize both the numerator and denominator of the left-hand side fraction and then investigate the result:

$$\frac{(x-1)(x+1)(x-2)}{(x-2)(x-3)} \leq 0 .$$

On the number axis we indicate all the critical points (roots) of the numerator and denominator taking into account that x can not receive the values 2 and 3, at which the fraction is meaningless.



Thus, from the main inequality sign ≤, the solution is the union of all negative intervals.

Thus, $x \in (-\infty, -1] \cup [1,2) \cup (2,3)$. ■

Exercises

1. Solve for x

   a) $x^2 - x < 0$; b) $x \leq 1$;

   c) $3 - 2x^2 > 2x + 3$.

2. Use factorization to solve for x

   a) $(2x-3)^2 < 4x + 2$; b) $x^2 + 5x - 6 > 0$;

   c) $x^2 - 5x - 50 \leq 0$.

3. Solve the following nonlinear inequalities

   a) $\dfrac{x-1}{x+2} > 0$; b) $\dfrac{x(x-1)^2}{(x+3)^3} \leq 0$;

   c) $(x+3)(x+2)(x-4)^2(5-x) > 0$.

4. Find the range of values of x for which $\frac{1}{x-2} < \frac{1}{x+6}$, expressing your answer as an interval.

Homework

1. Solve

a) $19 - x^2 \geq 1$;

b) $x^2 - 2x - 1 < 0$;

c) $\frac{x+2}{(x-1)^2} > 0$;

d) $\frac{x^2 - 5x - 14}{x+4} \leq 0$.

2. Find the difference between the largest and the smallest roots of the inequality

$$\frac{x^2 - 13x + 36}{x^4 + 25} \leq 0.$$

Branches of mathematics

Arithmetic deals with counting and the most basic mathematical operations.

Algebra in which symbols (usually letters) represent unknown numbers in mathematical expressions.

Applied mathematics the application of mathematics to various areas such as physics, chemistry, biology, medicine, engineering, economics and others.

Calculus (Mathematical Analysis) the study of functions with the help of limits.

Chaos theory the theory describing the complex and unpredictable motion or dynamics of systems that are sensitive to their initial conditions.

Combinatorics the study of solving problems of selection and positioning of elements of some, usually finite, set according to given rules.

Complex analysis a branch that investigates the functions of complex numbers.

Differential equations the study of mathematical equations for an unknown function of one or several variables which relates the values of the function itself and of its derivatives of various orders.

Functional analysis a branch of analysis which studies the properties of mappings of classes of functions from one topological vector space to another.

Game theory the mathematical analysis of any situation involving a conflict of interest, with the intent of indicating the optimal choices that, under given conditions, will lead to a desired outcome.

Geometry a branch of mathematics concerned with the properties of and relationships between points, lines, planes, and figures and with generalizations of these concepts.

Graph theory the mathematical study of the structure of graphs and networks.

Group theory a branch of mathematics concerned with groups and the description of their properties.

Harmonic analysis the study of functions given by a Fourier series or analogous representations, such as periodic functions and functions on topological groups.

Integral equations studies equations in which an unknown function appears under an integral sign.

Branches of mathematics

Linear (Nonlinear) optimization and Control the study of maximizing or minimizing a linear (nonlinear) function subject to given constraints which are linear (nonlinear) inequalities involving the variables.

Logic the formalized system of deductive logic, employing abstract symbols for the various aspects of natural language.

Linear algebra concerned with systems of linear equations, linear transformations, vectors and vector spaces, and related topics.

Mathematical economics a branch of mathematics that studies problems arising in the analysis of mathematical models of production, distribution, exchange and other economical processes.

Mathematics education the study of practices and methods of both teaching and learning of mathematics.

Mathematical Logic deals with the mathematical proof and questions of foundations of mathematics.

Mathematical Statistics the study of mathematical methods of systematization, processing and use of statistical data for the scientific and practical conclusions.

Number theory a branch of mathematics that deals with the properties and relationships of numbers.

Numerical analysis the study of approximation techniques for solving mathematical problems, taking into account the extent of possible errors.

Optimization deals with identifying optimal solutions among all possible outcomes.

Probability a branch of mathematics that deals with measuring or determining quantitatively the likelihood that an event or experiment will have a particular outcome.

Real Analysis deals with the set of real numbers and functions of real numbers.

Topology a branch of mathematics that studies the idea of continuity in mathematics.

Trigonometry a branch of mathematics that deals with the relationships between the sides and angles of triangles and with the properties and applications of the trigonometric functions of angles.

Science without religion is lame, religion without science is blind.
Albert Einstein (1879-1955), German-born American physicist

IV.4. Simultaneous nonlinear inequalities

Terms

1. **the simultaneous nonlinear inequalities** – nochiqizli tengsizliklar sistemasi | система нелинейных неравенств;
2. **radical sign** – ildiz belgisi | радикальный знак, корень;
3. **to denote** – belgilamoq | обозначать;
4. **to satisfy an equation** – tenglamani qanoatlantirmoq | удовлетворять уравнению;
5. **the empty set** – bo'sh to'plam | пустое множество;
6. **n-th root** – n-ildiz | n-й корень;
7. **to be equivalent** – teng kuchli bo'lmoq | быть равносильным;
8. **to correspond to** – mos bo'lmoq | соответствовать;
9. **to unite intervals** – oraliqlarni birlashtirmoq | объединить интервалы;
10. **thus** – shunday qilib | таким образом.

Learning Objectives

- to be able to identify simultaneous nonlinear inequalities;
- to know how to solve simultaneous nonlinear inequalities.

Irrational equation and its solution

$$\sqrt{P(x)} = Q(x) \Rightarrow \begin{cases} P(x) \geq 0 \\ Q(x) > 0 \\ P(x) = Q^2(x) \end{cases},$$

where $P(x)$ and $Q(x)$ are the algebraic expressions.

Irrational inequalities and their solutions

1) $\sqrt{P(x)} < Q(x) \Rightarrow \begin{cases} P(x) \geq 0 \\ Q(x) > 0 \\ P(x) < Q^2(x) \end{cases}$ or $\sqrt{P(x)} \leq Q(x) \Rightarrow \begin{cases} P(x) \geq 0 \\ Q(x) > 0 \\ P(x) \leq Q^2(x) \end{cases}$,

where $P(x)$ and $Q(x)$ are the algebraic expressions.

2) $\sqrt{P(x)} > Q(x) \Rightarrow \begin{cases} P(x) \geq 0 \\ Q(x) \geq 0 \\ P(x) > Q^2(x) \end{cases} \cup \begin{cases} P(x) \geq 0 \\ Q(x) < 0 \end{cases}$ or

$$\sqrt{P(x)} \geq Q(x) \Rightarrow \begin{cases} P(x) \geq 0 \\ Q(x) \geq 0 \\ P(x) \geq Q^2(x) \end{cases} \cup \begin{cases} P(x) \geq 0 \\ Q(x) < 0 \end{cases},$$

where $P(x)$ and $Q(x)$ are the algebraic expressions.

Examples

1. Solve the simultaneous nonlinear inequalities

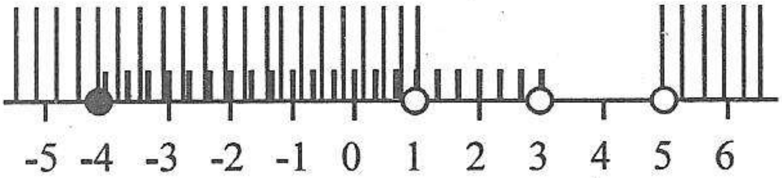
$$\begin{cases} \dfrac{x+4}{x-3} \leq 0 \\ (x-1)(x-5) > 0 \end{cases}.$$

▶ We will solve each inequality in turn to find particular solutions.

$$\begin{cases} \dfrac{x+4}{x-3} \leq 0 \\ (x-1)(x-5) > 0 \end{cases} \Rightarrow$$

$$\begin{cases} x \in [-4,3) - a\ particular\ solution \\ x \in (-\infty,1) \cup (5,+\infty) - a\ particular\ solution \end{cases}.$$

Now, the particular solutions can be indicated on the same number axis to find the intersecting region (if any).



Thus, the common part $x \in [-4; 1)$ is the general solution. ■

2. Solve the irrational inequality

$$\sqrt{2x^2 - x - 6} < x + 4 .$$

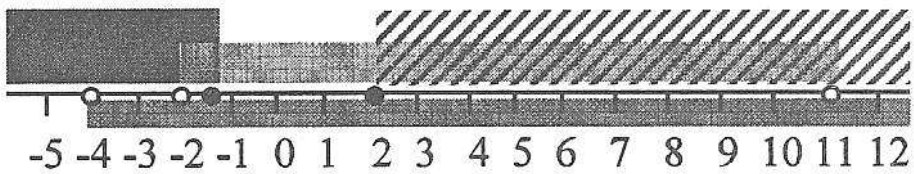
► The given irrational inequality corresponds to the following three simultaneous inequalities:

$$\begin{cases} 2x^2 - x - 6 \geq 0 \\ x + 4 > 0 \\ 2x^2 - x - 6 < (x+4)^2 \end{cases} \Rightarrow$$

$$\begin{cases} 2(x+1.5)(x-2) \geq 0 \\ x > -4 \\ (x+2)(x-11) < 0 \end{cases} \Rightarrow$$

$$\begin{cases} x \in (-\infty, -1.5] \cup [2. + \infty) \\ x \in (-4, +\infty) \\ x \in (-2, 11) \end{cases} .$$

Now, we display all these three particular solutions on the same number axis.



The region $(-2; 1.5]\cup[2; 11)$, where all three particular solutions intersect, is the general solution. ■

3. Solve the irrational inequality

$$\sqrt{x+8} > x+2.$$

▶ The irrational inequality corresponds to the two simultaneous inequalities. We will solve each of the two simultaneous inequalities and unite their solutions together at the end. ■

a) $\begin{cases} x+8\geq 0 \\ x+2\geq 0 \\ x+8>(x+2)^2 \end{cases} \Rightarrow \begin{cases} x\in[-8, +\infty) \\ x\in[-2, +\infty) \\ x\in(-4, 1) \end{cases} \Rightarrow x\in[-2, 1).$

b) $\begin{cases} x+8\geq 0 \\ x+2<0 \end{cases} \Rightarrow \begin{cases} x\in[-8, +\infty) \\ x\in(-\infty, -2) \end{cases} \Rightarrow x\in[-8, -2).$

Thus, $x\in[-8, -2)\cup[-2, 1)$ or $x\in[-8, 1)$ is the final solution.

Exercises

1. Solve for x

a) $\begin{cases} x^2\leq 9 \\ x>0 \end{cases}$; b) $\begin{cases} x^2+5x-6<0 \\ x^2+4x<0 \end{cases}$.

2. Solve the following irrational inequalities

a) $\sqrt{x+1}<2$; b) $x>\sqrt{2x-1}$;

c) $\sqrt{2x-1}>x$; d) $\sqrt{x+1}>\sqrt{3-x}$;

e) $\sqrt{x^2+x-12}<x$.

3. Solve

a) $\frac{2x}{x^2-1} \geq \frac{x}{x-1} - \frac{1}{x+1}$; b) $\sqrt{\frac{5-x}{x-10}} > -1$;

c) $\sqrt{x+2} > x$; d) $\sqrt{x^2+x-2} > x-2$.

4. Find the solution $\sqrt{x^2-55x+250} < x-14$.

5. Solve

a) $\sqrt{x}-4=12$; b) $\sqrt{x-3}=4$;

c) $\sqrt{x+2}=-3$; d) $\sqrt{x+2}=x$;

e) $\sqrt[3]{x+1}=4$; f) $\sqrt[4]{x-2}=3$.

6. Find the solutions

a) $\sqrt{2x-1}-\sqrt{x+2}=0$; b) $\sqrt{x}-\sqrt{x+3}=1$.

7. Find the roots

a) $\sqrt[3]{x}+\sqrt[6]{x}-2=0$; b) $\sqrt{x}+\sqrt[4]{x}-6=0$;

c) $x^2+x+\sqrt{x^2+x}=2$.

8. If x_0 is a root of $x-\sqrt{x+2}-10=0$, what is the value of $\sqrt{x_0+2}$?

9. Solve the equations

a) $\sqrt{16-\sqrt{x+2}}=4$; b) $\sqrt[3]{4-\sqrt[3]{x+3}}=2$.

10. Find x in $\sqrt{x^2}-\sqrt[3]{x^3}+\sqrt[4]{x^4}-\sqrt[5]{x^5}=7$.

Homework

1. Solve

a) $\begin{cases}(x+2)(2-x)<(x+3)(4-x)\\ \dfrac{3+x}{4}+\dfrac{1-2x}{6}\geq 1\end{cases}$;

b) $(x-1)\sqrt{x^2-x-2}\geq 0$.

2. How many whole roots does the inequality $\sqrt{8+2x-x^2}>6-3x$ have?

3. Solve

a) $\sqrt[4]{x-3}=2$; b) $\sqrt{x}=2-x$;

c) $x-1=\sqrt{x+5}$; d) $2\sqrt{x+5}=x+2$.

4. Find the roots

a) $\sqrt{x-3}+\sqrt{x-18}=1$; b) $\sqrt{\dfrac{x+1}{x-1}}=\sqrt{2x-1}$;

c) $\sqrt{16-\sqrt{x+1}}=3$.

5. Find the product of roots of $\sqrt{2x^2+17}=x^2+1$.

6. Solve the equations

a) $x^2+11+\sqrt{x^2+11}=42$; b) $\sqrt{x-5}+6=5\sqrt[4]{x-5}$;

c) $\sqrt{x+7}=\sqrt{3x-2}-1$.

Greek alphabet and Greek numbers

Lower case	Upper case	Pronunciation
α	Α	alpha
β	Β	beta
γ	Γ	gamma
δ	Δ	delta
ε	Ε	epsilon
ζ	Ζ	zeta
η	Η	eta
θ	Θ	theta
ι	Ι	iota
κ	Κ	kappa
λ	Λ	lambda
μ	Μ	mu
ν	Ν	nu
ξ	Ξ	xi
ο	Ο	omicron
π	Π	pi
ρ	Ρ	rho
σ	Σ	sigma
τ	Τ	tau
υ	Υ	upsilon
φ	Φ	phi
χ	Χ	chi
ψ	Ψ	psi
ω	Ω	omega

Numbers	Pronunciation
1	ena
2	dio
3	tria
4	tessera
5	pende
6	hexi
7	hepta
8	octo
9	enia
10	deca
11	endeka
12	dodeka

Doubt is the key to knowledge.
Persian Proverb

IV.5. Equations and inequalities involving absolute values

Terms

1. **absolute value** – absolyut qiymat | абсолютное значение;
2. **critical point** – kritik nuqta | критическая точка;
3. **modular equation** – modulli tenglama | модульное уравнение;
4. **symbol** – belgi, simvol | символ;
5. **to be identical to** – aynan o'xshash bo'lmoq | быть тождественным;
6. **to verify** – tekshirib ko'rmoq | проверить;
7. **example** – misol | пример;
8. **at the same time** – bir vaqtda | в то же время;
9. **rule** – qoida | правило;
10. **in front of** – oldi | перед, впереди.

Learning Objectives

- to be able to identify absolute value equations and inequalities and to find their critical points;
- to know how to solve equations and inequalities involving absolute values.

Absolute value

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}.$$

For example,

$$|3| = 3,\ |-3| = 3,\ |-15| = 15,\ |0| = 0,\ |2-5| = 3,\ |(1-3)^2| = 4.$$

Modular equations and their solutions

1) $$|P(x)|=Q(x)\Rightarrow\begin{cases}Q(x)\geq 0\\ P(x)\geq 0\\ P(x)=Q(x)\end{cases}\cup\begin{cases}Q(x)\geq 0\\ P(x)<0\\ -P(x)=Q(x)\end{cases},$$

where P (x) and Q (x) are the algebraic expressions.

2) $|P(x)|=Q(x)\Rightarrow P(x)=\pm Q(x)$,

where P (x) and Q (x) are the algebraic expressions. After solving the above equations, verify the found numbers against the original modular equation.

Modular inequalities and their solutions

1) $$|P(x)|<Q(x)\Rightarrow\begin{cases}P(x)<Q(x)\\ -P(x)<Q(x)\end{cases},$$

where P (x) and Q (x) are the algebraic expressions.

2) $|P(x)|>Q(x)\Rightarrow P(x)>Q(x)\cup -P(x)>Q(x)$,

where P (x) and Q (x) are the algebraic expressions.

Examples

1. Solve the modular equation $|x+3|=2x$.

▶ 1st method: We can expand the modular symbols, put negative and positive signs in front of the right-hand side expression, solve the two equations and then check the results.

$x+3=\pm 2x\Rightarrow$

1) $x+3=2x$ and 2) $x+3=-2x$

1) $x=3$ and 2) $x=-1$

But, only the first number satisfies the original equation, therefore $x=3$ is the final solution.

2^{nd} method: The critical value of the module is -3, which divides the number axis into two intervals. The absolute symbols of the original equation can be expanded at each interval and solved.

$$1)\ \begin{cases} 2x \ge 0 \\ x+3 \ge 0 \\ x+3 = 2x \end{cases} \quad \text{and} \quad 2)\ \begin{cases} 2x \ge 0 \\ x+3 < 0 \\ -(x+3) = 2x \end{cases}$$

$$1)\ \begin{cases} x \ge 0 \\ x \ge -3 \\ x = 3 \end{cases} \quad \text{and} \quad 2)\ \begin{cases} x \ge 0 \\ x < -3 \\ x = -1 \end{cases}$$

1) $x = 3$ and 2) No solution. ■

2. Solve the modular inequality $|7x+13| < 5x+21$.

▶ The inequality can be expressed as two simultaneous linear inequalities:

$$\begin{cases} 7x+13 < 5x+21 \\ -(7x+13) < 5x+21 \end{cases} \Rightarrow \begin{cases} x < 4 \\ x > -\dfrac{17}{6} \end{cases} \Rightarrow$$

$$x \in \left(-\frac{17}{6}, 4\right).\ ■$$

3. Solve the modular inequality $|2x-3| \ge x$.

▶ According to the relevant formula, the inequality will be identical to the following union of two inequalities:

$$2x-3 \ge x \cup -(2x-3) \ge x \Rightarrow$$

$$x \ge 3 \cup x \le 1 \Rightarrow$$

$$x \in (-\infty, 1] \cup [3, +\infty).\ ■$$

Exercises

1. Solve for x

   a) $|x|=3$; b) $3|x|=12$;

   c) $|x|=-2$; d) $|x-5|=3$;

   e) $|x+4|=2x$; f) $|x+1|=-3x$;

   g) $|2x-3|=x$; h) $|2x-3|=3-x$.

2. Solve the absolute value equations

   a) $|x|=|2x-6|$; b) $|x+3|+|2x-1|=8$;

   c) $x|x|+2x+1=0$.

3. Find the sum of all roots of

   a) $2-3|x-5|=-4$; b) $|3x-1|=|5-x|$.

4. Solve the equations

   a) $|x^2-1|+x=5$; b) $|x-6|=|x^2-5x+9|$.

5. Find the product of roots

   a) $|3-|2+x||=1$; b) $|1-|1-x||=0.5$.

6. Figure out the product of roots of $x^2-3|x|-40=0$.

7. Find the roots of

$$|x^2+7x+10|=|x^2+2x+5|+|x+9|.$$

8. Solve

a) $|x|<3$; b) $|x-3|\le 2$;

c) $|2x-3|<x-1$; d) $|x|>3$;

e) $|x-3|\ge 2$; f) $|x-3|>2x$.

9. Find the range of values of x for which $|x|\le 4$ and, at the same time $x^2-9>0$. Your answer should be given as the union of two intervals.

10. Find the range of values of x for which $3|x-2|>4+|x-2|$ and, at the same time $5|x-1|-10<20-|x-1|$. Your answer should be given as the union of two intervals.

11. Find the solutions

a) $|x^2+5x|<6$; b) $x^2-|5x+6|<0$.

Homework

1. Solve the equations

a) $2|x|-4=0$; b) $|x+1|=2x+1$;

c) $|3x-1|-2=0$.

2. What are the roots of

a) $|x|+|x-1|=1$; b) $x^2-\frac{1}{2}|x|=0$.

3. Solve for x

a) $|3x-2|<7$; b) $|x^2-8|<1$;

c) $|x+1|>2$; d) $|5-2x|>1$.

4. Find the range of values of x for which $12|x-1|>5+4|x-1|$, expressing your answer as a union of intervals.

5. Solve the equation $|x^2+5x-4|=3x-1$.

Prefixes

Atto-	10^{-18} = 0.000,000,000,000,000,001
Femto-	10^{-15} = 0.000,000,000,000,001
Pico-	10^{-12} = 0.000,000,000,001
Nano-	10^{-9} = 0.000,000,001
Micro-	10^{-6} = 0.000,001
Milli-	10^{-3} = 0.001
Centi-	10^{-2} = 0.01
Deci-	10^{-1} = 0.1
Hect-, hecto-	10^{2} = 100
Kilo-	10^{3} = 1,000
Myria-	10^{4} = 10,000
Mega-	10^{6} = 1,000,000
Giga-	10^{9} = 1,000,000,000
Tera-	10^{12} = 1,000,000,000,000
Penta-	10^{15} = 1,000,000,000,000,000
Eksa-	10^{18} = 1,000,000,000,000,000,000
Googol	10^{100}

IV.1. 1. d) **2.** b). **3.** a) $x>2$; b) $x<4$; c) $x>-2$. **4.** a) $x\in(-\infty, -2)$; b) $x\in(-\infty, -3]$; c) $x\in(-\infty, 2)$. **5.** a) $x<10$; b) $x<7$; c) $x>6$; d) $x<-142/19$. **6.** I and II. **7.** $p-q$. **8.** Between 32^oF and 39.2^oF. **9.** a) $(a-b)^2 \geq 0$; b) $(a-b)^2(a^2+b^2) \geq 0$. **10.** a) $x\in[-6, +\infty]$; b) $x\in(-\infty, -1]$.

Homework: 1. $x<w<y<z$. **2.** a) $x>5$; b) $x\geq-2$; c) $y>-6$. **3.** a) $x<42$; b) $x\leq-20/3$. **4.** $450<E<750$. **5.** 10

IV.2. 1. a) $x\in(3, 7)$; b) $x\in(2/3, 2)$; c) $x\in(-\infty, 2/3)$; d) $x\in[3, 6)$. **2.** a); b) and c). **3.** 6. **4.** 3/2.

Homework: 1. –4. **2.** 3.

IV.3. 1. a) $x\in(0, 1)$; b) $x\in[-1, 1]$; c) $x\in(-1, 0)$. **2.** a) $x\in(0.5, 3.5)$; b) $x\in(-\infty, -6)\cup(1, +\infty)$; c) $x\in[-5, 10]$. **3.** a) $x\in(-\infty, -2)\cup(1, +\infty)$; b) $x\in(-3, 0]\cup\{1\}$; c) $x\in(-\infty, -3)\cup(-2, 4)\cup(4, 5)$. **4.** $x\in(-6, 2)$.

Homework: 1. a) $x\in[-\sqrt{18}, \sqrt{18}]$; b) $x\in(1-\sqrt{5}, 1+\sqrt{5})$; c) $x\in(-2, 1)\cup(1, +\infty)$; d) $x\in(-\infty, -4)\cup[-2, 7]$. **2.** 5.

IV.4. 1. a) $x\in(0, 3]$; b) $x\in(-4, 0)$. **2.** a) $x\in[-1, 3)$; b) $x\in[0.5, 1)\cup(1, +\infty)$; c) $x\in\emptyset$; d) $x\in(1, 3]$; e) $x\in(3, 12)$. **3.** a) $x\in(-1, 1)$; b) $x\in[5, 10)$; c) $x\in[-2, 2]$; d) $x\in(-\infty, -2)\cup[1, +\infty)$. **4.** $x\in[50, +\infty)$; **5.** a) 256; b) 19; c) $\emptyset$; d) 2; e) 63; f) 83. **6.** a) 3; b) $\emptyset$. **7.** a) –8; 1; b) 4; c) $(-1\pm\sqrt{5})/2$. **8.** 4. **9.** a) –2; b) 67. **10.** – 1.75.

Homework: 1. a) $x\in(-8, -1]$; b) $x\in[2, +\infty)$. **2.** $x\in(1, 4]$. **3.** a) 19; b) 1; c) 4; d) 4. **4.** a) $\emptyset$; b) 2; c) 48. **5.** –4. **6.** a) ±5; b) $\emptyset$; c) 9.

IV.5. 1. a) ±3; b) ±4; c) $\emptyset$; d) 8; 2; e) 4; f) –1/4; g) 1; 3; h) 0; 2. **2.** a) 2; 6; b) –10/3; 2; c) $1-\sqrt{5}$. **3.** a) 10; b) –2; 1.5. **4.** a) –3; 2; b) 1; 3. **5.** a) 0; b) –0.9375. **6.** –64. **7.** 1. **8.** a) (–3, 3); b) [1, 5]; c) (2, 3); d) $(-\infty, -3)\cup(3, +\infty)$; e) $(-\infty, 1)\cup(5, +\infty)$; f) $(-\infty, 1)$. **9.** $[-4, -3)\cup(3,4]$. **10.** $(-4, 0)\cup(4, 6)$. **11.** a) $(-6, -3)\cup(-2, 1)$; b) (–6, –1).

Homework: 1. a) ±2; b) 0; c) –1/3; 1. **2.** a) [0, 1] b) ±0.5; 0. **3.** a) (– 5/3, 3); b) $(-3, -\sqrt{7})\cup(\sqrt{7}, 3)$; c) $(-\infty, -3)\cup(1, +\infty)$; d) $(-\infty, 2)\cup(3, +\infty)$. **4.** $(-\infty, 3/8)\cup(13/8, +\infty)$. **5.** $\sqrt{21}-4$; 1.

It is possible to make a mistake by keeping silence.
English proverb

CHAPTER V. PROGRESSIONS

V.1. Arithmetic progressions

Terms

1. **a sequence of numbers** – sonlar ketma-ketligi | последовательность чисел;
2. **arithmetic progression** – arifmetik progressiya | арифметическая прогрессия;
3. **common difference (of arithmetic progression)** – (arifmetik progressiya) ayirmasi | разность (арифметической прогрессии);
4. **increasing (decreasing)** – o'suvchi (kamayuvchi) | возрастающая (убывающая);
5. **n-th term (common term)** – n-had (umumiy had) | n-й член (общий член);
6. **bisector** – bissektrisa | биссектриса;
7. **angle** – burchak | угол;
8. **altitude** – balandlik | высота;
9. **a two-digit number** – ikki xonali son | двузначное число;
10. **degree** – daraja, gradus | степень, градус.

Learning Objectives

- to be able to recognize and construct an arithmetic progression (AP);
- to know how to use proper formulae to find the n^{th} term, the common difference and the sum of an arithmetic progression.

Arithmetic progression

$a_1, a_1 + d, a_1 + 2d, ...$

n^{th} term of an arithmetic progression

$a_n = a_1 + d(n-1)$.

Sum of the first n terms of an arithmetic progression

$$S_n = \frac{a_1 + a_n}{2} \cdot n \quad or \quad S_n = \frac{2a_1 + d(n-1)}{2} \cdot n.$$

Arithmetic mean of two terms of an arithmetic progression

$$\frac{a_m + a_n}{2} = a_{\frac{m+n}{2}}.$$

(Note: The formula above is useful only when $(m + n)/2$ is a natural number).

Examples

1. Find the first term and the sum of the first fifteen terms of the arithmetic progression, whose fifth and seventeenth terms are 18 and 54, respectively.

▶ We will express the given terms through the n^{th} term formula of an arithmetic progression and then solve the simultaneous equations

$$\begin{cases} a_5 = 18 \\ a_{17} = 54 \end{cases} \Rightarrow \begin{cases} a_1 + d \cdot (5-1) = 18 \\ a_1 + d \cdot (17-1) = 54 \end{cases} \Rightarrow$$

$$\left.\begin{cases} a_1 + 4d = 18 \\ a_1 + 16d = 54 \end{cases}\right| (-)\uparrow \Rightarrow$$

$$\begin{cases} 12d = 36 \\ a_1 + 16d = 54 \end{cases} \Rightarrow \begin{cases} d = 3 \\ a_1 = 6 \end{cases}.$$

$$S_{15}=\frac{a_1+a_{15}}{2}\cdot 15=\frac{a_1+a_1+14d}{2}\cdot 15=$$

$$\frac{6+6+14\cdot 3}{2}\cdot 15=27\cdot 15=405.\ ■$$

2. What is the value of the fraction

$$\frac{10+20+30+...+200}{30+60+90+...+600}.$$

► The sequences of numbers 10, 20, 30, ..., 200 and 30, 60, 90, ..., 600 at the numerator and denominator form arithmetic progressions. Therefore, we first find how many terms there are at the numerator and denominator and use the sum formula to find the two sums.

$$d=a_2-a_1=20-10=10\Rightarrow$$

$$a_n=a_1+d(n-1)=200\Rightarrow$$

$$10+10(n-1)=200\Rightarrow n=20.$$

So, $S_{20}=\frac{a_1+a_{20}}{2}\cdot 20=\frac{10+200}{2}\cdot 20=2{,}100$.

Similarly,

$$d^*=b_2-b_1=60-30=30\Rightarrow$$

$$b_n=b_1+d^*(n^*-1)=600\Rightarrow$$

$$30+30(n^*-1)=600\Rightarrow n^*=20.$$

$$S_{20}^*=\frac{b_1+b_{20}}{2}\cdot 20=\frac{30+600}{2}\cdot 20=6{,}300.$$

Finally, $\frac{S_{20}}{S_{20}^*}=\frac{2{,}100}{6{,}300}=\frac{1}{3}$. ■

Exercises

1. Identify arithmetic progressions

   a) 0, 4, 8, 12, ... b) 3, 15, 27, 40, ...

   c) 53, 42, 31, 20, ... d) 2, 4, 8, 16,... .

2. In the arithmetic progression 12, 15, ... find

   a) 5^{th} term; b) 30^{th} term.

3. Write down the 6^{th} term of the arithmetic progression 19, 15, ...
4. For the arithmetic progression $a_1 = 8$ and $d = 4$. Calculate a_{10}.
5. $a_{10} = 192$ and $d = 2$ belong to an arithmetic progression. What is the value of a_1?
6. $a_{10} = 13$ and $a_5 = 18$ are given for an arithmetic progression. Find its common difference.
7. If an arithmetic progression has $a_1 = 1$ and $d = 4$, is the number 10091 a term of this progression?
8. Given that $a_1 = 8$ and $d = 4$ in an arithmetic progression, find S_{16}.
9. What is the arithmetic mean of the first 1,000 natural numbers?
10. The first row of a cinema contains 21 seats. In every other row there are two more seats than in the previous row. How many seats are there in the 40^{th} row?
11. 7, a_2, a_3, a_4, a_5, 22 are the first 6 terms of an arithmetic progression. Find the sum $a_2 + a_3 + a_4 + a_5$.
12. In an arithmetic progression $a_6 = 10$ and $S_{16} = 200$. Find a_{11}.
13. Let ABC be a triangle whose angles, measured in degrees, are in arithmetic progression. Let A be the least of these angles and C the largest. If the internal bisectors of A and C meet at I, calculate the angle $\angle AIC$. (Hint: The sum of internal angles of a triangle is always equal to 180^{o}).
14. In an arithmetic progression $a_1 = 5$, $d = 10$ and the sum of the first n terms is 32,000. Find n.
15. For the arithmetic progression the following holds true: $a_1+a_2+...+a_{16}+a_{17}=136$. Find $a_6 + a_{12}$.

16. The 6th term of an arithmetic progression is four times less than the 9th term and their sum is 20. Find the sum of the first nine terms.

17. The numbers 3/2, 9/2, 15/2, 21/2 etc. are in arithmetic progression. Find five consecutive terms of this progression whose sum is 187.5.

18. An arithmetic progression consists of 21 terms; the sum of the three terms in the middle is 129, and the sum of the last three terms is 237. Find the arithmetic progression.

19. A grandfather gave a total of 1,000,000 soums (one million Uzbek soums) to his five grandchildren. Starting with the youngest, each got 20,000 soums more than the next younger one. In other words, the youngest got one sum, the next got 20,000 soums more, and so on. How much did the youngest grandchild get?

20. Evaluate the following sum

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\ldots+\left(\frac{1}{100}+\frac{2}{100}+\ldots+\frac{99}{100}\right).$$

21. Find the value of

$$\frac{1\cdot2\cdot4+2\cdot4\cdot8+3\cdot6\cdot12+\ldots+20\cdot40\cdot80}{1\cdot3\cdot9+2\cdot6\cdot18+3\cdot9\cdot27+\ldots20\cdot60\cdot180}.$$

22. In much the same way as $(1+x)^2=1+2x+x^2$, it is known that if we expand the expression $(1+x)^{100}$ as a polynomial in ascending powers of x, we obtain an identity of the form

$$(1+x)^{100}=a_0+a_1x+a_2x^2+\ldots+a_{99}x^{99}+a_{100}x^{100}$$

where $a_0, a_1, a_2, \ldots a_{99}, a_{100}$ are suitable numbers (which you are not required to find). Determine the value of the sum

$$a_0+a_1+a_2+\ldots+a_{99}+a_{100}.$$

23. Farrukh put 5 soums in a box on 1st January 2000, then 10 soums on 1st February 2000, 15 soums on 1st March 2000, 20 soums on 1st April 2000 and so on (increasing the amount by 5

soums every month) until 1st of August 2008. How much money was in the box immediately after 1st August 2008?

Homework

1. In the arithmetic progression 5, 12, 19 ... find the 10^{th} and 50^{th} terms.
2. Find the sum of all two-digit natural numbers.
3. If the eleventh term of an arithmetic progression is 24, find S_{21}.
4. Let ABC be a triangle whose angles, measured in degrees, are in arithmetic progression whose common difference is 20. Let A be the least of these angles and C the largest. If the altitude from A meets the internal bisector of C at the point D, calculate the angle $\angle ADC$. (Hint: The sum of internal angles of a triangle is always equal to 180°).
5. Find $S_5 - 3S_4 + 3S_3 - S_2$, if S_n is the sum of the first n terms of an arithmetic progression.
6. How many terms of the arithmetic progression 4, 10, 16 ... must be added to get 198?
7. Find N if $1 + 2 + 3 + 4 + ... + N = 1275$.

Number systems

Binary (base 2)	Octal (base 8)	Decimal (base 10)	Hexadecimal (base 16)
0	0	0	0
1	1	1	1
10	2	2	2
11	3	3	3
100	4	4	4
101	5	5	5
110	6	6	6
111	7	7	7
1000	10	8	8
1001	11	9	9
1010	12	10	A
1011	13	11	B
1100	14	12	C
1101	15	13	D
1110	16	14	R
1111	17	15	F
10000	20	16	10
10001	21	17	11
10010	22	18	12
1100100	144	100	64
11111010111	3727	2007	7D7

Word play: There are 10 kinds of people in the world. Those who understand binary and those who don't.

Every truth passes through three stages before it is recognized. In the first, it is ridiculed, in the second it is opposed, in the third it is regarded as self-evident.
Arthur Schopenhauer (1788-1860), German philosopher

V.2. Geometric progressions

Terms

1. **geometric progression** – geometrik progressiya|геометрическая прогрессия;
2. **ratio (quotient) of a geometric progression** – geometrik progressiyaning mahraji| знаменатель геометрической прогрессии;
3. **an infinitely descending geometric progression** – cheksiz kamayuvchi geometrik progressiya| бесконечно убывающая геометрическая прогрессия;
4. **consecutive** – ketma-ket|последовательный;
5. **to form** – tashkil etmoq|образовать;
6. **to plug into a formula** – formulaga qo'ymoq|подставлять в формулу;
7. **to check** – tekshirmoq|проверять;
8. **to practice** – mashq qilmoq|практиковать;
9. **to round a number** – sonni yaxlitlamoq|округлять число;
10. **to analyze** – tahlil qilmoq|анализировать.

Learning Objectives

- to be able to recognize and construct a geometric progression;
- to know how to use formulae to find the n^{th} term, the ratio and the sum of a geometric progression.

Geometric progression

$b_1,\ b_1q,\ b_1q^2,...$

n^{th} **term of a geometric progression** $b_n = b_1q^{n-1}$.

Sum of the first n terms of a geometric progression

$$S_n = \frac{b_1(q^n - 1)}{q-1}.$$

Geometric mean of two terms of geometric progression

$$\sqrt{b_m \cdot b_n} = b_{\frac{m+n}{2}}.$$

(Note: The formula above is useful only when $(m + n)/2$ is a natural number).

Sum of all terms of an infinitely descending geometric progression:

$$S = \frac{b_1}{1-q}.$$

Examples

1. Find the first term and the sum of the first 11 terms of the geometric progression; whose fifth and tenth terms are 48 and 1,536, respectively.

► We will express the given terms through the n^{th} term formula of a geometric progression.

$$\begin{cases} b_5 = b_1 \cdot q^{5-1} = 48 \\ b_{10} = b_1 \cdot q^{10-1} = 1{,}536 \end{cases} \Rightarrow \begin{cases} b_1 \cdot q^4 = 48 \\ b_1 \cdot q^9 = 1{,}536 \end{cases} \Bigg| \uparrow (:) \Rightarrow .$$

$$S_{11} = \frac{b_1\left(q^{11}-1\right)}{q-1} = \frac{3\cdot\left(2^{11}-1\right)}{2-1} = 3\cdot(2{,}048-1) = 6{,}141. \blacksquare$$

2. Find the ratio of the infinitely descending geometric progression, whose sum is 3 times the sum of its first three terms.

▶ The following equation represents the given condition:

$$S=3\cdot S_3 \Rightarrow \frac{b_1}{q-1}=3\cdot\frac{b_1\left(q^3-1\right)}{q-1}\Rightarrow$$

$$\frac{-(1-q)}{3\cdot(1-q)}=q^3-1\Rightarrow q^3=1-\frac{1}{3}\Rightarrow q=\sqrt[3]{\frac{2}{3}}.\ ■$$

Exercises

1. Identify geometric progressions

   a) 2, 4, 8, 16, ... b) 64, 32, 16, ...

   c) 1, 5, 10, ... d) 2, − 2, 2, − 2, ...

2. In the geometric progression 5, 10, 20, ... find b_{10} and S_{10}.
3. In the geometric progression b_2 = 6 and b_5 = 162 compute b_1 and q.
4. What is the sum 2 + 4 + 8 + ... + 128 + 256 + 512 equal to?
5. The first term of a geometric progression is 150 and the fourth is 1.2. Find the fifth term.
6. In a geometric progression $q = -2$ and S_5 = 5.5. Find b_5.
7. Insert three numbers between the numbers 1 and 256 so that all five numbers form a geometric progression.
8. The difference between the first and sixth terms of a geometric progression is 1,210 and q = 3. Find S_5.
9. In a geometric progression b_1 = 2 and q = 2. How many terms of the progression are needed to make their sum equal to 1,022?
10. $b_1 = \sqrt{2}$ and q = 0.5 in an infinitely descending geometric progression. Find S.
11. b_1 = 2 and q = − 0.5 of an infinitely descending geometric progression. Find b_4 and S.
12. Find the sum 512 + 256 + ... + 2. Compare your solution with the one in question 4 above.

13. If $b_1 = 2$ and $S = 5$, find q.
14. $b_3 = 2$ and $b_6 = 0.25$ in an infinitely descending geometric progression. Find S.
15. Solve the equation $\sqrt{x\sqrt{x\sqrt{x...}}} = 2009$.
16. The sum of an infinitely descending geometric progression is 32 and the sum of the first five terms is 31. Find the common ratio of the progression.
17. Find the common ratio of an infinitely descending geometric progression, whose first term is five times greater than the sum of all consecutive terms.
18. Calculate the sum $2 + \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + \cdots$.
19. If $q = 0.5$ and $S_6 = \frac{63}{2}$, find b_5.
20. Can three numbers form both arithmetic and geometric progression at the same time?

Homework

1. Write down the 6^{th} and 20^{th} terms of the geometric progression 2, 6, 18 ... .
2. $b_3 = 0.1$ and $b_7 = 8.1$ in a geometric progression. Find b_1 and S_4.
3. It is known that A, B, C are the a^{th}, b^{th}, c^{th} terms of a geometric progression, respectively. Show that $A^{b-c} \cdot B^{c-a} \cdot C^{a-b} = 1$.
4. Is the number 72.9 a term of the geometric progression 0.1, 0.3, 0.9...? If yes, what term is it?
5. $b_2 = 8$ and $q = 0.5$ of an infinitely descending geometric progression. Find S.
6. What is the sum 432 + 72 + 12 + 2 + ...?
7. The sum of the first four terms of an infinitely descending geometric progression is $\frac{80}{27}$ and $q = \frac{1}{3}$. Find the first term.
8. What is the sum of the first 10 terms of the geometric progression 1, − 2, 4...?

Mathematical symbols:

Symbol	Date	Meaning	Author
$+$	1489	plus or positive	Y. Widman
$-$		minus or negative	
$\sqrt{}$	1525	square root	K. Rudolf
$=$	1557	equal to	P. Record
$\neq$		not equal to	
$\times$	1631	multiplied by	W. Oughtred
$>$		greater than	Th. Harriot
$<$		less than	
log	1632	logarithm	B. Cavalieri
$\perp$	1634	perpendicular to	P. Herigone
∞	1655	infitity	J. Wallis
∂	1675	differential	G. Leibniz
$\int$		integral	
$\parallel$	1677	parallel to	W. Oughtred
$:$	1684	is to (ratio)	G. Leibniz
$\div$		divided by	
$*$	1698	multiplied by	
π	1706	ratio of circumference to its diameter	W. Jones
$f(x)$	1734	function	L. Euler
e	1736	base of natural logarithm	
sin	1748	sine	
cos		cosine	
tan	1753	tangent	
Δ	1755	difference	
$\sum$		sum of	
$f'(x),\ y'$	1770	derivative	J. Lagrange
$!$	1808	factorial	Ch. Kramp
$\prod$	1812	product of	G. Gauss
$\|x\|$	1841	absolute value	K. Weierstras
lim	1853	limit	W. Hamilton
$\equiv$	1857	identically equal to	B. Rieman
$\approx$	1882	approximately eqiual to	A. Gunter
$\cup$	1888	union of	J. Peano
$\cap$		intersection of	
$\in$	1895	belonging	

V.1. 1. a) and c). **2.** a) 24; b) 99. **3.** –1. **4.** 44. **5.** 174. **6.** –1. **7.** No. **8.** 604. **9.** 500.5. **10.** 99. **11.** 58. **12.** 15. **13.** 120°. **14.** 80. **15.** 16. **16.** 0. **17.** 11; 12; 13; 14; 15. **18.** a_1=3; d=4. **19.** 160,000. **20.** 2,475. **21.** 8/27. **22.** 2^{100}. 1st method: use geometric progression referring to the Pascal triangle; 2nd method: plug the number 1 for x in $(1+x)^{100}$. **23.** 27,300.

Homework: 1. 68; 348. **2.** 4,905. **3.** 504. **4.** 130°. **5.** 0. **6.** 25. **7.** 50.

V.2. 1. a); b) and d). **2.** 2,560; 5,115. **3.** 2; 3. **4.** 1,022. **5.** 0.6. **6.** 8. **7.** 4, 16, 64 or – 4, 16, – 64. **8.** 605. **9.** 9. **10.** $2\sqrt{2}$. **11.** – 0.25; 4/3. **12.** 1022. **13.** 0.6. **14.** 16. **15.** 2,009. **16.** 0.5. **17.** 1/6. **18.** $2.\dot{3}\dot{5}$. **19.** 1. **20.** Yes, if the numbers are all equal.

Homework: 1. 486; $2\cdot 3^{20}$. **2.** 3; 4/9 and – 3; – 2/9. **3.** Denote $A=b_1\cdot q^{a-1}$; $B=b_1q^{b-1}$; $C=b_1\cdot q^{c-1}$. **4.** Yes, 7th. **5.** 16. **6.** 518.4. **7.** 2. **8.** 1,023.

Discovery consists of seeing what everybody has seen and thinking what nobody has thought.
Albert Gyorgyi (1893-1986), Hungarian-born American biochemist

CHAPTER VI. Functions

VI.1. Linear functions

Terms

1. **straight line** – to'g'ri chiziq | прямая линия;
2. **the Cartesian system of coordinates** – Dekart koordinatalar sistemasi | Декартого система координат;
3. **gradient, a slope** – burchak koeffitsiyenti | градиент, угловой коэффициент;
4. **an increasing (decreasing) function** – o'suvchi (kamayuvchi) funktsiya | возрастающая (убывающая) функция;
5. **the graph of a function** – funktsiya grafigi | график функции;
6. **a quadrant (a quarter)** – kvadrant (koordinata choragi) | квадрант (координатная четверть);
7. **function domain (function range)** – funktsiyaning aniqlanish (qiymatlar) sohasi | область определения (область значения) функции;
8. **to draw (sketch, plot) a graph** – grafik chizmoq | начертить график;
9. **to create a table of values** – qiymatlar jadvalini tuzmoq | создать таблицу значений;
10. **to plot points** – nuqtalarni joylashtirmoq (belgilamoq) | отмечать точки.

Learning Objectives

- to know how to draw the graph of a linear function;
- to be able to determine if two lines are parallel or perpendicular to each other and to identify an increasing or decreasing function.

Linear function

$y = ax + b$,

where a is the slope and b is an intercept.

Gradient of a line

$\tan\varphi = a$,

where φ is the angle between the line and the x-axis and a is the gradient.

Equation of the straight line passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

For $y_1 = k_1x + b_1$ *and* $y_2 = k_2x + b_2$, if

1) $k_1 = k_2$, *then* $y_1 \,||\, y_2$ (These lines are parallel).

2) $k_1k_2 = -1$, *then* $y_1 \perp y_2$ (These lines are perpendicular).

For $y = kx + b$, if

1) $k > 0$, *then* $y \uparrow$ (This function is an increasing function);

2) $k < 0$, *then* $y \downarrow$ (This function is a decreasing function);

3) $k = 0$; $y = const$ (This function is a constant function).

Examples

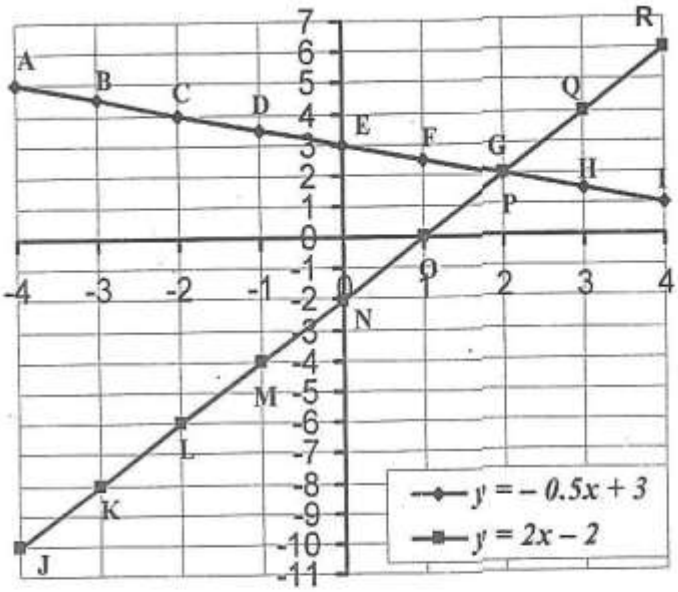
1. Draw the graph of the linear functions and show the point of intersection clearly:
   $x + 2y = 6$ and $2x - y = 2$

► We create tables of values of x and y to identify some points of the straight lines. (Two points for each are enough).

x	$y = -0.5x + 3$	Points
-4	$(-0.5)(-4) + 3 = 5$	$A\,(-4, 5)$
-3	$(-0.5)(-3) + 3 = 4.5$	$B\,(-3, 4.5)$
-2	4	$C\,(-2, 4)$
-1	3.5	$D\,(-1, 3.5)$
0	3	$E\,(0, 3)$
1	2.5	$F\,(-1, 2.5)$
2	2	$G\,(-2, 2)$
3	1.5	$H\,(-1, 1.5)$
4	1	$I\,(-2, 1)$

x	$y = 2x - 2$	Points
-4	$2(-4) - 2 = -10$	$J\,(-4, -10)$
-3	$2(-3) - 2 = -8$	$K\,(-3, -8)$
-2	-6	$L\,(-2, -6)$
-1	-4	$M\,(-1, -4)$
0	-2	$N\,(0, -2)$
1	0	$O\,(1, 0)$
2	2	$P\,(2, 2)$
3	4	$Q\,(3, 4)$
4	6	$R\,(4, 6)$

Now, we plot the points on the plane.



The graph above shows that the point of intersection is G (2, 2) or P (2, 2). ■

2. Find the equation of the straight line, which passes through the points $A(-2,3)$ and $B(2,-1)$.

▶ 1st method: Use the formula to determine the equation.

$$\frac{y-3}{-1-3}=\frac{x-(-2)}{2-(-2)}\Rightarrow\frac{y-3}{-4}=\frac{x+2}{4}\Rightarrow$$

$$y-3=-x-2\Rightarrow y=-x+1$$

2nd method: The equation of a straight line is $y=ax+b$. If a point belongs to a line, then the coordinates of the point satisfy the equation of that line. So, if we plug the coordinates of the given two points into the general equation we end with the following simultaneous linear equations:

$$\begin{cases}3=a\cdot(-2)+b\\-1=a\cdot 2+b\end{cases}\Rightarrow\begin{cases}3=-2a+b\\-1=2a+b\end{cases}\Bigg|(+)\Rightarrow$$

$$\begin{cases}2=2b\\-1=2a+b\end{cases}\Rightarrow\begin{cases}b=1\\-1=2a+1\end{cases}\Rightarrow\begin{cases}a=-1\\b=1\end{cases}$$

Thus, $y=-x+1$. ■

Exercises

1. Indicate the following points on the Cartesian plane

   A (2, 3); B (1, −4); C (−3, 1); D (−2, −3).

2. Identify linear function

   a) $y = 2x + 5$; b) $y = x^2 + 3x - 4$;

   c) $4x + 2y = 5$; d) $x + y = x^3$.

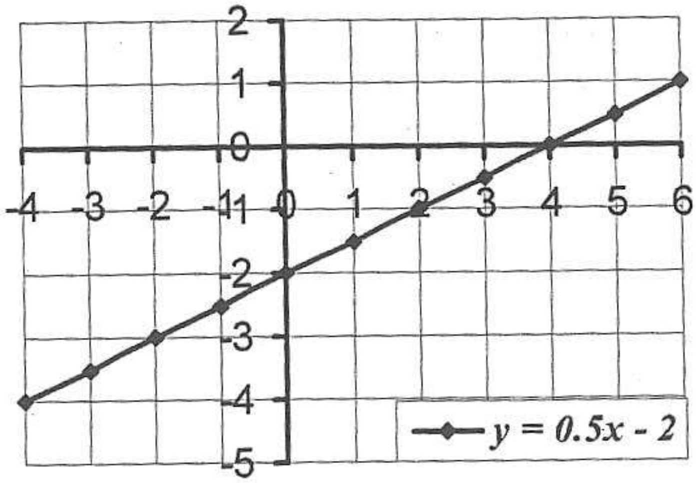
3. Draw the graph of the following linear functions

   a) $y = 3x - 2$; b) $y = -x + 1$;

   c) $y = -2$; d) $x = -3$.

4. Draw the line $y = x + 1$, if the function domain is $[-2, 2]$.
5. From the graph below point out the range of values of x for which the function obtains

   a) negative values b) values no less than -2.



6. In which quadrants is the graph of $y = ax + b$ located if

   a) $a > 0$ and $b < 0$; b) $a < 0$ and $b > 0$;

   c) $a > 0$ and $b = 0$; d) $a = 0$ and $b < 0$?
7. For what value of k does the graph of $y = kx - 5$ pass through the point

   a) A (4, 3); b) B (– 2, 6)?
8. Write down the equation of the line which passes through the points

   a) (2, 2) and (5, 5); b) (1, 3) and (4, 12);

   c) (– 2, – 3) and (5, 11).
9. Solve the simultaneous linear equations graphically

   $$\begin{cases} y = 2x + 1 \\ y = -2 - x \end{cases}.$$
10. On what values of t do the points A (3, 8), B (9, t) and C (– 5, 0) lie on the same straight line?
11. Investigate whether the two lines are parallel or perpendicular

    a) $y_1 = 2x + 3$; $y_2 = 2x - 2$;

b) $y_1 = 4x + 1$; $y_2 = -0.25x - 2$.

12. For what values of k are the lines $y = 2x/3 - 2$ and $y = kx + 2$

a) parallel;

b) perpendicular?

13. Find the values of k for which

$f(x) = (k-1)x + k^2 - 3, \quad x \in (-\infty, +\infty)$:

a) increases; b) decreases;

c) is constant.

14. Find the equation of the straight line which passes through

a) (3, 7) and has a gradient of 1;

b) (2, 8) and has a gradient of 3.

15. A firm that manufactures calculators experienced a decline in calculator sales for a 5-month period, as shown in the table below.

Month	Sales (in 1,000,000 soums)
July	18.6
August	17.4
September	16.2
October	15.0
November	13.8

a) Express the relationship between time (t) and sales (s) as a linear function.

b) If sales continue to decline at the same rate, what would be the company's sales, in million soums, for December?

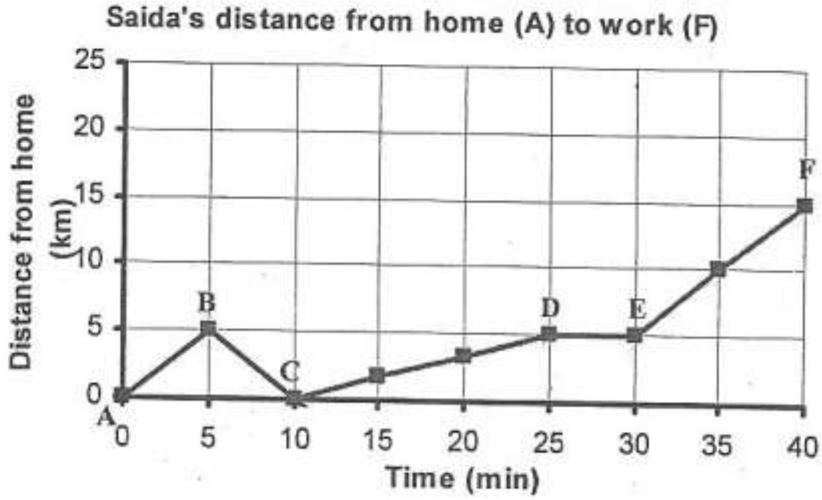
16. The cost of sending a package is 30 cents per ounce in addition to a basic fee of 5 dollars. If integer x represents the weight in ounces of a certain package, which of the following represents the cost, in dollars, of sending the package?

a) $5x + 0.30$; b) $(5 + x) \cdot 0.30$;

c) $5 - 0.30x$; d) $0.30x + 5$;

e) $0.70x + 5$.

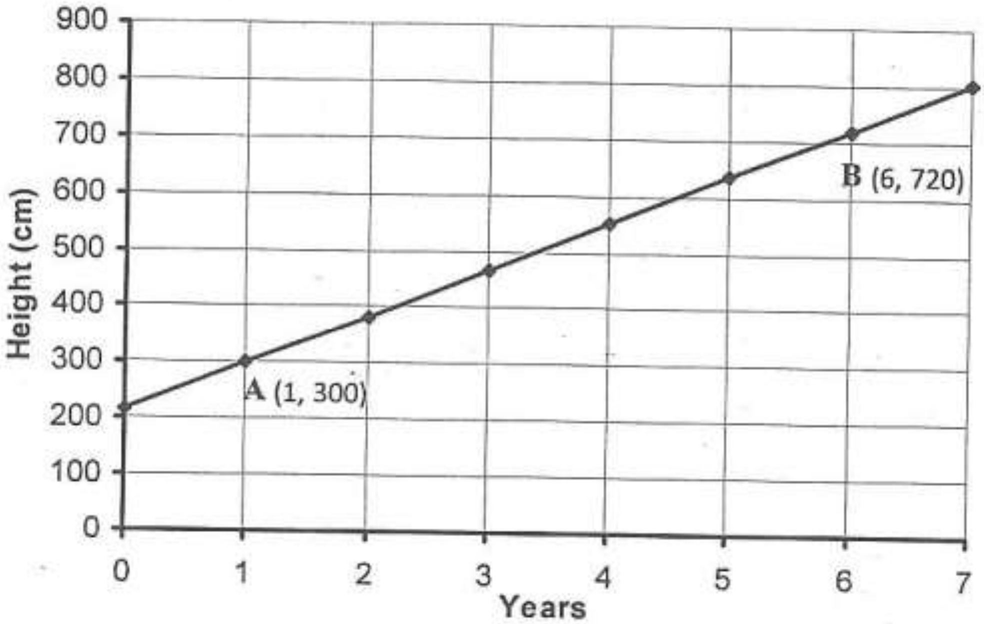
17. The graph below shows Saida's distance from home (A) to work (F) at various times during her drive.



a) Saida left her briefcase at home and had to return to get it. State which point represents when she turned back around to go home and explain how you arrived at that conclusion.

b) Saida also had to wait at the railroad tracks for a train to pass. How long did she wait?

18. The graph below shows the increase in height of a tree over a number of years. Express the relationship between time and height of the tree in terms of a linear function.



19. In an experiment, the following measurements were made:

x	5	10	15	30	40	60	90
y	30	50	70	130	200	250	370

Theoretical predictions indicate that x and y are related linearly. It is also thought that one measurement is incorrect. Draw a suitable graph to

a) find the wrong point;

b) predict the value of y for $x = 75$.

Homework

1. Draw the graph of the linear functions

   a) $y=2x-1$; b) $y=-x/3+1$; c) $y = 0$.

2. Complete this table for the line $y=4x-5$

x	-3	-2	-1	0	1	2	3
y							

3. What is the gradient of the line joining the points

   a) (3, 5) and (5, 9); b) (− 6, 4) and (− 3, 1).

4. In what quadrant do the lines $y=2x+1$ and $y=-2-x$ meet?

5. Find the equation of the straight line which passes through

   a) (4, − 4) and has a gradient of − 1;

   b) points A (− 1, 3) and B (3, 1).

6. For what values of a and b is there no solution to

$$\begin{cases} ax-5y=-1 \\ 6x+15y=b+3 \end{cases}?$$

7. Check if

   a) the lines $y=2x-1$ and $y=-\frac{x}{2}+3$ are parallel or perpendicular;

   b) the lines $y=15x+23$ and $y=-28x+14$ are increasing or decreasing.

8. Nigora's monthly salary is determined by the formula s=85,000+0.1x, where x is the total amount of her monthly sales, both expressed in soums. If the total of Nigora's sales for July was 400,000 soums, what was her salary that month?

9. Rahimjon has 60,000 soums at present and earns 1,000 soums per hour. Write down an equation of the linear function of his budget in terms of hours worked.

10. In an experiment the following values have been found

x	0	0.5	1.5	2	2.5
y	5	6.5	10.5	11	12.5

It is thought that these values fit a linear formula y = Ax + B, but one of the values in the table is wrong. Plot the values of the table to find the wrong value and thus find the value of the constants A and B.

SI.

The International System of Units (or Systeme International d'Unites – SI) is the current form of the metric system that has been in use since 1960.

The seven base SI units:

Unit	Symbol	Quantity
meter	m	length/distance
kilogram	kg	mass
ampere	a	electric current
Kelvin	K	thermodynamic temperature
candela	cd	luminosity
second	s (or sec)	time
mole	mol	amount of substance

In questions of science, the authority of a thousand is not worth that humble reasoning of a single individual.

Galileo Galilee (1564-1642), Italian physicist and astronomer

VI.2. Quadratic functions

Terms

1. **quadratic function** – kvadrat funktsiya | квадратичная функция;
2. **parabola** – parabola | парабола;
3. **coordinates** – koordinatalar | координаты;
4. **vertex of parabola** – parabola uchi | вершина параболы;
5. **the x-axis (y-axis)** – abtsissa (ordinata) o'qi | ось абсциссы (ординаты);
6. **symmetrical axis of parabola** – parabolaning simmetriya chizig'i | ось симметрии параболы;
7. **to be located (situated) at** – joylashmoq | распологаться, находиться;
8. **a charactersitic point** – xarakteristik nuqta | характеристическая точка;
9. **x-intercept (y-intercept)** – x o'qini (y o'qini) kesish nuqtasi | точка пересечения с осью x (с осью y);
10. **distance** – masofa | расстояние, дистанция.

Learning Objectives

- to identify a quadratic function;
- to know how to find the coordinates of the vertex of the parabola and to draw the graph of a quadratic function.

Quadratic function

$y = ax^2 + bx + c$,

where a, b and c are constant numbers.

The branches of a parabola $y = ax^2 + bx + c$ look

1) up $\cup$, if the coefficient $a > 0$

2) down $\cap$, if the coefficient $a < 0$.

Vertex of a parabola $y = ax^2 + bx + c$

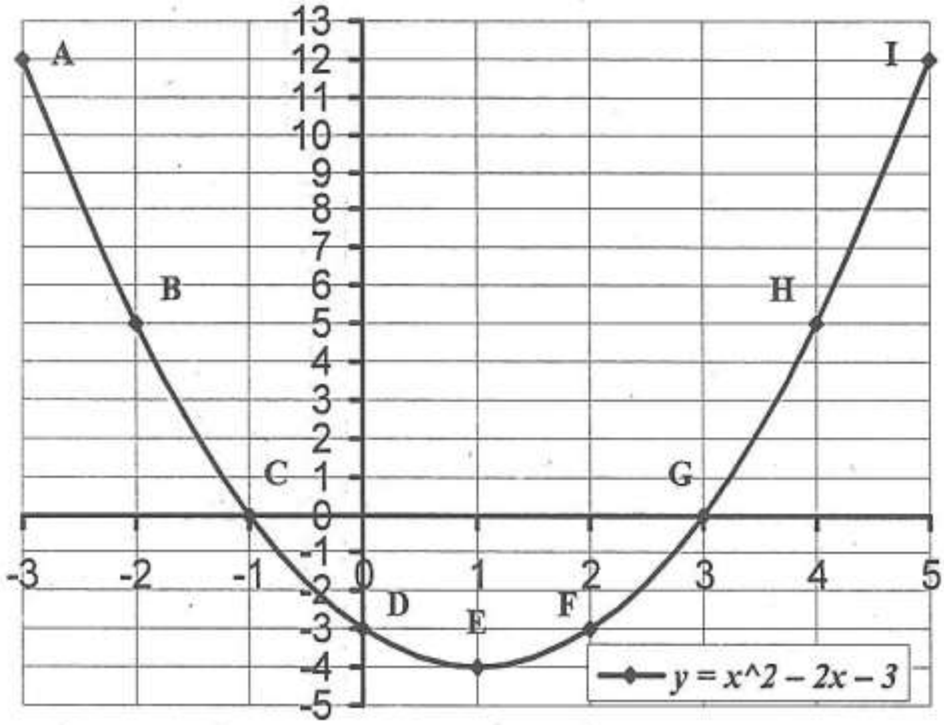
$$V\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right).$$

Examples

1. Draw the curve of the parabola $y = x^2 - 2x - 3$.

► 1[st] method. Create a table of values of x and y, indicate the points on the plane and draw the curve (parabola) through these points.

x	$y = x^2 - 2x - 3$	**Points**
-3	$(-3)^2 - 2(-3) - 3 = 12$	$A(-3, 12)$
-2	$(-2)^2 - 2(-2) - 3 = 5$	$B(-2, 5)$
-1	$(-1)^2 - 2(-1) - 3 = 0$	$C(-1, 0)$
0	$0^2 - (2)(0) - 3 = -3$	$D(0, -3)$
1	$1^2 - (2)(1) - 3 = -4$	$E(1, -4)$
2	$2^2 - (2)(2) - 3 = -3$	$F(2, -3)$
3	$3^2 - (2)(3) - 3 = 0$	$G(3, 0)$
4	$4^2 - (2)(4) - 3 = 5$	$H(4, 5)$
5	$5^2 - (2)(5) - 3 = 12$	$I(5, 12)$



2^{nd} method. Find all characteristic points of the parabola, which enable us to spot the parabola more efficiently. We focus on the three characteristic points:

a) x-intercept $y = 0$:

$$x^2 - 2x - 3 = 0 \Rightarrow x_1 = -1;\ x_2 = 3 \Rightarrow$$

$A\ (-1, 0)$; $B\ (3, 0)$.

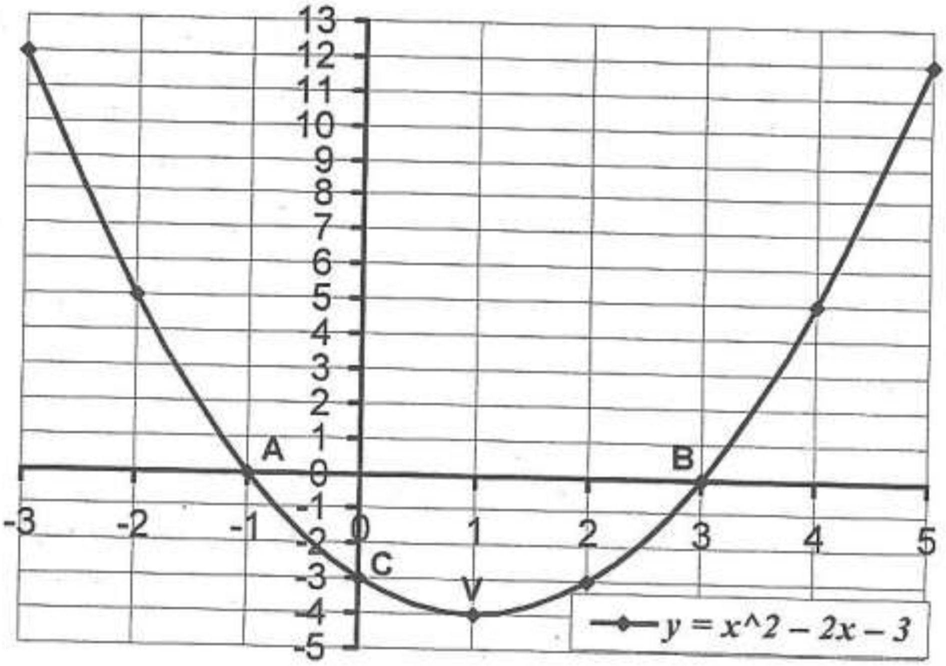
b) y-intercept $x = 0$:

$$y = 0^2 - 2\cdot 0 - 3 = -3 \Rightarrow$$

$C\ (0, -3)$.

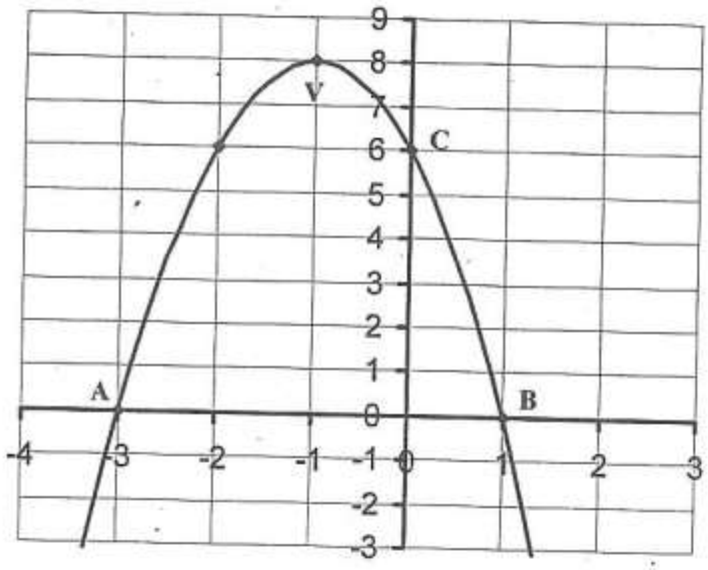
c) Vertex of the parabola:

$$V\left(-\frac{b}{2a},\ c - \frac{b^2}{4a}\right) = V\left(-\frac{-2}{2\cdot 1},\ -3 - \frac{(-2)^2}{4\cdot 1}\right) = V(1, -4).$$



■

2. The graph of a quadratic function is given below. Find the equation of the function that represents this graph.



▶ The general form of the quadratic equation is $y = ax^2 + bx + c$. The four characteristic points are A (– 3, 0), B (1, 0) C (0, 6) and V (–1, 8). These points belong to the parabola, which means their coordinates satisfy the equation of the parabola. So,

$$\begin{cases} a\cdot(-3)^2+b\cdot(-3)+c=0 \\ a\cdot(1)^2+b\cdot(1)+c=0 \\ a\cdot(0)^2+b\cdot(0)+c=6 \end{cases} \Rightarrow \begin{cases} 9a-3b+c=0 \\ a+b+c=0 \\ c=6 \end{cases} \Rightarrow$$

$$\begin{cases} 9a-3b+6=0 \\ a+b+6=0 \\ c=6 \end{cases} \Rightarrow \begin{cases} a=-2 \\ b=-4 \\ c=6 \end{cases}.$$

So, $y=-2x^2-4x+6$. ■

Exercises

1. Identify quadratic functions

   a) $y = 4x^2 + 5x + 5$; b) $7x - 5x^2 - 11 + y = 0$;

   c) $5x + 2y = 3y + 2x + 11$; d) $y = x - x^2$.

2. If $f(x)=x^2-2x+3$, find

   a) $f(3)$; b) $f(-1)$; c) $f(\sqrt{2}+1)$.

3. For what value of k will the following be true if $f(x)=4x^2+2kx+k-2$?

   a) $f(-1)=-2$; b) $f(2)=-4$.

4. Draw the graph of

   a) $y=x^2$; b) $y=5x^2$;

   c) $y=x^2-1$; d) $y=-x^2+1$;

   e) $y=x^2+2x+2$; f) $y=x^2+5x-6$;

   g) $y = (3x-9)(x+1)$; h) $y = (2x-1)^2 + 1$.

5. For what value of k does the graph of $f(x)=kx^2+3$ pass through the point

a) A (1, 4); b) B (2, 5)?

6. Find the smallest value of the function $y=x^2-6x+7$.

7. The point M (2, 5) belongs to the parabola $y=-x^2+ax+5$. Find the y coordinate of its vertex.

8. For what value of b does the parabola $y=x^2+bx+4$ touch the x-axis?

9. The graph of $y = 4(x - 2)(x + 3)$ cuts the x-axis at two points P and Q. What is the length of the line segment PQ?

10. Determine the range of the function $y=-x^2+6x-12$.

11. Find the distance between the origin and the axis of symmetry of the parabola $y=x^2-5x+19$.

12. If the point A (2, 3) is the vertex of the parabola $y=x^2+bx+c$, find b and c.

13. Show the interval for which the functions below increase

a) $y=x^2+1$; b) $y=-x^2+4$;

c) $y=x^2+2x+3$.

14. For what value of t does the parabola $y=3x^2-4x+5t$ have two common points with the x-axis?

15. For what values of m does the graph of

$$y=(m+4)x^2-2(m+2)x+1$$

lie above the x-axis?

16. Find the intersection points of the graphs $y=-x^2+2x$ and $y=x-2$.

17. An architect is designing a museum entranceway in the shape of a parabolic arch represented by the equation $y = -x^2 + 20x$, where $0 \le x \le 20$ and all dimensions are expressed in feet. Sketch a graph of the arch and determine its maximum height, in feet. (1 foot = 12 inches = $30.48cm$).

Homework

1. If $f(x)=x^2+x+1$, find f (3.5).

2. Find the coordinates of the vertex of the following parabolas

   a) $y=x^2-6x+7$; b) $y=x^2+4x+4$.

3. Find the greatest value of $y=1+2x-x^2$.

4. Draw the graphs of

   a) $y=2(x-2)^2-1$; b) $y=-(x+1)^2$.

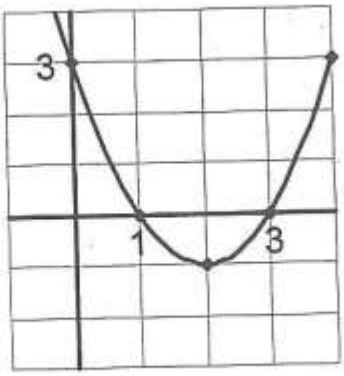
5. For what values of x will the values of the function $y=x^2$ be greater than 9?

6. In what quadrants is the graph of the function $y=-2x^2+4x-8$ located at?

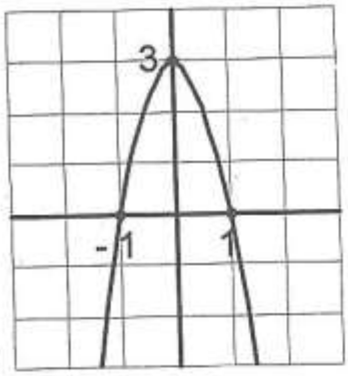
7. Show the interval where the graph of the function decrease:

   a) $y=x^2+4x+5$; b) $y=x^2+2x$.

8. Find the equation of the function from the graphs below



a)



b)

9. Farkhod is in a car at the top of a roller-coaster ride. The distance, d, of the car from the ground as the car descends is determined by the equation $d = 144 - 16t^2$, where t is the number of seconds it takes the car to travel down the ride. How many seconds will it take Farkhod to reach the ground?

Conversion formulas

Conversion from imperial to metric:

Length:	1 inch (in) = 2.54cm 1 foot (ft) = 30.48cm 1 yard (yd) = 91.4cm 1 mile (mi) = 1.61km
Weight:	1 ounce (oz) = 28.35 grams 1 pound (lb) = 0.45 kilograms
Temperature:	Fahrenheit (^oF), Celsius (^oC), Kelvin (^oK). $^oF = (^oC\text{x}1.8) + 32$, $^oC = {^oK} - 273.16$

I know not with what weapons World War III will be fought, but World War IV will be fought with sticks and stones.
Albert Einstein (1879-1955), German-born American physicist

VI.3. Polynomial and exponential functions

Terms

1. **polynomial (power) function** – darajali funktsiya | степенная функция;
2. **curve** – egri chiziq | кривая;
3. **asymptote** – assimptota | ассимптота;
4. **hyperbola** – giperbola | гипербола;
5. **exponential function** – ko'rsatkichli funktsiya | показательная функция;
6. **symmetrical** – simmetriyaviy | симметричный;
7. **property** – xossa | свойство;
8. **graphical (analytical) method of solution** – grafik (analitik) yechish usuli | графический (аналитический) метод решения;
9. **table of values of x and y** – x va y ning qiymatlar jadvali | таблица значений x и y;
10. **plane** – tekislik | плоскость.

Learning Objectives

- to recognize the degree of a function and exponential functions;
- to know how to draw the graph of a polynomial function and an exponential function;
- to be able to solve equations and inequalities containing exponential functions.

Polynomial function

$y = ax^n$,

where a and n are constant numbers.

Exponential function

$y=a^x$,

where $a>0$ and $a\neq 1$.

Properties of exponential functions $y=a^x$

1) *If* $a>1$, then it is an increasing function;

2) *If* $0<a<1$, then it is a decreasing function.

Exponential inequalities and their solutions

1) $a^{P(x)}<a^{Q(x)} \Rightarrow \begin{cases} P(x)<Q(x), \text{ if } a>1 \\ P(x)>Q(x), \text{ if } 0<a<1 \end{cases}$,

where $P(x)$ and $Q(x)$ are the algebraic expressions.

2) $a^{P(x)}\leq a^{Q(x)} \Rightarrow \begin{cases} P(x)\leq Q(x), \text{ if } a>1 \\ P(x)\geq Q(x), \text{ if } 0<a<1 \end{cases}$,

where $P(x)$ and $Q(x)$ are the algebraic expressions.

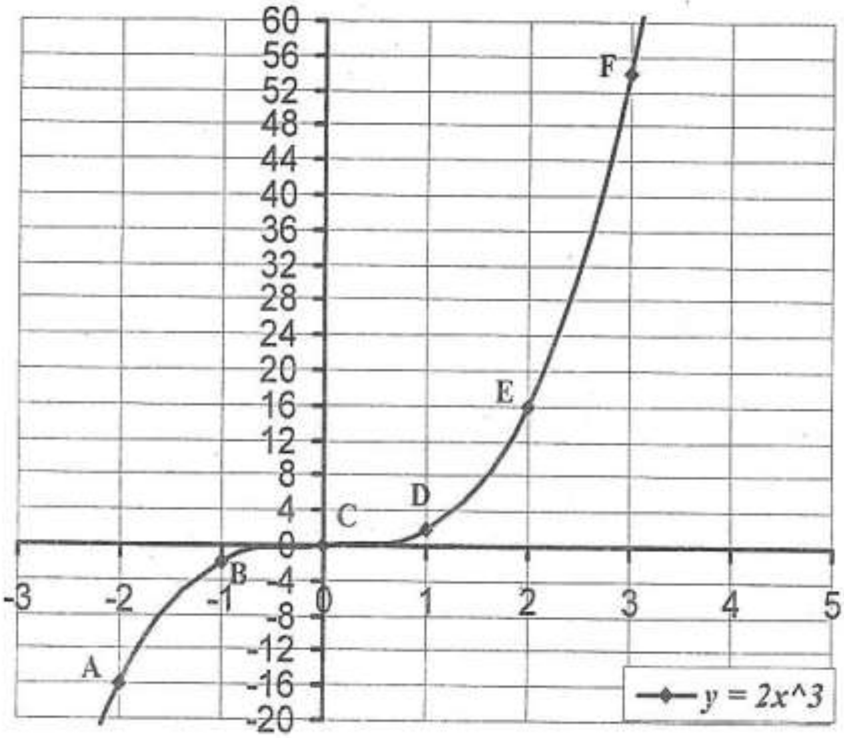
Examples

1. Draw the graph of the third degree polynomial function $y=2x^3$ for $x\in[-2;3]$.

► We create a table of values of x and y, where the range of x is between − 2 and 3.

x	$y=2x^3$	Points
−2	$2(-2)^3=-16$	$A(-2,-16)$
−1	$2(-1)^3=-2$	$B(-1,-2)$
0	$2(0)^3=0$	$C(0,0)$
1	$2(1)^3=2$	$D(1,2)$
2	$2(2)^3=16$	$E(2,16)$
3	$2(3)^3=54$	$F(3,54)$

We then plot the points on the plane and sketch the graph through these points.



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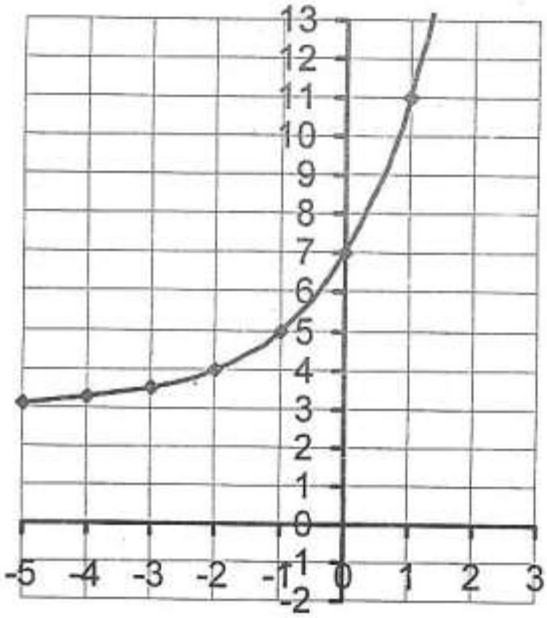
2. The graph of the function $y = a \cdot 2^x + b$ is given below. Find the values of a and b.



▶ The points $A(0, 7)$ and $B(1, 11)$ lie on the graph, therefore the coordinates of the points must satisfy the equation:

$$\begin{cases} 7 = a\cdot 2^0 + b \\ 11 = a\cdot 2^1 + b \end{cases} \Bigg| (-) \Rightarrow \begin{cases} -4 = -a \\ 11 = 2a + b \end{cases} \Rightarrow \begin{cases} a = 4 \\ b = 3 \end{cases} \Rightarrow$$

$$y = 4\cdot 2^x + 3 \Rightarrow y = 2^{x+2} + 3. \blacksquare$$

3. Solve the inequality $\left(\frac{1}{3}\right)^{x^{3-9x}} > 1$.

▶ Express both sides in the same base and use the formula.

$$\left(\frac{1}{3}\right)^{x^{3-9x}} > 1 \Rightarrow \left(\frac{1}{3}\right)^{x^{3-9x}} > \left(\frac{1}{3}\right)^0 \Rightarrow x^3 - 9x < 0 \Rightarrow$$

$$x(x-3)(x+3) < 0 \Rightarrow x \in (-\infty; -3) \cup (0; 3). \blacksquare$$

Exercises

1. For $y = 3x^3$, find

   a) $f(1)$; b) $f(-3)$; c) $f(\sqrt[3]{5})$.

2. Draw the graph of the following functions

   a) $y = 2$; b) $y = 3x$; c) $y = 2x^2$;

   d) $y = \frac{3}{x}$; e) $y = x^3$; f) $y = x^4$.

3. The domain of $f(x) = x^3 + 2$ is $-1 < x < 3$. Find its range.

4. For what value of k does the graph of the function $y = kx^3 + 2$ pass through the point B (–2, 10)?

5. Draw the graph of the exponential functions

   a) $y = 2^x$; b) $y = \left(\frac{1}{2}\right)^x$; c) $y = 3^x$.

6. Solve

a) $2^x > 2^3$; b) $\left(\frac{1}{3}\right)^x > \left(\frac{1}{3}\right)^3$; c) $3^x > 9^x$;

d) $2^{9x-x^3} > 1$; e) $(0.3)^{x^2-3} < (0.09)^x$.

7. Solve the equations

a) $5^x = 125$; b) $\left(\frac{2}{3}\right)^x \cdot \left(\frac{9}{8}\right)^x = \frac{27}{64}$;

c) $3^x\left(3^x - 12\right) = -27$; d) $2^{2x} - 5 \cdot 2^x = 24$.

8. Solve the equation $\frac{1}{2^{x-4-x^2}} = 4^{2x-1}$.

9. Find the range of values of x for which $2^{4x-x^3} > 1$, expressing your answer as the union of 2 intervals.

10. Solve the equations

a) $7^{2x} - 9^{2x} = 6 \cdot 7^{2x} - 6 \cdot 9^{2x}$; b) $2^{x^2-3} \cdot 5^{x^2-3} = 0.01 \cdot \left(10^{x-1}\right)^3$.

Homework

1. Draw the graph of

a) $y = x^3 - 1$; b) $y = x^4 + 1$.

2. Draw the graphs of the functions $y = \frac{1}{2}x$ and $y = \frac{2}{x}$ on the same axes. From the graph indicate the coordinates of intersection points.

3. Draw the graph of

a) $y = 4^x$; b) $y = 2^x + 1$.

4. Solve the equations

a) $2^x + 3 \cdot 2^{x+2} = 6.5$; b) $2^{x-1} + 2^{x-2} + 2^{x-3} = 448$.

5. Solve the equation $5^{3x} + 9 \cdot 10^{3x} = 10^{3x} + 9 \cdot 5^{3x}$.

6. Find the range of values of x that satisfy

$5^{4|x|-2} \cdot 5^{-3|x|-3} \leq 5^{11-|x|}$.

7. Find the sum of roots $\left(\frac{\sqrt{5}}{3}\right)^{2x^2-5x} = 1.8$.

Famous problems

1. **Gauss's problem** about 8 chess queens. Is it possible to locate eight queens on a chessboard so that none of them stands under attack from another? Gauss found 76 solutions; in fact there are 92.

2. **Division of circle (circumference) into n equal parts.** An ancient problem of dividing a circle (circumference) with only ruler and compass.

3. **Commisvoyageur's problem (the problem of the traveling trader).** Simplest case: There are n cities and the distance between every two cities is known. A trader must depart from one city, visit the other n-1 cities and return to the original city. In what sequence should the trader visit the cities (once each) so that the total distance is the least? There are many types of such problems: products delivery to stores, stretching electroenergic lines to customers, etc. In general, the method of solving such problem is to check all options.

4. **Narayana's problem (India, 14th century).** A cow bears a calf every year. The calf grows and at three years old it starts to bear a calf itself. How many cattle will descend from a cow in 20 years?

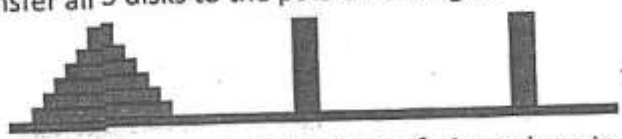
5. **Ananiya's problem (Armenia, 7th century).** There is a pool in Athens, which has three trenches supplying water. The first trench fills the pool in one hour, the second in two hours and the third in three hours. How much of the pool will be filled in one hour if all three trenches are opened?

Answers: 4. 2,475. **5.** 6/11th part of the pool.

Famous problems

6. **Pythagoras' students.** When Pythagoras was asked to state the number of his students, he replied: "Half of my students study mathematics, one fourth study the natural science, one seventh spend their time in quiet and the rest are three girls". How many students were there?

7. **Tower of Hanoi.** According to legend, the priests have to transfer from one place to another a tower of 64 different sized golden disks, one disk at a time ending with the largest and the smallest disks at the bottom and top respectively. The disks are fragile therefore a larger disk can never be placed on a smaller disk. There is an additional location where the disks can be temporarily placed. It is said when the priests complete their task the temple will collapse and the world will vanish. The simplified version of the problem can be solved with 5 disks as shown below. Find the least number of moves required to transfer all 5 disks to the pole on the right.



8. **Officer problem.** How can a delegation of six regiments, each of which sends a colonel, a lieutenant-colonel, and major, a captain, a lieutenant, and a sub-lieutenant be arranged in a regular 6x6 array such that no row or column duplicates a rank or a regiment?

9. **Apollonius Pursuit Problem.** Given a ship with a known constant direction and speed v, what course should be taken by a chase ship in pursuit (traveling at speed V) in order to intercept the other ship in as short a time as possible?

10. **Archimedes' Cattle Problem.** The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown. Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown; the number of the black, one quarter plus one fifth the number of the spotted greater than the brown; the number of the spotted, one sixth and one seventh the number of the white greater than the brown. Among the cows, the number of white ones was one third plus one quarter of the total black cattle; the number of the black, one quarter plus one fifth the total of the spotted cattle; the number of spotted, one fifth plus one sixth the total of the brown cattle; the number of the brown, one sixth plus one seventh the total of the white cattle. What was the composition of the herd? (Weisstein E 2002, p. 114).

Answers: 6. 28. **7.** 31 (in general, 2^n-1, where n is a number of disks). **8.** No such arrangement is possible. **9.** Find all points which can be simultaneously reached by both ships. **10.** White=10,366,482; Black=7,460,514; Spotted=7,358,060; Brown=4,149,387.

Only the educated are free.
Epictetus (55-135), Greek philosopher

VI.4. Absolute value functions

Terms

1. **absolute value (modular) function** – modulli funktsiya|модульная функция;
2. **plane** – tekislik|плоскость;
3. **the maximum (minimum) value** – maksimum (minimum) qiymat|максимальное (минимальное) значение;
4. **largest (smallest) value** – eng katta (eng kichik) qiymat|наибольшее (наименьшее) значение;
5. **crossing point (point of intersection)** – kesishish nuqtasi|точка пересечения;
6. **ray** – nur|луч;
7. **to turn to zero** – nolga aylanmoq|обращаться в нуль;
8. **to solve graphically (to solve by graphical method)** – grafik usulda yechmoq|решать графическим способом;
9. **to reduce** – qisqartirmoq|сокращать;
10. **to cancel** – qisqartirmoq, yo'qotmoq|сокращать.

Learning Objectives

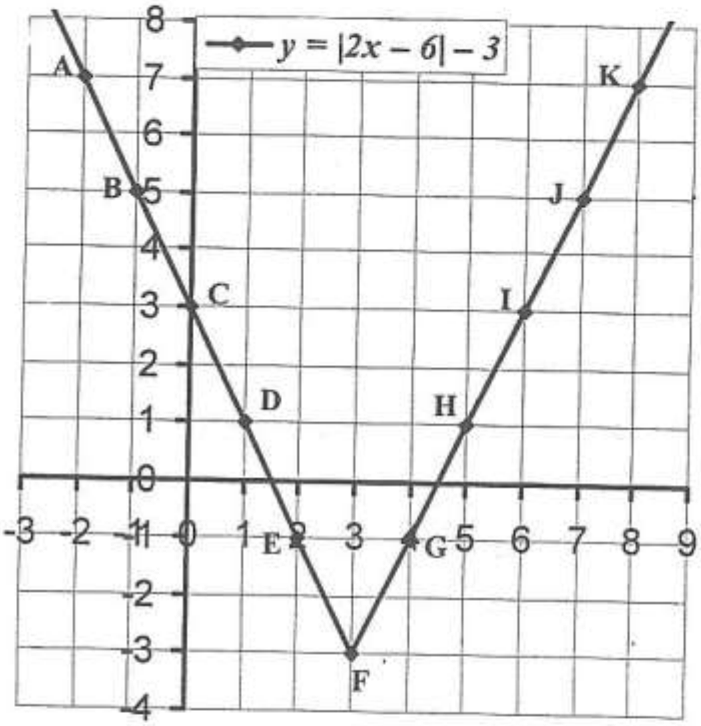
- to know how to draw the graph of an absolute value function;
- to know how to solve absolute value (modular) equations and inequalities using graphs.

Examples

1. In what quadrant is the graph of the function $y=|2x-6|-3$ located?

▶ The expression under the absolute value turns to zero when x = 3, so this is the critical value of the function. Define a table of values of x and y around this critical value and use the table to draw the graph of the function.

x	$y = \|2x-6\| - 3$	Points
-2	$\|2(-2)-6\| - 3 = 7$	$A(-2, 7)$
-1	$\|2(-1)-6\| - 3 = 5$	$B(-1, 5)$
0	$\|2(0)-6\| - 3 = 3$	$C(0, 3)$
1	$\|2(1)-6\| - 3 = 1$	$D(1, 1)$
2	$\|2(2)-6\| - 3 = -1$	$E(2, -1)$
3	$\|2(3)-6\| - 3 = -3$	$F(3, -3)$
4	$\|2(4)-6\| - 3 = -1$	$G(4, -1)$
5	$\|2(5)-6\| - 3 = 1$	$H(5, 1)$
6	$\|2(6)-6\| - 3 = 3$	$I(6, 3)$
7	$\|2(7)-6\| - 3 = 5$	$J(7, 5)$
8	$\|2(8)-6\| - 3 = 7$	$K(8, 7)$



It is obvious from the graph above that the graph is situated at the 1st, 2nd and 4th quadrants. ■

2. Draw the graph of the function $y=|x+1|+|x-2|$.

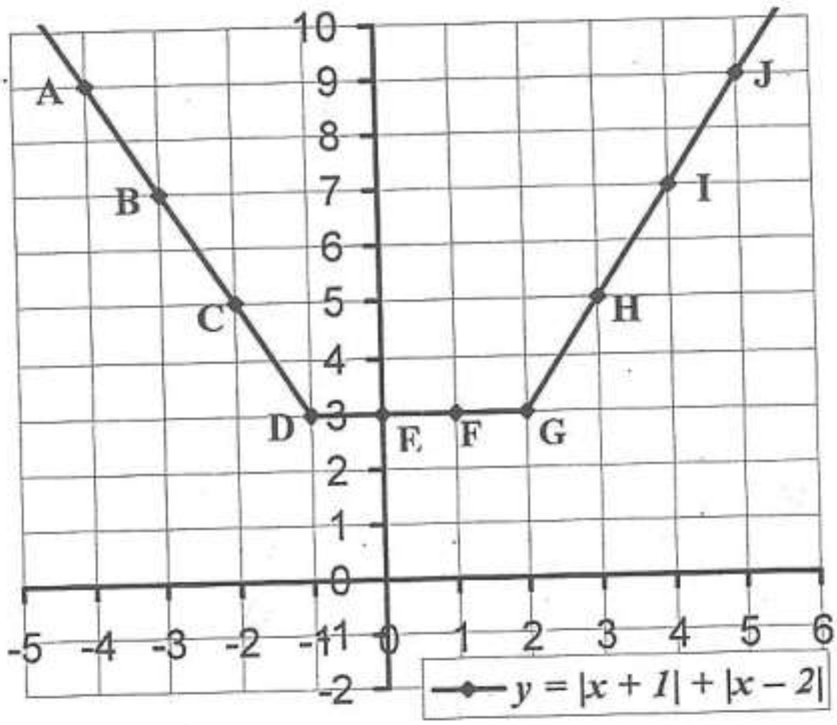
▶ There are two critical values $x = -1$ and $x = 2$, at which the modules turn to zero. We will expand the absolute values of the given function at each of the three intervals defined by the two critical values on the number axis.

1) $\begin{cases} x<-1 \\ y=-(x+1)-(x-2)=-2x+1 \end{cases};$

2) $\begin{cases} -1\le x<2 \\ y=x+1-(x-2)=3 \end{cases};$

3) $\begin{cases} x\ge 2 \\ y=x+1+x-2=2x-1 \end{cases}.$

Now, we draw the graph of the linear functions at each interval, which together will make up the general graph of the original modular function.



■

Exercises

1. Draw the graphs

   a) $y = |x|$; b) $y = |x-1|$; c) $y = |x| + 1$;

   d) $y = -|x| + 1$; e) $y = |x^2 - 6x|$.

2. For what value of x does the function $y = \dfrac{5x}{2|x+1|-5}$ equal 2?

3. Find the intersection point of the graphs of the functions $y = |x-2|+1$ and $y = 5$.

4. Find the function domain of $y = \dfrac{|x-1|}{x-1} - 2$.

5. What quadrants is the graph of $y = \dfrac{x}{|x|}$ located in?

6. Find the largest and the smallest values of the functions in the segment [–2, 3]

   a) $f(x) = |3x-6|$; b) $f(x) = |3x-6| - |x-4| + |2x+4|$.

7. Solve graphically

   a) $|x-4| = x$; b) $|x-1| + 2x - 5 = 0$;

   c) $|x-4| > |x+4|$.

Homework

1. Draw the graphs

   a) $y = |x^2 - 1| + 1$; b) $y = |x| + |x-1|$.

2. Find the largest value of the functions

   a) $y = |x-3| - |x-1|$; b) $y = |3x-6|$.

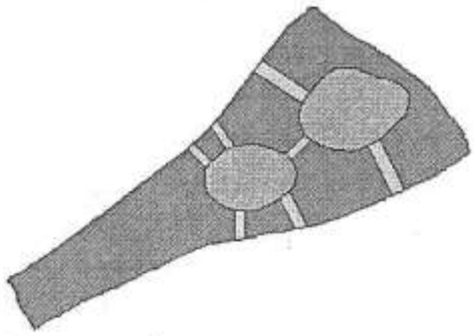
Famous problems

1. Fermat's Last Theorem (1630). No solution exists to the equation $x^n+y^n=z^n$ for positive integers x, y and z when n is greater than 2. Fermat formulated the problem in the margin of Diophantus's book "Arithmetic" as follows: "I have discovered a truly remarkable proof which this margin is too small to contain". In his papers a proof for $n=4$ has been found. A proof for $n=3$ has been attributed to L. Euler (1770). In 1839, G. Lame proved the theorem for $n=7$. The general solution was shown in 1994 by Wiles incompletely, but in 2004 completely.

2. Fermat's Minor theorem (1640). If p is a prime number and a is a whole number indivisible by p, then $a^{p-1}-1$ is divisible by p (that is $a^{p-1}\equiv 1 \pmod p$).

3. Four Color Problem. Is it possible to paint any map with four colors so that no two fields with a common border are painted with the same color? In 1976 the statement was confirmed by means of a large-scale computer. Some mathematicians don't accept the solution, because it was not checked by hand. However, to verify the result by hand requires too much human effort.

4. Euler's problem (on Königsberg bridges, 1736). Is it possible to cross the seven bridges over the Pregel River, connecting two islands and the mainland, without crossing over any bridge twice? (See Figure below). Euler proved that it was not possible, giving rise to a new branch of mathematics, graph theory.



5. Fibbonacci's problem. Seven old women are going to Rome. Each of them has seven donkeys, each donkey carries seven sacks, each sack contains seven loaves, and with each loaf were seven knives, and each knife was placed in seven cases. How many cases are there?"

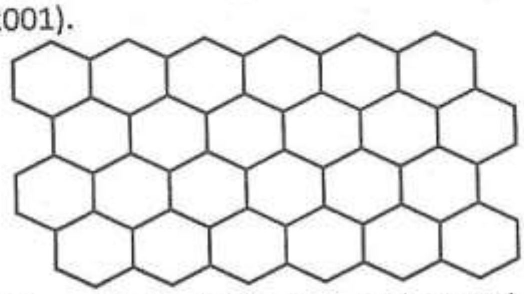
Answer: 5. $7^6 = 117,649$.

Famous problems

6. **Goldbach's problem (1792).** Can any whole number greater than or equal to 6 be expressed as a sum of three prime numbers? Goldbach posed this problem in his letter to Euler, who noticed that it would be enough to prove that any even number can be expressed as a sum of two prime numbers. However, both of these problems have not been solved yet. In 1975 it was proved for sufficiently large numbers. The problem of expressing an even number as a sum of two primes is not solved yet.

7. **Packing problems.** a) Fit as many circles as possible of n cm diameter into a strip of dimensions a cm x b cm. b) How many spheres (often oranges) of given diameter d can be packed into a box of size axbxc?

8. **Honeycomb conjecture.** Any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal grid (i.e. the honeycomb illustrated below). The conjecture was finally proven by Hales (1999, 2001).



9. **Brahmagupta's problem.** Solve the Pell equation $x^2-92y^2=1$ for integer values.

10. **Archimedes' Problem.** Cut a sphere by a plane in such a way that the volumes of the spherical segments have a given ratio.

Answer: 9. The smallest solution is $x = 1151$; $y = 120$.

Science is one thing, wisdom is another. Science is an edged tool, with which men play like children, and cut their own fingers.
Sir Arthur Eddington (1882-1944), British astronomer and physicist

VI.5. Irrational functions

Terms

1. **irrational function** – irratsional funktsiya | иррациональная функция;
2. **scale** – o'lchov birligi | единица измерения;
3. **the horizontal (vertical) axis** – gorizontal (vertical) o'q | горизонтальная и вертикальная ось;
4. **the coordinate system** – koordinatalar sistemasi | система координат;
5. **to solve analytically (to solve by analytical method)** – analitik usulda yechish | решать аналитическим методом.

Learning Objectives

- to learn how to draw the graph of an irrational function;
- to be able to solve equations and inequalities containing irrational functions.

Domain of an irrational function $y=\sqrt{P(x)}$

The range of x for which $P(x)\geq 0$.

Examples

1. Find the function domain of $y=\sqrt{-5x^2+3x+2}$.

▶ A function's domain is the range of x for which the function is meaningful. The radical sign is meaningful unless a negative number appears under it. So, we have to eliminate the values of x for which the expression under the root receives a negative value.

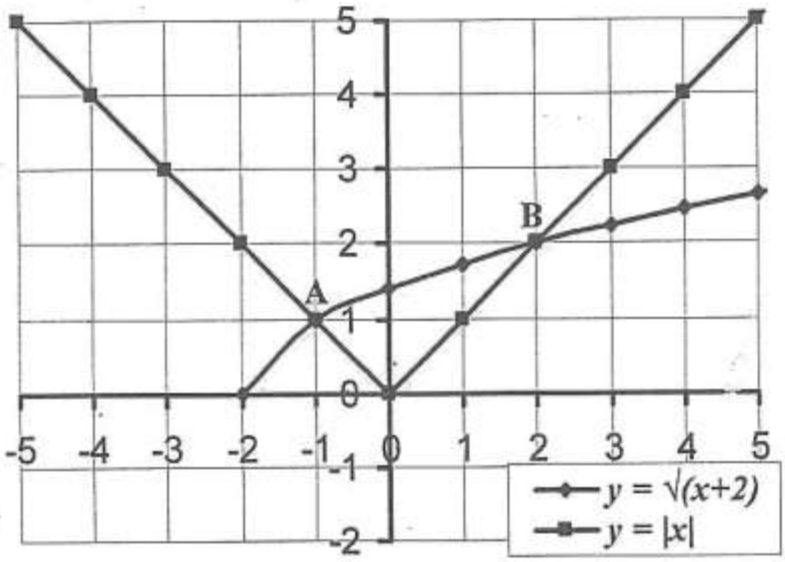
$$-5x^2+3x+2\geq 0 \Rightarrow -5(x-1)\left(x+\frac{2}{5}\right)\geq 0 \Rightarrow x\in\left[-\frac{2}{5},1\right].$$

2. Draw the graphs of the functions $y=\sqrt{x+2}$ and $y = |x|$. Show the intersection point(s) of these graphs. ■

▶ The domain of the irrational function is $x\in[-2;+\infty)$. The domain of the modular function is all real numbers. If we create a table of values of x and y for each function and draw the graphs through the found points, we will also hopefully spot the points of intersection.

x	$y=\sqrt{x+2}$	Points
-2	$\sqrt{-2+2}=0$	$(-2, 0)$
-1	$\sqrt{-1+2}=1$	$(-1, 1)$
0	$\sqrt{0+2}\approx 1.4$	$(0, 1.4)$
1	$\sqrt{1+2}\approx 1.7$	$(1, 1.7)$
2	$\sqrt{2+2}=2$	$(2, 2)$
3	$\sqrt{3+2}\approx 2.2$	$(3, 2.2)$

x	$y=\lvert x\rvert$	Points
-3	$\lvert -3\rvert = 3$	$(-3, 3)$
-2	$\lvert -2\rvert = 2$	$(-2, 2)$
-1	$\lvert -1\rvert = 1$	$(-1, 1)$
0	$\lvert 0\rvert = 0$	$(0, 0)$
1	$\lvert 1\rvert = 1$	$(1, 1)$
2	$\lvert 2\rvert = 2$	$(2, 2)$
3	$\lvert 3\rvert = 3$	$(3, 3)$



Intersection points are A $(-1, 1)$ and B $(2, 2)$. ■

Exercises

1. Draw the graphs

 a) $y=\sqrt{x}$; b) $y=\sqrt{x-1}$;

 c) $y=-\sqrt{x}-1$; d) $y=\sqrt[3]{x}$.

2. Find the function domain of

 a) $y=\sqrt{x^2-2x-15}$; b) $y=\sqrt{-12x^2-4x+5}$.

3. Find the value of k for which the graph of $y=\sqrt{x^2-k}+3$ passes through the point A (3, 5).

4. Determine the function domain of

 a) $y=\sqrt{5-x-\frac{6}{x}}$; b) $y=\sqrt{x+7}+\sqrt{11-x}$;

 c) $y=\sqrt{x^2-|x|}+\frac{1}{\sqrt{9-x^2}}$.

5. Identify the interval of increase of the functions

 a) $y=\sqrt{x-4}$; b) $y=\sqrt{x^2+4x+3}$.

6. Find the largest and the smallest value of the function $y=\sqrt{100-x^2}$ for x in the segment $x\in[-6, 8]$.

Homework

1. Draw the graph of $y=\sqrt{x^2-1}$.

2. Find the function domain of

 a) $y=\sqrt{\frac{x^2-3x-10}{x^4-9x^2}}$; b) $y=\sqrt{3^x-4^x}$;

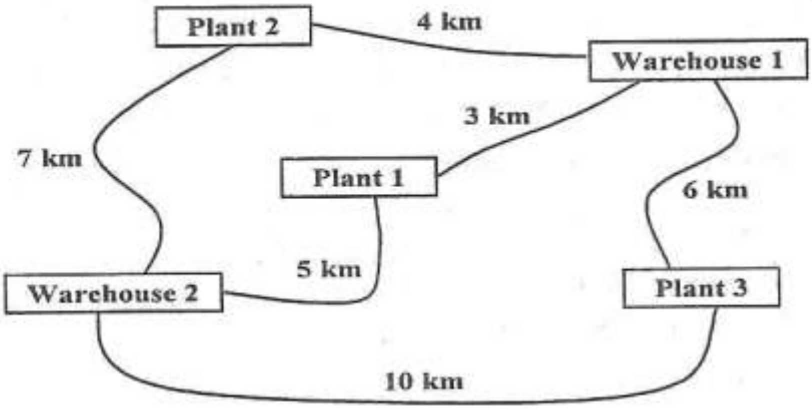
 c) $y=\sqrt{\frac{x^2}{|x|-3}}$; d) $y=\frac{1}{\sqrt{x^2+4x}}$.

3. Find the function range of $y=\sqrt{x^2-6x+11}$.

4. Solve graphically the inequality $\sqrt{x} \geq x-6$.

Famous problems

1. **Optimal way** Find the least costly way to deliver a certain raw material to the three plants from the two warehouses. Warehouse 1 and warehouse 2 have 20 tons and 25 tons and plant 1, plant 2 and plant 3 have a need for 10 tons, 15 tons and 20 tons of the raw material, respectively (i.e. find the least general ton-kilometer indicator). Diagram not to scale.



2. **Magic tree** A gardener grew 25 bananas and 30 oranges on a miraculous tree in the garden. He gathers two fruits every day; and a new fruit appears in their place everyday. If the gardener gathers two identical fruits, then an orange appears. If the gardener gathers two different fruits, then a banana appears. What will be the last fruit on the tree?

3. **Sun-Moon** A father and son were observing a solar eclipse and talking about the Sun and the Moon. "Father" asked the son, "How many times is the Sun further from us than the Moon?". "If I am not mistaken" answered the father, "387 times". "Oh, now, I can find how many times the Sun's volume is greater than that of the Moon". "Sure you can", replied the father after a while. Well, what is the answer?

Answers: 1. Warehouse 1 sends all its material to Plant 3. **2.** Banana. **3.** 387^3=57,960,603 times.

Famous problems

4. **Cards** Anvar used cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and placed them in pairs on a table in the ratio 1:2:3:4:5. In the evening he wanted to show this interesting result to his dad, but could not find the card numbered 0. Nevertheless, he thought a while and he made another set of five numbers in the ratio 1:2:3:4:5. How did Anvar arrange the cards the first and the second time?

5. **Domino chips** The traditional domino game has 28 chips. If the points on the chips were not from 0 to 6, but were from 0 to 4, then there would be 15 chips. (Check it). How many chips would there be if the points were from 0 to 12?

6. **Alhazen's Billiard problem** In a given circle, find an isosceles triangle whose legs pass through two given points inside the circle. The problem was first formulated by Ptolemy in 150 AD, and was named after the Arab scholar Alhazen, who discussed it in his work on optics. (Weisstein E 2002, p. 54).

7. **Castillo's problem** Inscribe a triangle in a circle such that the sides of the triangle pass through three given points A, B and C.

8. **Brachistochrone problem** Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time.

9. **Dido's problem** Find the figure bounded by a line which has the maximum area for a given perimeter.

Answers: 4. 1st set: 18, 36, 54, 72, 90; 2nd set: 9, 18, 27, 36, 45. **5.** 91. **8.** A segment of cycloid. Found by Leibniz, L'Hospital, Newton and the two Bernoullis. **9.** A semicircle.

Study as if you were going to live forever;
live as if you were going to die tomorrow.
Maria Mitchell (1818-1889), American astronomer

If I feel unhappy, I do mathematics to become happy.
If I am happy, I do mathematics to keep happy.
Alfred Renyi

VI.6. Inverse and composite functions

Terms

1. **inverse function** – teskari funktsiya|обратная функция;
2. **composite function** – murakkab funktsiya|сложная функция;
3. **function argument** – funktsiya argumenti|аргумент функция;
4. **even (odd) function** – juft (toq) funktsiya|четная (нечетная) функция;
5. **monotonous function** – monoton funktsiya|монотонная функция;
6. **billion** – milliard|миллиард;
7. **to obtain an inverse function** – teskari funktsiyani topmoq|находить обратную функцию;
8. **to rearrange** – almashtirmoq|преобразовать;
9. **elementary function** – elementar funktsiya|элементарная функция;
10. **interval of monotonic increase** – monoton o'sish oralig'i|интервал монотонного возрастания.

Learning Objectives

- to know how to obtain an inverse and a composite function of an elementary function;
- to be able to identify an odd and an even function.

Inverse function of the given function

$y = f(x)$ **with domain** $D(f(x))$ **and range** $E(f(x))$

$y = f^{-1}(x)$ with domain $D(f^{-1}(x)) = E(f(x))$ and range $E(f^{-1}(x)) = D(f(x))$.

Composite function is a superposition of several functions

$z = f(g(x))$ or $f(g(h(x)))$, etc

Examples

1. Find the inverse function of $y = \dfrac{2x-1}{x+2}$, $x \in [0; 3]$.

▶ The domain and range of the given function will be

$$D(y) = [0; 3] \text{ and } E(y) = \left[-\frac{1}{2}; 1\right].$$

To find the equation of the inverse function, it is necessary to switch x and y in the equation of the given function and solve it for y.

$$y = \frac{2x-1}{x+2} \Rightarrow x = \frac{2y-1}{y+2} \Rightarrow x(y+2) = 2y-1 \Rightarrow$$

$$xy - 2y = -2x - 1 \Rightarrow y = \frac{2x+1}{2-x}.$$

This is the inverse function, whose domain will be the range of the original function. Thus, the final answer is

$$y = \frac{2x+1}{2-x},\ x \in \left[-\frac{1}{2}; 1\right]. \blacksquare$$

2. Evaluate $f(g(5))$, given that the elementary functions

$$f(x) = x^2 + x \text{ and } g(x) = x + 1.$$

▶ First make up the composite function.

$$f(g(x))=f(x+1)=(x+1)^2+(x+1)=x^2+3x+2.$$

Now insert the number five into the composite function.

$$f(g(5))=5^2+3\cdot5+2=42. ∎$$

Exercises

1. Find the inverse function of the following functions

   a) $f(x)=x+5$; b) $y=\frac{5-x}{2}$;

   c) $y=\frac{4}{2-x}-3$; d) $f(x)=x^2-2$;

   e) $y=\frac{6x+2}{x}$.

2. Rearrange the following formulae to make y the new subject

   a) $x(y-1)=y$; b) $x=\frac{y^2+1}{2y^2-1}$.

3. Find the inverse function of $y=2x+1$, $x\in[0;3]$.

4. Show the interval where the functions below monotonically increase

   a) $f(x)=5x+6$ b) $y=x^2$;

   c) $y=x^3+1$; d) $y=3.5^x$.

5. Identify the odd (even) functions

   a) $y=x$; b) $y=x^2$; c) $y=3x-x^3$;

   d) $y=2^x+2^{-x}$; e) $y=x+\frac{1}{x}$; f) $y=x+1$

6. If $f(x)=x^2$ and $g(x)=x-1$, define the following composite functions

   a) $f(g(x))$; b) $g(f(x))$;

   c) $f(f(x))$; d) $g(g(x))$.

7. If $f(x)=\frac{x^2-1}{x^2}$ and $g(x)=\frac{1}{x^2}$, calculate $f(g(2))$.

8. If $f(x)=2x^2$ and $g(x)=x+1$, find the values of x for which $f(g(x))=g(f(x))$.

9. Find $f(x)$, if

   a) $f(x+1)=x^2-3x+2$; b) $f(3x-1)=x^2+3x-2$.

10. What is the value of $f(0)$, if $f\left(\frac{3x-2}{2}\right)=x^2-x-1$?

Homework

1. Find the inverse function of

   a) $y=\frac{x-1}{2-3x}$; b) $y=\frac{2x+3}{5x-2}$.

2. If $f(x)=x^2-8$, $x\in[0;+\infty]$, find the function domain of its inverse function.

3. Find the intervals of monotonic increase and decrease of the functions

   a) $f(x)=|x-4|$; b) $f(x)=\sqrt{x-1}$;

   c) $f(x)=2^{-x}$.

4. Which function is odd (even)

   a) $y=3x^2+x^4$; b) $y=\frac{x^3+x}{x^3-x}$;

   c) $y=\frac{x^4+1}{2x^3}$; d) $y=|x|$?

5. If $f(x)=\sqrt{x^3-1}$, what is the value of $f\left(\sqrt[3]{x^2+1}\right)$?

6. If $f(x)=2x^2+1$, find the coordinates of the vertex of the parabola $y=f(x-1)$.

7. Compute $f\left(\sqrt{3}\right)$, if $f(x+2)=x^3+6x^2+12x+8$.

Famous problems

1. Problem taken from the Chinese "Nine book mathematics" (2nd century BC) There is a city in the shape of a square. The sides are unknown. In the middle of each side there is a gate. There is a pole 20*bus* (1*bu*=1.6*km*) far from the northern gate (outside of the city). If one walks out of the southern gate for 14*bus* and 1775*bus* to the west, then he can see the pole. What is the size of the city?"

2. Boat problem A boat sails from Gorkiy to Astrakhan in 5 days and from Astrakhan to Gorkiy in 7 days. How many days will a raft drift for from Gorkiy to Astrakhan by the stream?

3. River transfer A farmer needs to transfer a wolf, a goat and a cabbage to the other side of a river. The boat is so small that it can only fit the man and one of the three objects (either wolf or goat or cabbage). However, it is no good to leave the wolf with the goat (the wolf will eat the goat) or the goat with the cabbage (the goat will eat the cabbage). What should the man do?

4. Napoleon's problem (The French Emperor Napoleon Bonaparte was fond of mathematics and would find time to enjoy mathematics): On the sides of a triangle external equilateral triangles are built with the sides as bases. It is required to prove that the triangle with vertices on the centers of the equilateral triangles will also be an equilateral triangle.

5. Pulling a rope One day four friends, Alexander, Genghiskhan, Napoleon and Temurlane decided to play (the game of pulling a rope for fun). Temurlane and Genghiskhan could easily defeat Alexander and Napoleon. But Alexander and Gengizkhan could hardly beat Temurlane and Napoleon. When Gengizkhan and Napoleon fought against Alexander and Temurlane, a draw resulted. Rank the four friends, according to their strength.

Answer: 1. 250*bus*. **2.** 35 days. **3.** Goat; cabbage; goat; wolf; goat. **5.** Gengizkhan, Temurlane, Alexander, Napoleon.

Famous problems

6. Birthday One man told his friend "The day before yesterday I was 10 years old and next year I will turn to 13". Can this be true?

7. Diophantus's riddle On the ancient mathematician Diophantus' tombstone the following is written: "Hey, passenger! Under the tombstone lie the ashes of old Diophantus. Diophantus's youth lasts 1/6 of his life. He grew a beard after 1/12 more of his life. After 1/7 more of his life, Diophantus married. Five years later, he had a son. The son lived exactly half as long as his father and Diophantus died just four years after his son's death. All of this totals the years Diophantus lived." How long did Diophantus live?

8. Chinese remainder problem We have a number of things, but do not know exactly how many. If we count them by threes, we have two left over. If we count them by fives, we have three left over. If we count them by sevens, we have two left over. How many things are there?

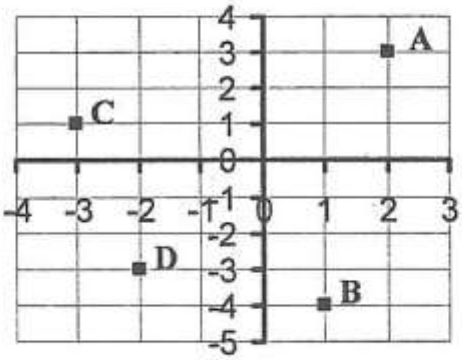
9. Josephus problem Given a group of n men arranged in a circle under the edict that every m-th man will be executed going around the circle until only one remains, find the position $P(n; m)$ in which you should stand in order to be the last survivor. [The original Josephus problem consisted of a circle of 41 men with every third man killed (n=41, m=3) and required positions that should be taken by two men in order to survive?]

10. Fermat's problem In a given acute triangle ABC, locate a point whose distances from A, B and C have the smallest possible sum.

Answers: 6. Yes, the man's birthday is 31st December and they spoke on the 1st January. **7.** 84. **8.** 23. **9.** 31 (last) and 16 (last but one). **10.** The point from which each side subtends an angle of 120°.

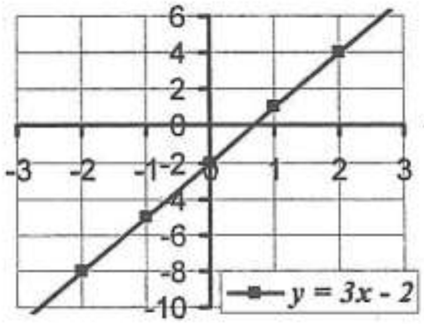
Chapter VI Answers. Functions

VI.1.

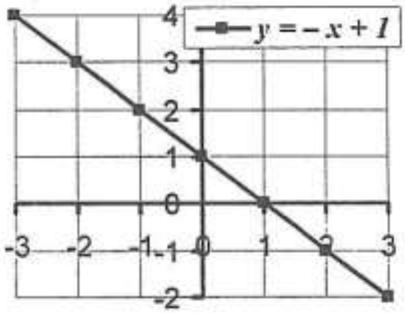


1.

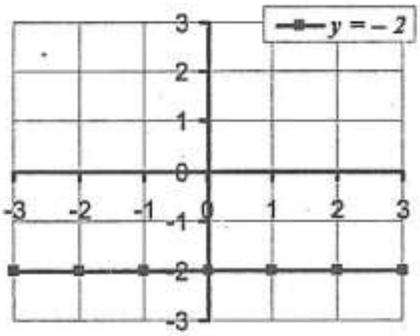
2. a) and c).



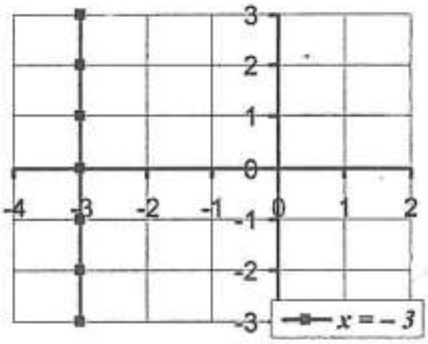
3. a)



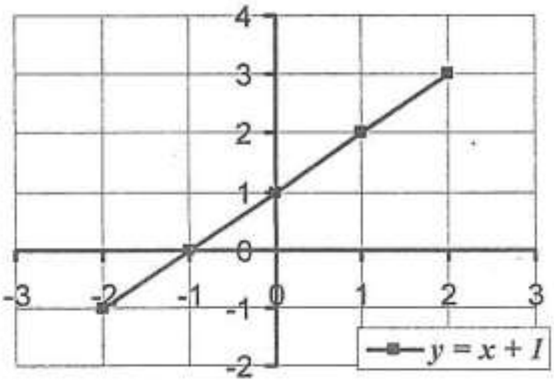
b)



c)



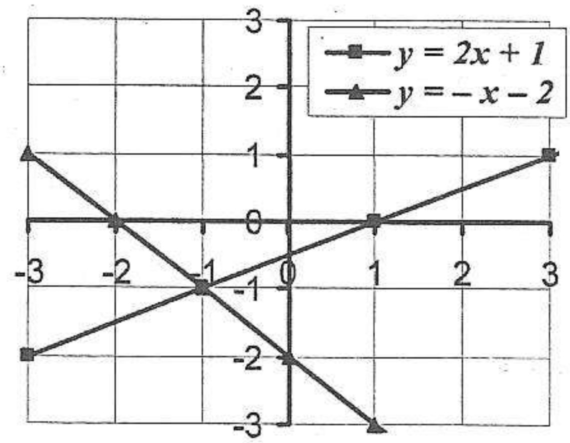
d)



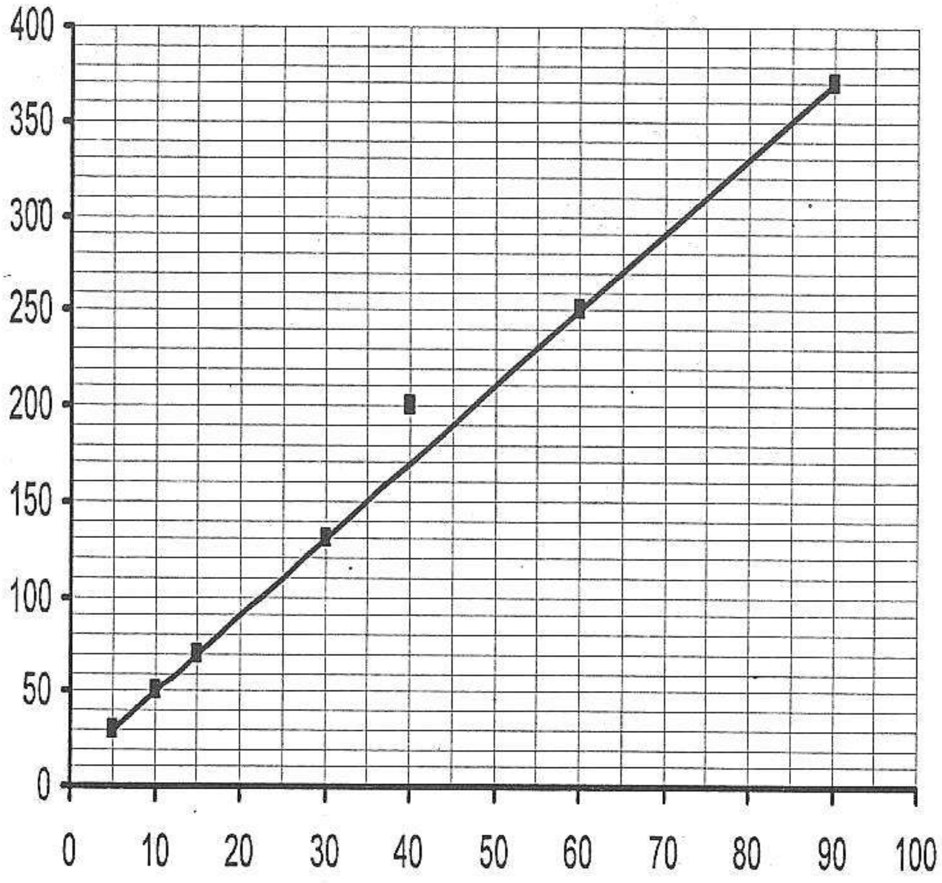
4.

5. a) $(-\infty, 4)$; b) $[0, +\infty)$.

6. a) I, II and III; b) I, II and IV; c) I and III; d) III and IV. **7.** a) 2; b) –5.5. **8.** a) $y=x$; b) $y=3x$; c) $y=2x+1$.



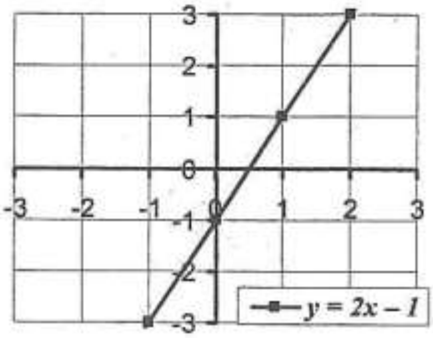
9. **10.** 14. **11.** a) parallel; b) perpendicular. **12.** a) 2/3; b) –3/2. **13.** a) (1, +∞); b) (–∞, 1); c) 0. **14.** a) $y=x+4$; b) $y=3x+2$. **15.** a) $S=18.6–1.2t$; b) 12.6. **16.** d). **17.** a) B; the distance gets shorter; b) For 5 minutes between the points D and E. **18.** h (height) = 84t (time) + 216. **19.**



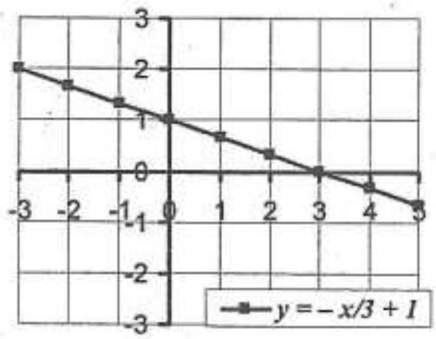
a) the wrong point is (40, 200); b) y (75) = 4*75 + 10 = 310.

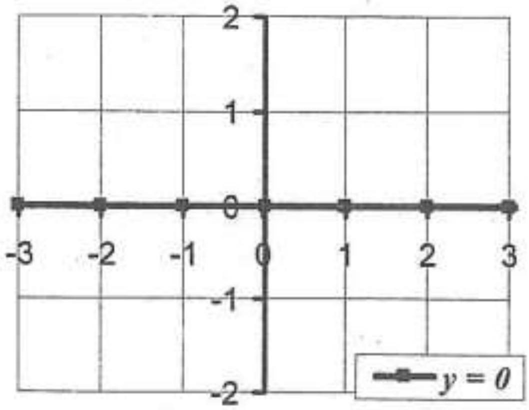
Homework:

1. a)



b)



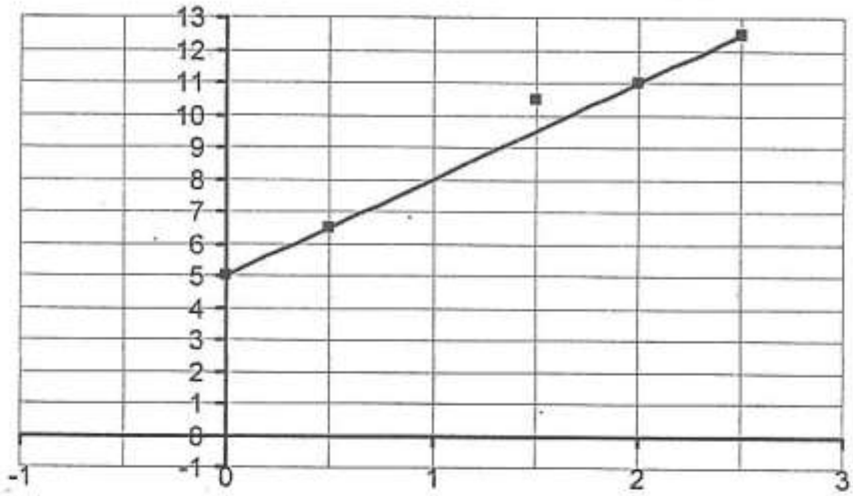


c)

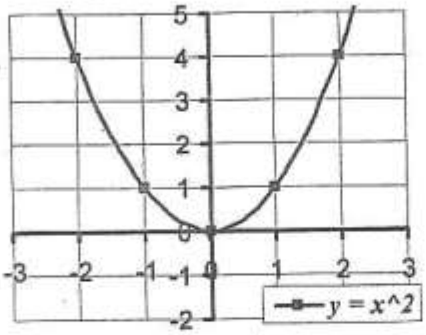
2.

x	−3	−2	−1	0	1	2	3
y	−17	−13	−9	−5	−1	3	7

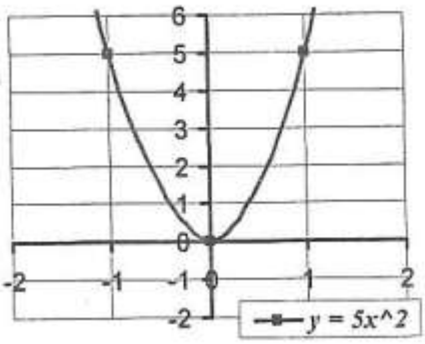
3. a) 2; b) −1. **4.** III. **5.** a) $y = -x$; b) $y = -0.5x + 2.5$. **6.** $a = -3$; $b \neq 0$. **7.** a) perpendicular; b) increasing; decreasing. **8.** 125,000. **9.** Budget=60,000+1,000h. **10.** $y = 3x + 5$. The wrong point is (1.5, 10.5).



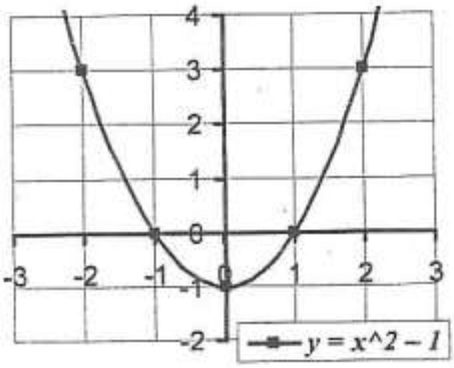
VI.2. 1. a); b) and d). **2.** a) 6; b) 6; c) 4. **3.** a) 4; b) –3.6.



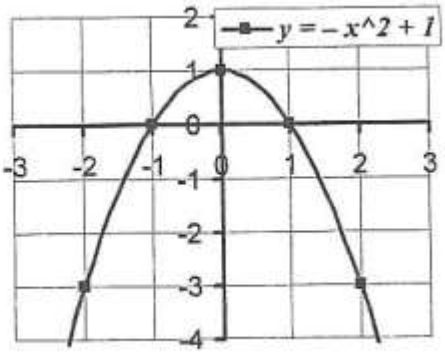
4. a)



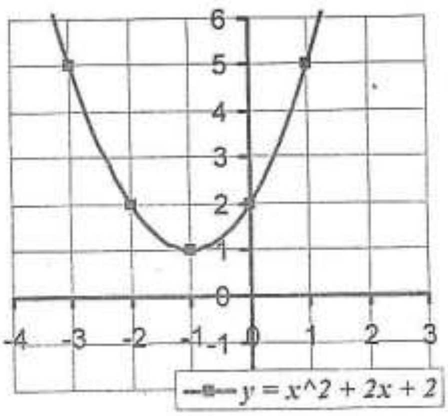
b)



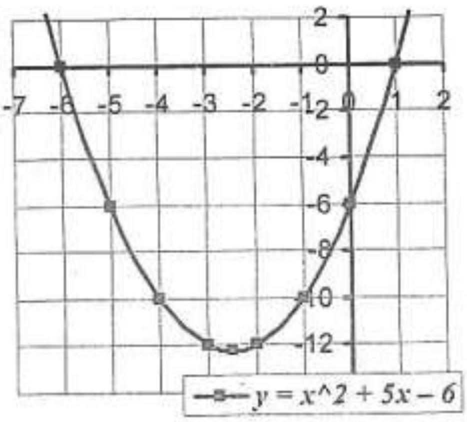
c)



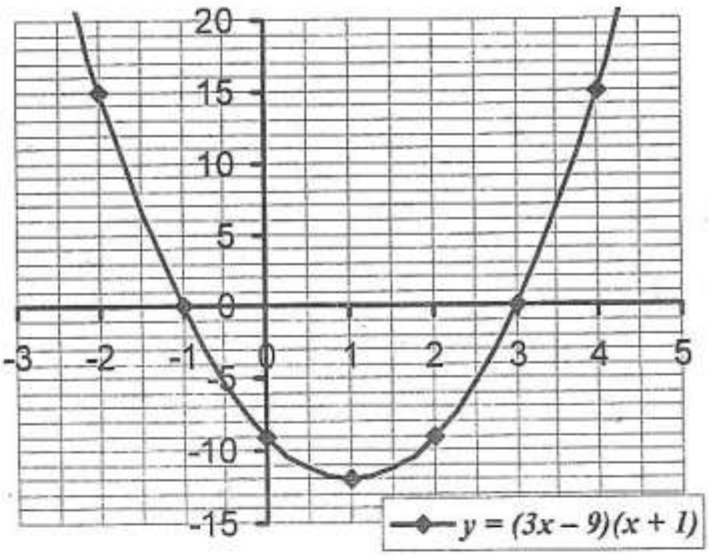
d)



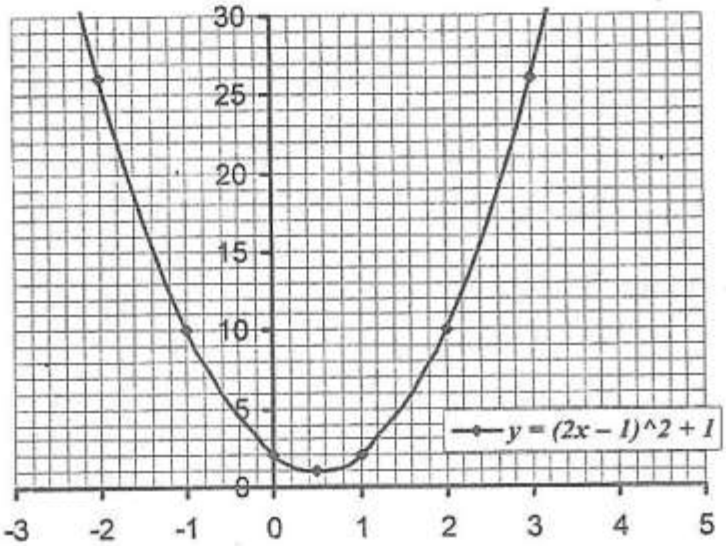
e)



f)

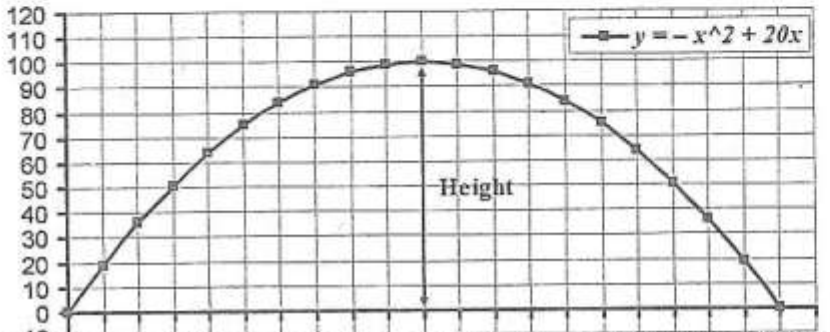


g)

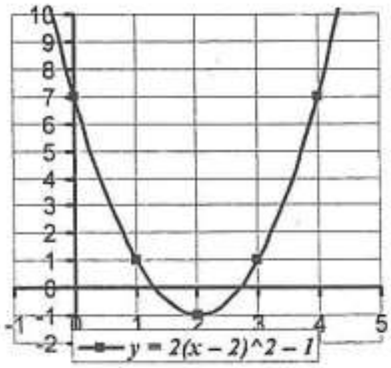


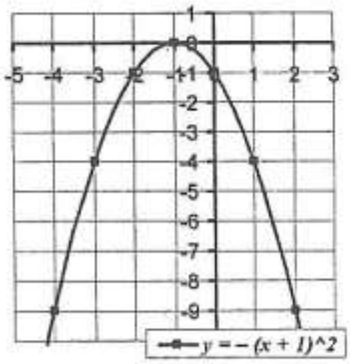
h)

5. a) 1; b) 0.5. **6.** 3. **7.** 6. **8.** ±4. **9.** 5. **10.** $(-\infty, -3]$. **11.** 2.5. **12.** –4; 7. **13.** a) $(0, +\infty)$; b) $(-\infty, 0)$; c) $(-1, +\infty)$. **14.** $(-\infty, 4/15)$. **15.** $(-3, 0)$. **16.** $(2, 0)$; $(-1, -3)$. **17.** 100 feet.



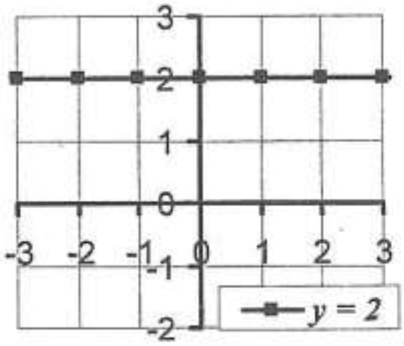
Homework: 1. 16.75. **2.** a) (3, –2); b) (–2, 0). **3.** 2.

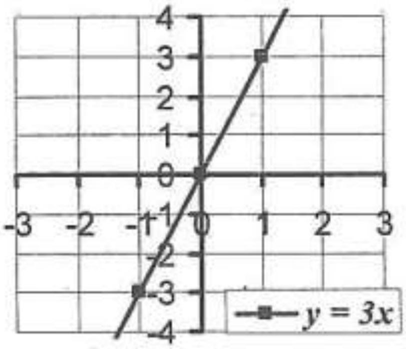
4. a) 

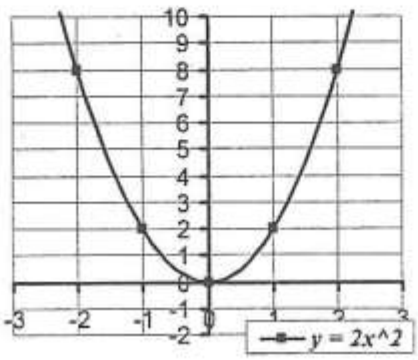
b) 

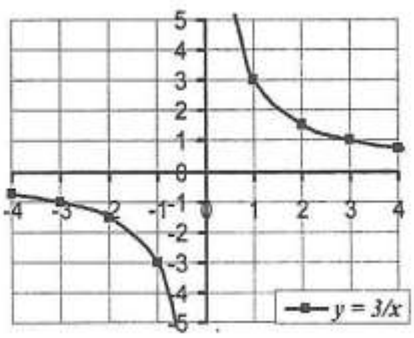
5. (–∞, –3)U(3, +∞). **6.** III and IV. **7.** a) (–∞, –2); b) (–∞, –1). **8.** a) $y=x^2-4x+3$; b) $y=-3x^2+3$. **9.** 3 seconds.

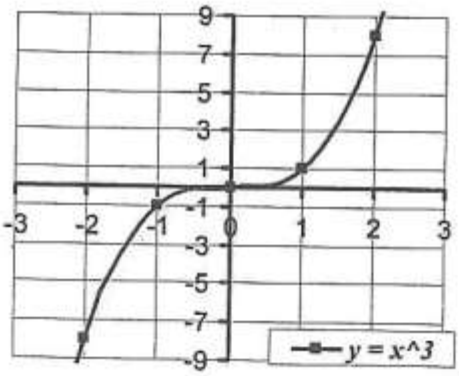
VI.3. 1. a) 3; b) –81; c) 15.

2.a) 

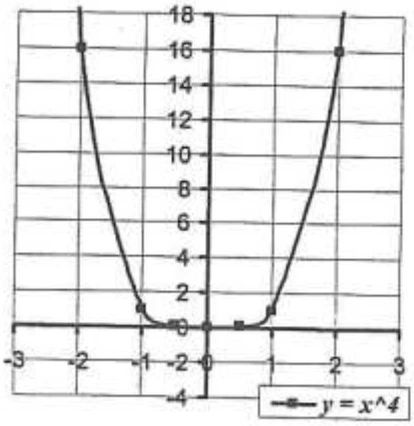
b) 

c) 

d) 



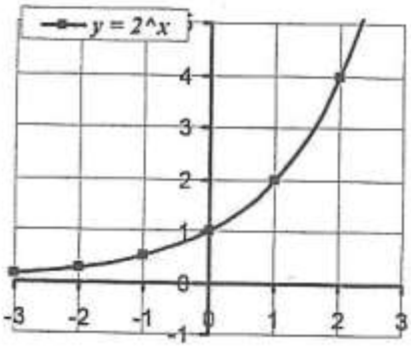
e)



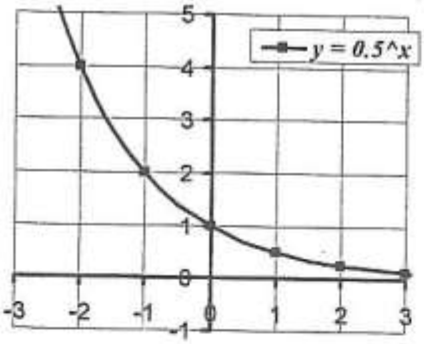
f)

3. (1, 29). **4.** –1.

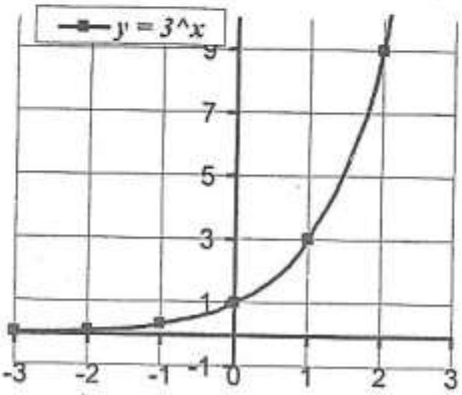
5. a)



b)



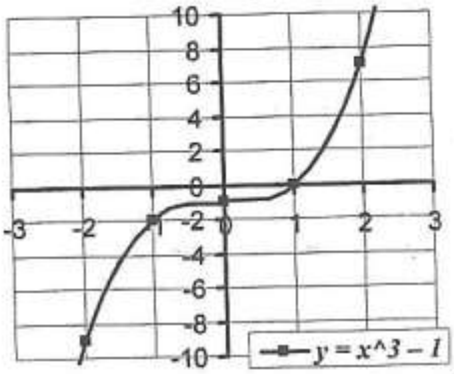
c)



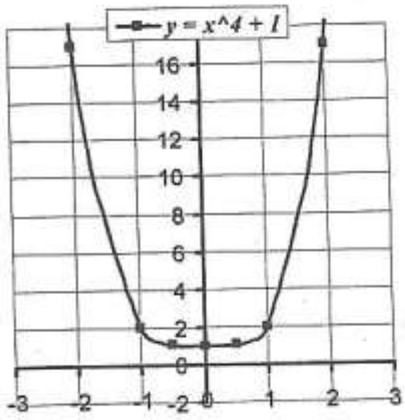
6. a) (3, +∞); b) (–∞, –3); c) (–∞, 0); d) (–∞, –3)U(0, 3); e) (–∞, –1)U(3, +∞). **7.** a) 3; b) 3; c) 1; 2; d) 3. **8.** 3; 2. **9.** (–∞, –2)U(2, +∞). **10.** a) 0; b) 2; 1.

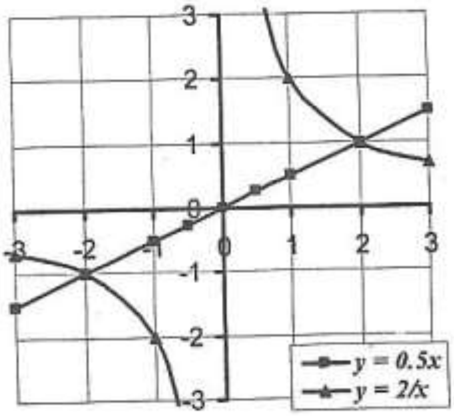
Homework:

1. a)



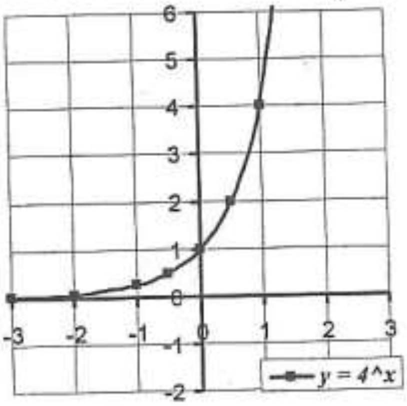
b)



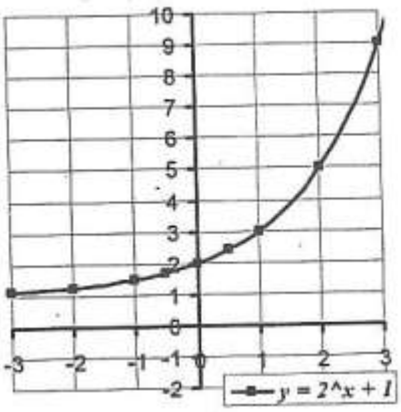


2. (– 2, –1); (2, 1).

3. a)



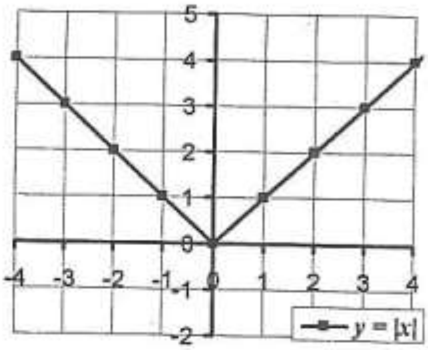
b)



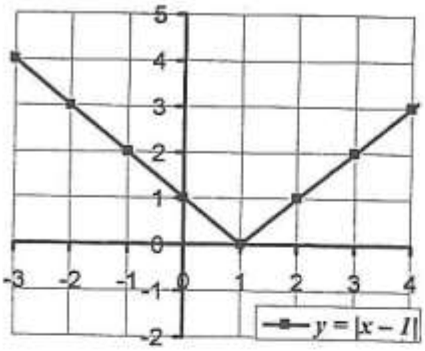
4. a) –1; b) 9. **5.** 0. **6.** [–8, 8]. **7.** 0.5; 2.

VI.4.

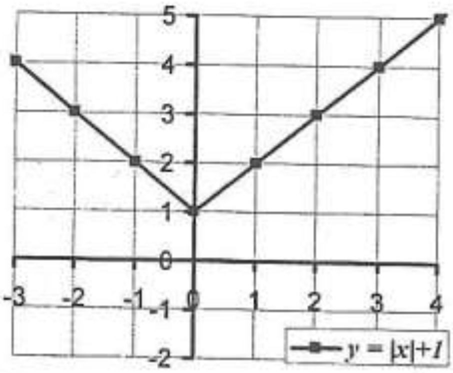
1. a)



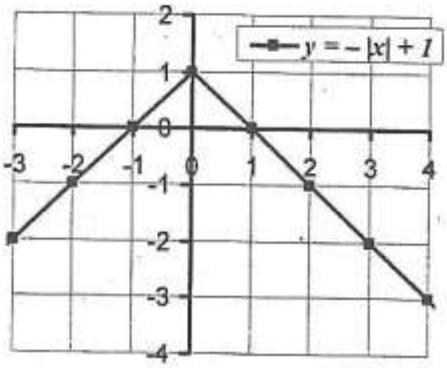
b)



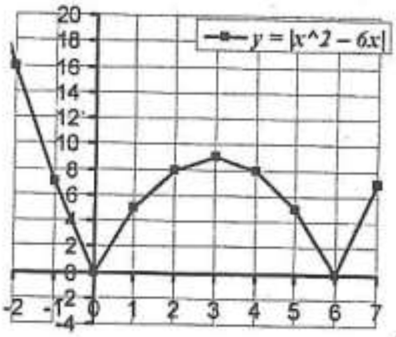
c)



d)

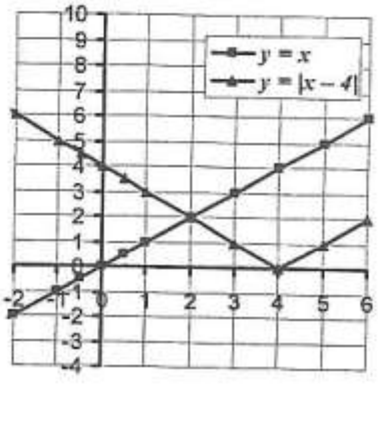


e)

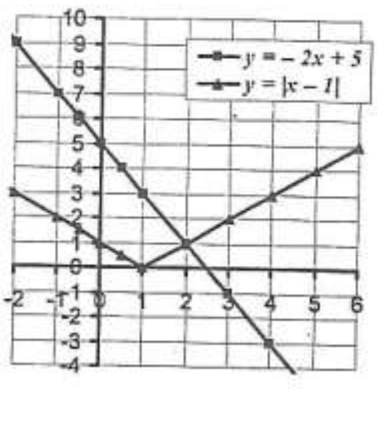


2. –14/9. **3.** –2; 6. **4.** {–3; –1}. **5.** I and III.

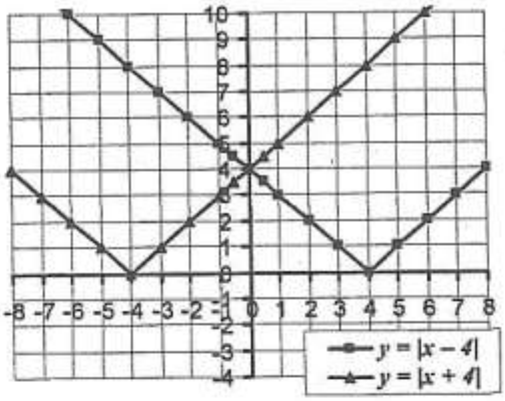
6. a) 0; 12. b) 6; 12. **7.**



a) x=2.

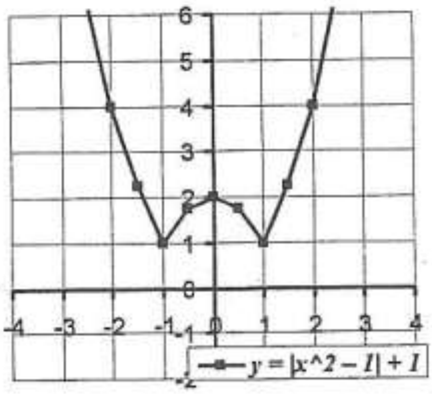


b) x=2

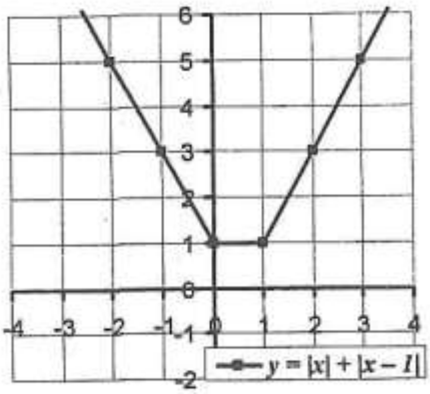


c) (0, +∞).

Homework: 1.

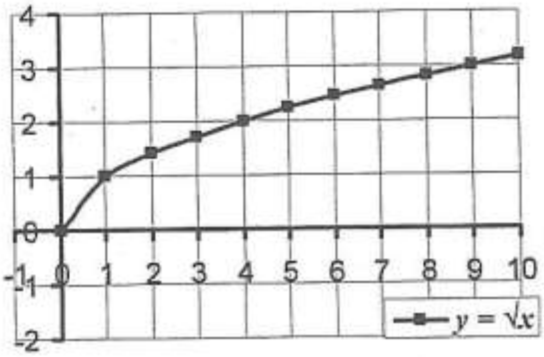


a)

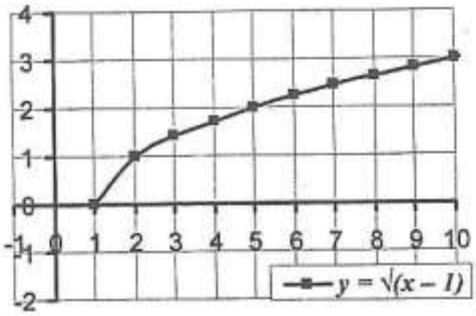


b)

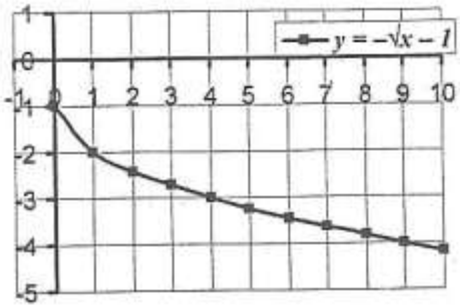
2. a) 2; b) +∞.



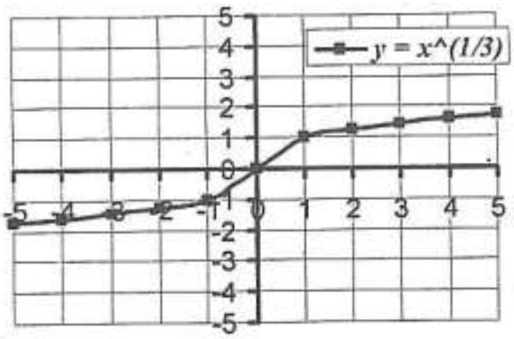
VI.5. **1.** a)



b)



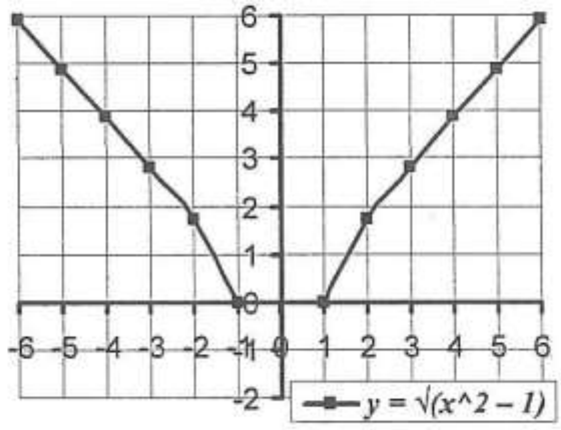
c)



d)

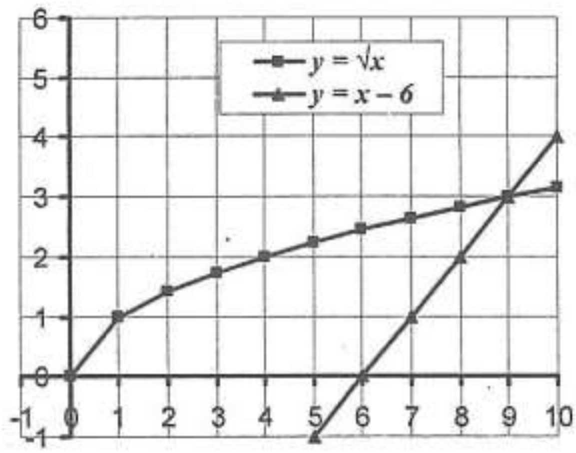
2. a) (−∞, −3]∪[5, +∞); b) [−1/2, 5/6]. **3.** 5. **4.** a) (−∞, 0)∪[2, 3]; b) [−7, 11]; c) (−3, −1]∪{0}∪[1, 3). **5.** a) [4, +∞); b) [−2, −1]. **6.** y_{max}=10; y_{min}=6.

Homework:



1.

2. a) (−∞, −3)U[−2, 0)U(0, 3)U[5, +∞); b) (−∞, 0]; c) (−∞, −3)U{0}U(3, +∞); d) (−∞, −4)U(0, +∞). **3.** [$\sqrt{2}$, +∞). **4.** [6, 9], because here the graph of the irrational function lies above the graph of the linear function.



§6.6. 1. a) $y=x-5$; b) $y=5-2x$; c) $y=(2x+4)/(x+3)$; d) $y=\sqrt{x+2}$; e) $y=2/(x-6)$. **2.** a) $y=x/(x-1)$; b) $y=\sqrt{(x+1)/(2x-1)}$. **3.** $y=0.5x-0.5$; $x\in[1, 7]$. **4.** a) (−∞, +∞); b) [0, +∞); c) (−∞, +∞); d) (−∞, +∞). **5.** a) odd; b) even; c) odd; d) even; e) odd; f) not even, not odd. **6.** a) $y=(x-1)^2$; b) $y=x^2-1$; c) $y=x^4$; d) $y=x-2$. **7.** −15. **8.** $x = -0.25$. **9.** a) $y=x^2-5x+6$; b) $y=x^2/9+11x/9-8/9$. **10.** −11/9.

Homework: 1. a) $y=(2x+1)/(3x+1)$; b) $y=(2x+3)/(5x-2)$. **2.** [−8, +∞). **3.** a) [4, +∞) – increases; b) [1, +∞) – increases; c) (−∞, +∞) – decreases. **4.** a) even; b) even; c) odd; d) even. **5.** $y=x$. **6.** (1, 0). **7.** $3\sqrt{3}$.

A scholar who loves comfort is not fit to be called a scholar.
Confucius (551-479 BC), Chinese philosopher

Chapter VII. Trigonometry and Logarithms

VII.1. Basic concepts and formulae

Terms

1. **degree** – daraja | градус;
2. **radian** – radian | радиан;
3. **trigonometric function** – trigonometrik funktsiya | тригонометрическая функция;
4. **sine (cosine, tangent, cotangent)** – sinus (kosinus, tangens, kotangens) | синус (косинус, тангенс, котангенс);
5. **angle** – burchak | угол;
6. **identity** – ayniyat | тождество;
7. **formula of reduction** – keltirish formulasi | формула приведения;
8. **formula of addition** – qo'shish formulasi | формула сложения;
9. **axiom** – aksioma | аксиома;
10. **to state** – ta'kidlamoq | утверждать.

Learning Objectives

- to know how to change an angle from a degree to a radian measure and vice-versa;
- to learn how to identify and calculate the value of a trigonometric function.

Conversion formulas

1) 1 radian = $\dfrac{180^\circ}{\pi} \approx 57^\circ$.

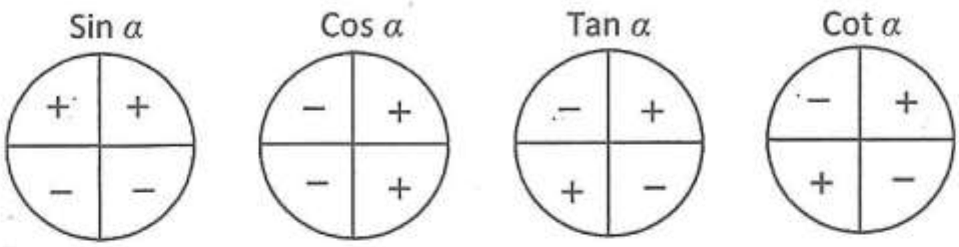
2) $\rho = \dfrac{\pi}{180^\circ} \cdot \alpha^\circ$ (from a degree to a radian).

3) $\alpha^\circ = \dfrac{180^\circ}{\pi} \cdot \rho$ (from a radian to a degree).

Some values of trigonometric functions

Angles	Sin α	Cos α	Tan α	Cot α
0	0	1	0	meaningless
$30^\circ = \dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{3}}$	$\sqrt{3}$
$45^\circ = \dfrac{\pi}{4}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	1	1
$60^\circ = \dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$	$\dfrac{1}{\sqrt{3}}$
$90^\circ = \dfrac{\pi}{2}$	1	0	meaningless	0
$180^\circ = \pi$	0	-1	0	meaningless
$270^\circ = \dfrac{3\pi}{2}$	-1	0	meaningless	0
$360^\circ = 2\pi$	0	1	0	meaningless

Signs of trigonometric functions



Main trigonometric identities

1) $\sin^2\alpha+\cos^2\alpha=1$;

2) $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}$;

3) $\cot\alpha=\dfrac{\cos\alpha}{\sin\alpha}$;

4) $\tan\alpha\cdot\cot\alpha=1$;

5) $1+\tan^2\alpha=\dfrac{1}{\cos^2\alpha}$;

6) $1+\cot^2\alpha=\dfrac{1}{\sin^2\alpha}$.

Formulae of addition

1) $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$;

2) $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$;

3) $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$;

4) $\cot(\alpha\pm\beta)=\dfrac{1\mp\tan\alpha\tan\beta}{\tan\alpha\pm\tan\beta}$.

Formulae of reduction

Express the argument of the trigonometric function as a sum (difference) of two angles so that one of them lies on either the x-axis or the y-axis (0°; 90°; 180°; 270°; ...). If the angle lies on the x-axis, then the function remains, the other angle is appointed to it and the sign of the trigonometric function is determined for the original function. If the angle lies on the y-axis, then the function changes to an opposite trigonometric function, the other angle is appointed and the sign of the trigonometric function is determined for the original function.

For example,

$\sin 330^\circ=\sin(270^\circ+60^\circ)=-\cos 60^\circ=-\dfrac{1}{2}$ or

$\sin 330^\circ=\sin(360^\circ-30^\circ)=-\sin 30^\circ=-\dfrac{1}{2}$.

Examples

1. Convert the following angles from degrees (radians) to radians (degrees)

   a) $15°$; b) $22.5°$; c) $\frac{\pi}{20}$; d) $\frac{12\pi}{5}$.

► The given angles must be multiplied by the corresponding scaling factor.

a) $15° = 15° \cdot \frac{\pi}{180°} = \frac{\pi}{12}$; b) $22.5° = 22.5° \cdot \frac{\pi}{180°} = \frac{\pi}{8}$;

c) $\frac{\pi}{20} = \frac{\pi}{20} \cdot \frac{180°}{\pi} = 9°$; d) $\frac{12\pi}{5} \cdot \frac{180°}{\pi} 432°$. ■

2. Evaluate

$$\sin^2 45° + \cos 240° + \tan 15° - \tan 23° \cdot \tan 67° .$$

► We need to express the angles in terms of those present in the table of values of trigonometric functions.

a) $\sin^2 45° = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

b) $\cos 240° = \cos(180° + 60°) = -\cos 60° = -\frac{1}{2}$.

c) $\tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \cdot \tan 30°} =$

$$\frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3} + 1} =$$

$$\frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} .$$

d) $\tan 23° \cdot \tan 67° = \tan(90° - 67°) \cdot \tan 67° =$

$\cot 67° \cdot \tan 67° = 1$.

Thus, $\frac{1}{2} + \left(-\frac{1}{2}\right) + 2 - \sqrt{3} - 1 = 1 - \sqrt{3}$. ■

Exercises

1. Express in radians the angles

   a) $45°$; b) $36°$; c) $180°$;

   d) $240°$; e) $216°$.

2. Change from radians to degrees

   a) $\frac{\pi}{3}$; b) $\frac{\pi}{5}$; c) π;

   d) $\frac{2\pi}{3}$; e) $\frac{5\pi}{4}$; f) $-\frac{\pi}{9}$.

3. What is the value of

   a) $\sin 0° + \cos\frac{\pi}{2} + \sin^2\frac{\pi}{2} + 6\sin\frac{\pi}{6} - 2\cos 0° + \tan^2\frac{\pi}{3}$?

   b) $3\sin\frac{\pi}{6} + 2\cos\pi + \cos^2\frac{\pi}{6} - \cos 0° + \sin 270°$?

4. If $\cos\alpha = \frac{1}{3}$, $(0° < \alpha < 90°)$, find the values of the rest of the trigonometric functions. (Hint: Use the main trigonometric identities).

5. Find the values of the following (using the formulae of addition)

   a) $\sin(30° + 30°)$; b) $\sin 75°$;

   c) $\cos 15°$; d) $\tan 105°$.

6. Calculate $\sin 1050° - \cos(-90)° + \cot 660° + \cos 405° + \sin 315°$.

7. Simplify as much as possible the expression

$$1 - \frac{\cos^2 x}{1 + \sin x} - \sin x + (5\sin y + 4\cos y)^2 + (4\sin y - 5\cos y)^2.$$

8. Compute

a) $\dfrac{\cos 68^\circ \cdot \cos 8^\circ - \cos 82^\circ \cdot \cos 22^\circ}{\cos 53^\circ \cdot \cos 23^\circ - \cos 67^\circ \cdot \cos 37^\circ}$;

b) $\sin^2\left(\dfrac{\pi}{8}\right) + \cos^2\left(\dfrac{3\pi}{8}\right) + \sin^2\left(\dfrac{5\pi}{8}\right) + \cos^2\left(\dfrac{7\pi}{8}\right)$.

9. Find the sum of the numbers

$\cos 1^\circ$, $\cos 2^\circ$, $\cos 3^\circ$, ..., $\cos 177^\circ$, $\cos 178^\circ$, $\cos 179^\circ$.

10. Prove the identity $\sin\left(\dfrac{\pi}{4} + \alpha\right) = \cos\left(\dfrac{\pi}{4} - \alpha\right)$.

Homework

1. Convert to the radians (degrees):

a) 22.5°; b) 18°; c) 75°; d) 300°;

e) $\dfrac{\pi}{18}$; f) $\dfrac{\pi}{10}$; g) $\dfrac{\pi}{5}$; h) $\dfrac{4\pi}{3}$.

2. What is the value of

a) $\sin 120^\circ$; b) $\cos 135^\circ$;

c) $\sin 315^\circ$; d) $\tan 15^\circ$?

3. Find the value of

a) $\cos 45^\circ \cos 15^\circ - \sin 45^\circ \sin 15^\circ$;

b) $\sin 180^\circ + \sin 270^\circ - \cot 90^\circ + \tan 180^\circ - \cos 90^\circ$;

c) $\tan\dfrac{\pi}{12} + \cot\dfrac{\pi}{12}$.

4. Simplify

a) $\dfrac{(\sin\alpha + \cos\alpha)^2 - \sin 2\alpha}{\cos 2\alpha + 2\sin^2\alpha}$;

b) $\cos\alpha + \cos(120^\circ + \alpha) + \cos(120^\circ - \alpha)$.

5. Calculate $\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ$.

Axioms/Postulates/Principles

1. Law of Third Excluded Only one of the two statements "A" and "not A" is valid.

2. Trichotomy Law Every real number is negative, 0, or positive. The law is sometimes stated as "For arbitrary real numbers a and b, exactly one of the relations $a<b$, $a=b$ or $a>b$ holds."

3. Law of Contradiction Logical law stating that no statement can be valid simultaneously with its rejection.

4. Axiom of Eudoxus-Euclid Some multiple of the smaller of two homogeneous quantities a and b will be greater than the larger (if $a<b$, then there exists a natural n such that $na>b$).

5. Axiom of Empty Set There is a set such that no set is a member of it.

6. Things which are equal to the same thing are also equal to one another.

7. If equals be added to equals, the wholes are equal.

8. If equals be subtracted from equals, the remainders are equal.

9. Things which coincide with one another are equal to one another.

10. The whole is greater than the part.

11. It is possible to produce a finite straight line continuously in a straight line.

12. It is possible to describe a circle with any center and distance.

13. Dirichlet principle If $m>n$, then distribution of each of m items to any of n classes results in at least one class with no less than two items.

Axioms/Postulates/Principles

14. Euclid's postulates 1. A straight line segment can be drawn joining any two points. 2. Any straight line segment can be extended indefinitely in a straight line. 3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center. 4. All right angles are congruent. 5. If two lines are drawn which intersect a third in such as way that the sum of the inner angles on one side is less than two right angles, then two lines inevitably must intersect each other on that side if extended far enough. (Also known as the parallel postulate).

15. Hilbert's axioms The 21 assumptions which underlie the geometry published in Hilbert's classic text "Grundlagen der Geometrie" consist of: The eight 'incidence axioms' about collinearity and intersection and include the first of Euclid's postulates; The four 'ordering axioms' about the arrangement of points; the five 'congruence axioms' about geometric equivalence, and the three 'continuity axioms' about continuity. There is also a single parallel axiom equivalent to Euclid's 'parallel postulate'.

16. Playfair's axiom Through any point in space, there is exactly one straight line parallel to a given straight line.

17. Cantor-Dedekind axiom The points on a line can be cut into a one-to-one correspondence with the real numbers.

18. Dedekind's axiom For every partition of all the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.

19. Fano's axiom The three diagonal points of a complete quadrilateral are never collinear.

20. Field axioms The field axioms are generally written in additive and multiplicative pairs.

Name	Addition	Multiplication
Commutativity	$a+b=b+a$	$ab=ba$
Associativity	$(a+b)+c=a+(b+c)$	$(ab)c=a(bc)$
Distributivity	$a(b+c)=ab+ac$	$(a+b)c=ac+bc$
Identity	$a+0=a=0+a$	$a*1=a=1*a$
Inverses	$a+(-a)=0=(-a)+a$	$a*a^{-1}=1=a^{-1}*a$, if $a\neq 0$

Science is organized knowledge. Wisdom is organized life.
Immanuel Kant (1724-1804), German philosopher

VII.2. Formulae of halved and doubled argument

Terms

1. **radius** – radius|радиус;
2. **circle** – doira|круг;
3. **circumference** – aylana|окружность;
4. **period** – davr|период;
5. **periodic function** – davriy funktsiya|периодическая функция;
6. **particular period** – xususiy davr|частный период.
7. **sine curve (sinusoid)** – sinuosoida|синусоида;
8. **tangent curve** – tangensoida|тангенсоида;
9. **doubled argument formula** – ikkilangan burchak formulasi|формула двойного аргумента;
10. **halved argument formula** – yarim burchak formulasi|формула половинного аргумента;

Learning Objectives

- to know how to draw the graph of a trigonometric function;
- to be able to use the formulae of doubled and halved argument (angle).

Formulae of doubled argument

1) $\sin 2x = 2\sin x \cos x$;

2) $\cos 2x = \cos^2 x - \sin^2 x$;

3) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

Formulae of halved argument

1) $\sin^2 x = \frac{1-\cos 2x}{2}$;

2) $\cos^2 x = \frac{1+\cos 2x}{2}$;

3) $\tan x = \frac{\sin 2x}{1+\cos 2x}$.

Periods of trigonometric functions

Period of the sine and cosine functions is 2π and period of the tangent and cotangent functions is π.

Formula for calculating the period of

1) $y = A\sin(\omega x+\varphi)$ or $y = A\cos(\omega x+\varphi)$: $T = \frac{2\pi}{\omega}$,

   where A, ω and φ are constant numbers.

2) $y = A\tan(\omega x+\varphi)$ or $y = A\cot(\omega x+\varphi)$: $T = \frac{\pi}{\omega}$,

   where A, ω and φ are constant numbers.

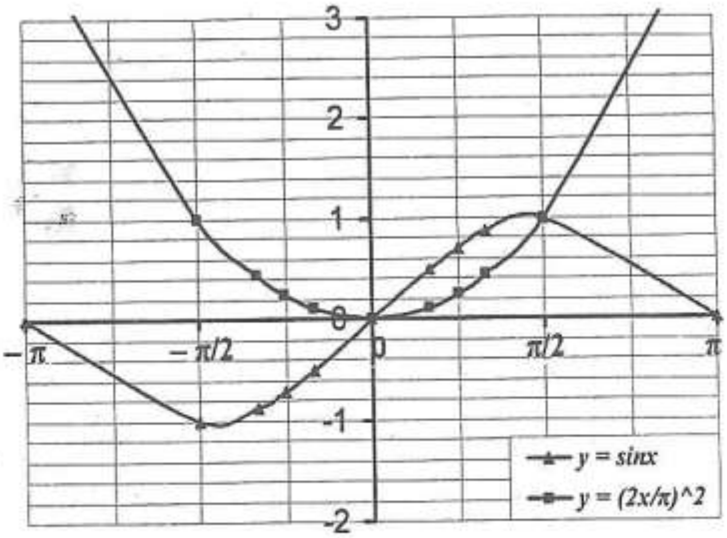
If $y=f(x)$ has a period of T_1 and $y=g(x)$ has a period of T_2, then the period of $y=f(x)+g(x)$ is the smallest possible number, which if divided by T_1 and T_2 results in a whole number.

Examples

1. Sketch the graphs of the functions $y=\sin x$ and $y=\left(\frac{2x}{\pi}\right)^2$ and indicate the intersection point(s).

▶ Domain of both functions is all real numbers. We will create a table of values of x and y.

x	$y = \sin x$	$y = \left(\frac{2x}{\pi}\right)^2$
$-\pi$	0	4
$-\frac{\pi}{2}$	-1	1
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{4}{3}$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{4}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$	$\frac{1}{9}$
0	0	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{9}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{4}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{4}{3}$
$\frac{\pi}{2}$	1	1
π	0	4



The graphs intersect at the points $(0, 0)$ and $\left(\frac{\pi}{2}, 1\right)$. ■

2. Find the period of the function

   $y = \cos(2x + 45°) + \cot(0.5x - 15°)$.

▶ Let's find the particular periods:

$T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$ and $T_2 = \frac{\pi}{\omega} = \frac{\pi}{0.5} = 2\pi$.

The smallest number that can be divided by the particular periods above without remainder is 2π. ■

3. Calculate $8\cos 22.5° \cdot \sin^2 22.5°$.

▶ We can use the formulae of doubled and halved arguments.

$8\cos 22.5° \cdot \sin 22.5° \cdot \sin 22.5° =$

$4 \cdot 2 \cdot \sin 22.5° \cdot \cos 22.5° \cdot \sqrt{\frac{1-\cos(2 \cdot 22.5°)}{2}}$

$4\sin(2 \cdot 22.5°) \cdot \sqrt{\frac{1-\cos 45°}{2}} =$

$$4\sin 45°\cdot\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2-\sqrt{2}}}{2}=\sqrt{4-2\sqrt{2}}\ .\ ■$$

Exercises

1. Find the domain, the range and sketch the graph of

   a) $y=\sin x$; b) $y=2+\sin x$;

   c) $y=1+2\sin x$; d) $y=\cos x-1$;

   e) $y=\tan x$; f) $y=\cot x$.

2. Check whether the function is even or odd

   a) $y=x^2\cos x$; b) $y=x^3\sin x^2$;

   c) $y=x\cos x$; d) $y=\dfrac{\sin x}{x^2-1}$.

3. Evaluate

   a) $\sin 120°$; b) $\cos 120°$;

   c) $\tan 120°$; d) $\sin 15°$;

   e) $\cos 22.5°$; f) $\tan 67.5°$.

4. Simplify as much as possible

   a) $\dfrac{(\sin\alpha+\cos\alpha)^2-\sin 2\alpha}{\cos 2\alpha+2\sin^2\alpha}$; b) $\dfrac{\sin 3\alpha}{\sin\alpha}-\dfrac{\cos 3\alpha}{\cos\alpha}$;

   b) $2\cos^2 x-\cos^4 x+\sin^4 x$.

5. Find the period of the functions

   a) $y=\sin(5x-3)$; b) $y=\cos(2x)+\cot(2x-3)$;

   c) $y=2\sin x+\tan(\pi x-4)$.

Homework

1. Determine the function domain and range of

   a) $y=3\sin x-1$; b) $y=2\cos x+1$;

   c) $y=-\tan x$.

2. Find the period of the functions

a) $y=\cos\left(\frac{3}{2}x-18°\right)$; b) $y=\sin\frac{x}{7}+\tan\frac{x}{5}$;

c) $y=\sin(2x)+\cos(\pi x)$.

3. If $\cos 2\alpha=\frac{1}{4}$, find $\sin^2\alpha$.

4. Express in terms of $\cos\theta$ the expression $\cos 4\theta+4\cos 2\theta+3$.

5. Simplify a) $\frac{\sin 6\alpha}{\sin 2\alpha}+\frac{\cos(6\alpha-\pi)}{\cos 2\alpha}$;

b) $\frac{\sin\alpha-2\sin 2\alpha+\sin 3\alpha}{\cos\alpha-2\cos 2\alpha+\cos 3\alpha}-\tan 2\alpha$.

6. Calculate $\dfrac{1-\tan^2\frac{\pi}{12}}{\tan\frac{\pi}{12}}$.

Infor(mathe)matics

1. Fields Medal The mathematical equivalent of the Nobel Prize (there is no Nobel Prize in mathematics), this international prize for achievement in the field of mathematics is awarded every four years by the International Mathematical Union at the International Congress of Mathematicians. The award recognizes both existing work as well as the promise of future achievement and is presented to mathematicians under the age of 40. It was founded by J. Ch. Fields, a Canadian mathematician, in 1932. The prize constitutes 15,000 Canadian dollars (approximately 1,500 US dollars). In 1966 it was agreed that, in light of the great expansion of mathematical research, up to four medals could be awarded at each Congress. The Fields Medal is made of gold, and shows the head of Archimedes together with a quotation attributed to him: "Transire suum pectus mundoque potiri" ("Rise above oneself and grasp the world"). The reverse side bears the inscription: "Congregati ex toto orbe mathematici ob scripta insignia tribuere" ("the mathematicians assembled here from all over the world pay tribute for outstanding work").

2. One Hundred Eminent Mathematicians Walter Crosby Eells endeavored to determine the one hundred most eminent mathematicians living prior to 1905, and to list these men in order of eminence (Crosby W 1962). In this list Newton appears in first place, Leibniz in second place, Lagrange in third place, and Euler in fourth place.

Infor(mathe)matics

3. The Hundred Greatest Theorems At a mathematics conference in July 1999, Paul and Jack Abad presented their list of "The Hundred Greatest Theorems". Their ranking is based on the following criteria: "the place the theorem holds in the literature, the quality of the proof, and the unexpectedness of the result". Some of the theorems are "Irrationality of $\sqrt{2}$" at 1^{st} place, "The fundamental theorem of algebra" at 2^{nd}, "Pythagorean Theorem" at 4^{th}, "Sum of the Angles of a Triangle" at 27^{th}, "Sum of a Geometric Series" at 66^{th}, "Sum of an arithmetic series" at 68^{th}, "The Law of Cosines" at 94^{th}, "Descartes Rule of Signs" at 100^{th}.

4. Mathematics Contests There are several regular mathematics competitions available to students. The International Mathematical Olympiad (IMO) is the yearly world championship of mathematics for high school students and is held in a different country each year. The first IMO was held in 1959 in Romania. The William Lowell Putnam Competition is a North American math contest for college students. Each year over 2000 students spend six hours in two sittings trying to solve 12 problems. The majority of the problems are very difficult, in the sense that their solution may require a nonstandard and creative approach. The International Mathematical Contest in Modeling (MCM) is a competition that challenges teams of undergraduate students to clarify, analyze, and propose solutions to open-ended problems.

Give me six hours to chop down a tree and
I will spend the first four sharpening the axe.
Abraham Lincoln (1809-1865), 16th president of the United States

VII.3. Trigonometric equation and inequalities

Terms

1. **inverse trigonometric function** – teskari trigonometrik funktsiya|обратная тригонометрическая функция;
2. **the main value of an inverse trigonometric function** – teskari trigonometrik funktsiyaning asosiy qiymati|главное значение обратной тригонометрической функции;
3. **arctangent** – arktangens|арктангенс;
4. **formula of transformation of sums (differences) into product** – yig'indini (ayirmani) ko'paytmaga almashtirish formulasi|формула преобразования суммы (разности) в произведения;
5. **simple (complex) trigonometric equation** – oddiy (murakkab) trigonometrik tenglama|простое (сложное) тригонометрическое уравнение.

Learning Objectives

- to know how to calculate the values of an inverse trigonometric function;
- to be able to solve trigonometric equations and inequalities.

Formulae of transformation of sums and differences into product

1) $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$;

2) $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$;

3) $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$;

4) $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$.

Simple trigonometric equations and their solutions

1) $\sin x = a(-1 \le a \le 1)$; $x = (-1)^n \arcsin a + \pi n,\ n \in Z$;

2) $\tan x = a$; $x = \arctan a + \pi n,\ n \in Z$;

3) $\cos x = a\ (-1 \le a \le 1)$; $x = \pm\arccos a + 2\pi n,\ n \in Z$;

4) $\cot x = a$; $arc\cot a + \pi n,\ n \in Z$.

Examples

1. Evaluate

$$\arcsin\frac{1}{2} + \arccos\left(-\frac{\sqrt{3}}{2}\right) + \arctan 1 + \operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right).$$

▶ We will find the value of each arc function.

$$\arcsin\frac{1}{2} + \arccos\left(-\frac{\sqrt{3}}{2}\right) + \arctan 1 + \operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right) =$$

$$30^\circ + 150^\circ + 45^\circ + 60^\circ = 285^\circ.$$ ■

2. Solve the equation $\sin 3x = \sin 2x + \sin x$ for $90° \le x \le 180°$.

▶ We will transfer all terms to the left side and use the formula of transformation.

$(\sin x3x - \sin x) - \sin 2x = 0 \Rightarrow 2\cos 2x \sin x - 2\sin x \cos x = 0 \Rightarrow$

$\sin x \cdot (\cos 2x - \cos x) = 0$.

The above product is equal to zero when at least one of the two factors is equal to zero. Thus, each factor must be equated to zero.

1) $\sin x = 0 \Rightarrow x = \pi n,\ n \in Z$.

Taking into account the condition $90° \le x \le 180°$, we find

$x_1 = \pi$ or $x_1 = 180°$.

2) $2\cos^2 x - \cos x - 1 = 0$.

Denoting $\cos x = t$, we obtain the quadratic equation.

$2t^2 - t - 1 = 0 \Rightarrow t_1 = 1$ and $t_2 = -\frac{1}{2}$.

Now we substitute back to our notation $\cos x = t$.

a) $\cos x = 1 \Rightarrow x = \pm\frac{\pi}{2} + 2\pi n,\ n \in Z$.

However, there is no solution satisfying $90° \le x \le 180°$.

b) $\cos x = -\frac{1}{2} \Rightarrow x = \pm\frac{2\pi}{3} + 2\pi n,\ n \in Z$.

There are two solutions from the segment $90° \le x \le 180°$.

$x_2 = \frac{2\pi}{3}$ or $x_2 = 120°$. ■

3. Solve the inequality $\sin x + \cos x > 1$.

▶ Multiply both sides of the inequality by $\frac{\sqrt{2}}{2}$ before using the transformation formula.

$$\sin x + \cos x > 1 \Big| \cdot \frac{\sqrt{2}}{2} \Rightarrow$$

$$\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x > \frac{\sqrt{2}}{2} \Rightarrow$$

$$\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x > \frac{\sqrt{2}}{2} \Rightarrow$$

$$\sin\left(x + \frac{\pi}{4}\right) > \frac{\sqrt{2}}{2} \Rightarrow$$

$$\frac{\pi}{4} + 2\pi n < x + \frac{\pi}{4} < \frac{3\pi}{4} + 2\pi n \Rightarrow 2\pi n < x < \frac{\pi}{2} + 2\pi n \text{ or}$$

$$x \in \left(2\pi n,\ n \in Z;\ \frac{\pi}{2} + 2\pi n,\ n \in Z\right). \blacksquare$$

Exercises

1. Calculate

   $\arcsin 1 + \arccos\frac{1}{2} + \arctan\frac{\sqrt{3}}{3} + \arccos\left(-\frac{1}{2}\right)$.

2. Solve the trigonometric equation

   a) $\sin x = 0.5$; b) $\cos x - \sin 2x \cos x = 0$;

   c) $\sin x \cdot \cos 2x = 0$.

3. Find all angles x between 0° and 360° for which

   $2^{2\sin x - 1} = 3 \cdot 2^{\sin x} - 4$.

4. Find the solution of

   a) $\cos 2x - \cos 6x = 0$; b) $\sin x + \sin 3x = 0$.

5. Express $\dfrac{2\sin 2\theta + \sin 4\theta}{2(\cos\theta + \cos 3\theta)\tan 2\theta}$ in terms of $\cos\theta$, giving you answer in the simplest possible form.

6. Solve the inequalities

   a) $\sin x > 0.5$; b) $\cos x < 0.5$;

   c) $\sin x \cdot \cos x > \dfrac{\sqrt{2}}{4}$; e) $\tan x < 1$.

Homework

1. Calculate

   a) $\cos\left(\arctan\sqrt{3} + \arccos\dfrac{\sqrt{3}}{2}\right)$;

   b) $\cos\left(2\arcsin\dfrac{4}{5}\right)$.

2. Solve the equations

   a) $\cos^4 x - \sin^4 x = 1 - \cos 2x$;

   b) $2\sin x + 3\sin 2x = 0$;

   c) $\tan 3x - \tan x = 0$;

   d) $2^{\cos 2x} - \dfrac{1}{2\cdot 2^{\cos 2x}} = 0$;

   e) $2\cos^2 x + 2\sin^2 x = 5 + 2\sin x$.

3. Solve the inequalities

   a) $\sin 2x > 0$; b) $\sin x \le -1$;

   c) $\tan 2x > 0$; d) $|\cos x| < 0.5$;

   e) $\cos^2 x < \dfrac{\sqrt{2}}{2} + \sin^2 x$;

   f) $(\cos x + 2)\cdot|x-5|\cdot(x-2) \le 0$.

4. Find all angles x with $0 \le x \le 90°$ such that

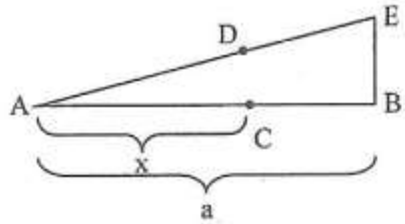
$$1+\cos 2x+4\left(2-3\sin^2 x\right)\cos x=0.$$

5. Find all angles x between $0°$ and $180°$ that satisfy the equation $\tan 2x \cdot \tan 3x = 1$. (Remember to reject all angles for which either $\tan 2x$ or $\tan 3x$ is meaningless).

Interesting numbers

1. Friendly numbers Two natural numbers, each of which is equal to the sum of the divisors of the other excluding the number itself. For example, 284 and 220 (284=1+2+4+5+10+11+20+22+44+55+110 and 220=1+2+4+71+142). Euler found about 60 such pairs. The use of computers has found a few hundred more pairs.

2. Golden mean (or harmonic division). A division of the line segment AB so that the bigger part AC is half proportional to all AB and smaller part BC. Algebraic solution of finding AB: let's denote $AB = a$, $AC = x$, then $a{:}x=x{:}(a-x)$.



The ratio of x to a can also be expressed approximately by the fractions 2/3, 3/5, 5/8, 8/13, 13/21, ..., where 2, 3, 5, 8, 13, 21, ... are Fibonacci numbers.

3. For given numbers $x_1, x_2, x_3, \ldots, x_n$,

Arithmetic mean: $\bar{x}=\dfrac{x_1+x_2+\ldots+x_n}{n}$;

Geometric mean $g=\sqrt[n]{x_1 x_2 \cdots x_n}$;

Harmonic mean: $h=\dfrac{n}{\dfrac{1}{x_1}+\dfrac{1}{x_2}+\ldots+\dfrac{1}{x_n}}$;

Quadratic mean: $s=\sqrt{\dfrac{x_1^2+x_2^2+\ldots+x_n^2}{n}}$.

4. Perfect proportion Since

$$\frac{2a}{a+b}=\frac{2ab}{(a+b)b},$$

it follows

$$\frac{a}{\frac{a+b}{2}}=\frac{\frac{2ab}{(a+b)}}{b}.$$

So $\frac{a}{A}=\frac{H}{b}$, where A and H are the arithmetic mean and harmonic mean of a and b. This relationship was purportedly discovered by Pythagoras.

5. Chain fraction One of the main methods of expressing numbers and functions. A chain fraction has the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{\dots+\cfrac{1}{a_n+\dots}}},$$

where $a_0, a_1, \dots, a_n$ are whole numbers.

For example:

$$1=2-\frac{2}{2}=2-\frac{1}{2-1}=2-\cfrac{1}{2-\cfrac{1}{2-1}}=\dots=2-\cfrac{1}{2-\cfrac{1}{2-\dots}};$$

$$\sqrt{2}=1+\left(\sqrt{2}-1\right)=1+\frac{1}{1+\sqrt{2}}=1+\frac{1}{1+\left(\sqrt{2}-1\right)}=$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\sqrt{2}}}=\dots=1+\cfrac{1}{1+\cfrac{1}{1+\dots}}.$$

Books are the compass and telescopes and sextants and charts which other men have prepared to help us navigate the dangerous seas of human life.
Jesse Lee Bennett (1907-2000)

VII.4. Logarithmic equations and inequalities

Terms

1. **common (denary, decimal) logarithm** – o'nli logarifm|десятичный логарифм;
2. **natural (hyperbolic) logarithm** – natural logarifm|натуральный логарифм;
3. **logarithmic base** – logarifm asosi|основание логарифма;
4. **logarithm of a to base b** – a ning b asosga ko'ra logarifmi|логарифм числа a по основанию b;
5. **to take the logarithm of a number** – sonning logarifmini topmoq| находить логарифм числа.

Learning Objectives

- to evaluate expressions containing logarithms;
- to know how to draw graphs of logarithmic functions;
- to solve logarithmic equations and inequalities.

Logarithm of b to the base a

$a^n = b \ (b > 0) \Rightarrow \log_a b = n$.

Common logarithm

$\log_{10} a = \lg a$

Natural logarithm

$\log_e a = \ln a$.

Note: $e \approx 2.718$.

Rules of logarithms

1) $\log_a a = 1$

2) $\log_a 1 = 0$

3) $\log_a (bc) = \log_a b + \log_a c$

4) $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$

5) $\log_a b^n = n \log_a b$

6) $\log_{a^m} b = \frac{1}{m} \log_a b$

7) $\log_a b = \frac{\log_c b}{\log_c a}$

8) $a^{\log_a b} = b$.

Logarithmic function

$y = \log_{p(x)} q(x)$,

where $p(x)$ and $q(x)$ are the algebraic expressions for which

$$\begin{cases} q(x) > 0 \\ p(x) > 0. \\ p(x) \neq 1 \end{cases}$$

Note: The above conditions constitute the logarithmic function domain.

Logarithmic equation and its solution

$$\log_a f(x) = \log_a g(x) \Rightarrow \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) = g(x) \end{cases}.$$

Logarithmic inequality and its solution

$$\log_a b(x) < \log_a c(x) \Rightarrow \begin{cases} 0 < b(x) < c(x), \ if \quad a > 1 \\ b(x) > c(x) > 0, \ if \ 0 < a < 1 \end{cases}.$$

Examples

1. Calculate $\dfrac{3^{2\log_9(\log_3 16)}}{\log_3 5\cdot\log_5 8-\log_3 2}$.

▶ We will try to express the logarithms through a common base.

$$\frac{\left(3^2\right)^{\log_9(\log_3 16)}}{\dfrac{1}{\log_5 3}\cdot\log_5 8-\log_3 2}=$$

$$\frac{9^{\log_9(\log_9 16)}}{\dfrac{\log_5 8}{\log_5 3}-\log_3 2}=\frac{\log_9 16}{\log_3 8-\log_3 2}=$$

$$\frac{\log_{3^2} 2^4}{\log_3\left(\dfrac{8}{2}\right)}=\frac{\dfrac{4}{2}\log_3 2}{\log_3 2^2}=\frac{2\log_3 2}{2\log_3 2}=1.\ ■$$

2. Solve the equation $x^{\log_4 x-2}=2^{3(\log_4 x-1)}$.

▶ It is possible to simplify the equation if we set $t=\log_4 x$ or $x=4^t$, so that

$$\left(4^t\right)^{t-2}=2^{3(t-1)}\Rightarrow 2^{2t(t-2)}=2^{3(t-1)}\Rightarrow$$

$$2t(t-2)=3(t-1)\Rightarrow 2t^2-7t+3=0\Rightarrow t_1=\frac{1}{2}\text{ and } t_2=3.$$

Hence, $\log_4 x=\dfrac{1}{2}\Rightarrow x_1=2$ and $\log_4 x=3\Rightarrow x_2=64$. ■

3. Solve the inequality $\log_{0.6}\log_{27} x>-1$.

▶ $\log_{0.6}\log_{27} x>\log_{0.6} 0.6^{-1}\Rightarrow$

$$\log_{27} x<0.6^{-1}\Rightarrow\log_{3^3} x<\left(\frac{3}{5}\right)^{-1}\Rightarrow$$

$$\frac{1}{3}\log_3 x < \frac{5}{3}\Big|\cdot 3 \Rightarrow \log_3 x < 5 \Rightarrow$$

$$\log_3 x < \log_3 3^5 \Rightarrow x < 3^5 \Rightarrow x < 243.$$ ■

4. What is the function domain of $y = \log_{4-x}(x^2 - 2x)$?

▶ According to the formula the function domain of the given logarithmic function is the solution of the following simultaneous inequalities

$$\begin{cases} x^2 - 2x > 0 \\ 4 - x > 0 \\ 4 - x \neq 1 \end{cases} \Rightarrow \begin{cases} x(x-2) > 0 \\ x < 4 \\ x \neq 3 \end{cases} \Rightarrow$$

$$\begin{cases} x \in (-\infty;\, 0) \cup (2;\, +\infty) \\ x \in (-\infty;\, 4) \\ x \in (-\infty;\, 3) \cup (3;\, +\infty) \end{cases} \Rightarrow$$

$$x \in (-\infty;\, 0) \cup (2;\, 3) \cup (3;\, 4).$$ ■

5. Compare the numbers 2^{30} and 3^{20}.

▶ We take logarithm to the base 2 (base 3 is also fine) of each number as follows.

$\log_2 2^{30}$ and $\log_2 3^{20} \Rightarrow$

30 and $20\log_2 3 \Rightarrow$

30 and $20 \cdot 1.58 \Rightarrow 30 < 31.6$.

Thus, the second number is greater. ■

Exercises

1. Calculate

a) $\log_2 16 + \log_3 \frac{1}{81} + \log_{17} 1 + \log_{\frac{1}{3}} 9 + \log_{\sqrt{5}} 1$;

b) $2^{\log_4 9} + \log_2 \log_5 \sqrt[8]{5} + \log_5 \ln e^5$;

c) $\log_3 5 \cdot \log_4 9 \cdot \log_5 2$.

2. Find the function domain of

   a) $y = \log_2 (x^2 - 2x)$; b) $y = \log_x (6 - x)$.

3. Draw the graph of the logarithmic functions

   a) $y = \log_2 x$; b) $y = \ln x$.

4. Solve the equations

   a) $\log_2 (x-2) = 4$; b) $\log_x (2x^2 - x - 2) = 2$;

   c) $\log_2 \log_3 x = 0$; d) $\log_{16} x + \log_4 x + \log_2 x = 7$;

   e) $3^{x-5} = 7$; f) $\log_x \sqrt[3]{625} - \log_x \sqrt{125} + \frac{1}{6} = 0$.

5. For what value of x is $2\log_3 (x-2) - 2\log_3 (4-x) = 0$?

6. Solve the trigonometric inequalities

   a) $\log_2 (2x-1) < 3$; b) $\log_{16} (3x+1) > 0.5$;

   c) $\log_{0.2} (2x^2 + 5x + 1) < 0$; d) $\log_x (x+2) > 2$.

Homework

1. Calculate

   a) $\frac{3\lg 2 + 3\lg 5}{\lg 1300 - \lg 13}$;

   b) $\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + \frac{1}{\log_{16} 4} + \frac{1}{\log_{32} 4}$;

   c) $\log_{128} \left((0.25)^{\log_{16} \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots \right)} \right)$;

   d) $\log_3^{-1} \sqrt[3]{\sqrt[3]{\sqrt[3]{3}}}$.

2. Solve the equation

   a) $\lg(3 + 2\lg(1+x)) = 0$; b) $3^{2\log_3 x} = 16$;

c) $\log_3(3^x - 8) = 2 - x$; d) $x^{1+\lg x} = 0.001^{-\frac{2}{3}}$.

3. Solve the inequality

a) $\log_5(2x-4) < \log_5(x+3)$; b) $\log_{0.5} x > -2$.

Interesting numbers

1. **Googol** A large number equal to 10^{100} (i.e. a 1 with 100 zeros following it).

2. **Googolplex** A large number equal to $10^{10^{100}}$ (i.e. 1 with a Googol number of zeros written after it.)

3. **Myriad** The Greek word for 10,000.

4. **Twins** Two prime numbers, whose difference is equal to two. It is still unclear if the set of twins is limited or not.

5. **Perfect number** A positive integer that is equal to the sum of all its positive, proper divisors. For example, 6=1+2+3; 28=1+2+4+7+14. Euclid (3rd century BC) has indicated the formula $2^{p-1}(2^p-1)$ (where 2^p-1 must be prime numbers) for even perfect numbers. The formula works for about 27 even perfect numbers. It is still unclear if there is any odd perfect number. It is clear that no odd perfect number exists in the interval from 1 to 10^{50}.

6. **Palindrome** A number that reads the same forwards and backwards. For example, 32423.

7. **Egyptian (aliquot) number** A number n is called an Egyptian number if it is the sum of the denominators in some unit fraction representation of a positive whole number not consisting entirely of 1s. For example, $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, so 2+3+6=11 is an Egyptian number. The numbers which are not Egyptian are 2, 3, 5, 6, 7, 8, 12, 13, 14, 15, 19, 21, and 23.

8. **Transcendental number** A number that is not a root of a polynomial equation with whole coefficients. For example, π (3,14159...), e (2,71828...), $ln2$.

9. **Mersenne prime number** A prime number of the form $2^n - 1$, where n is a prime number. For example, 3, 31, 1023. Note that if 2^n-1 is prime; then $(2^n-1)*2^{(n-1)}$ is a perfect number. For example, if n=3, $2^3-1=7$ is prime and 7*4=28; if n=5, $2^5-1=31$ and 31*16=496.

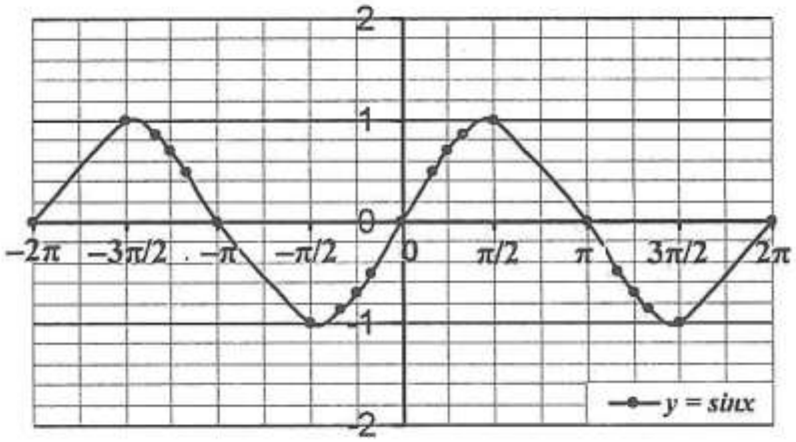
10. **Narcissistic number** An n-digit number which is the sum of the nth powers of its digits. The smallest example other than the trivial 1-digit number is $153=1^3+5^3+3^3$. The series of smallest narcissistic numbers are 0, 153, 1634, 54748, 548834, ...

Chapter VII Answers. Trigonometry and logarithms

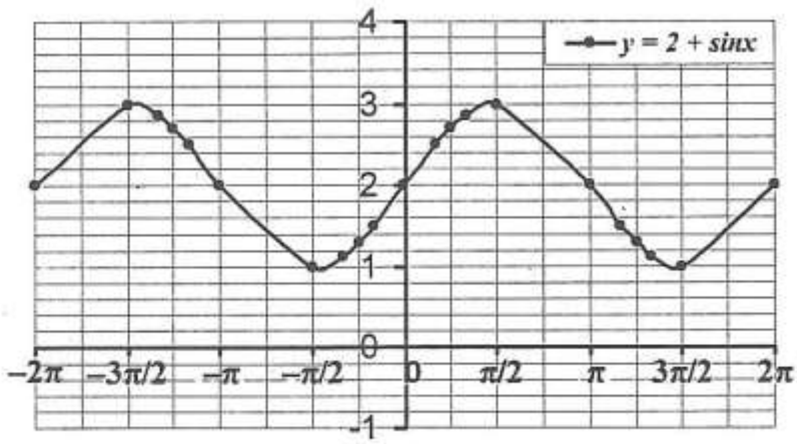
VII.1. 1. $\pi/4$; $\pi/5$; π; $4\pi/3$; $6\pi/5$. **2.** 60°; 36°; 180°; 120°; 225°; −20°. **3.** a) 5; b) −7/4. **4.** $\sin\alpha = \frac{2\sqrt{2}}{3}$; $\tan\alpha = 2\sqrt{2}$; $\cot\alpha = \frac{\sqrt{2}}{4}$. **5.** a) $\frac{\sqrt{3}}{2}$; b) $\frac{\sqrt{2}+\sqrt{6}}{4}$; c) $\frac{\sqrt{2}+\sqrt{6}}{4}$; d) $-2-\sqrt{3}$. **6.** $-\frac{1}{2}-\frac{\sqrt{3}}{3}$. **7.** 41. **8.** a) 1; b) 2. **9.** 0. **10.** Use the formula of addition.

Homework: 1. a) $\frac{\pi}{8}$; b) $\frac{\pi}{10}$; c) $\frac{5\pi}{12}$; d) $\frac{5\pi}{3}$; e) $10°$; f) $18°$; g) $36°$; h) $240°$. **2.** a) $\frac{\sqrt{3}}{2}$; b) $-\frac{\sqrt{2}}{2}$; c) $-\frac{\sqrt{2}}{2}$; d) $2-\sqrt{3}$. **3.** a) 1/2; b) −1; c) 4. **4.** a) 1; b) 0. **5.** 1.

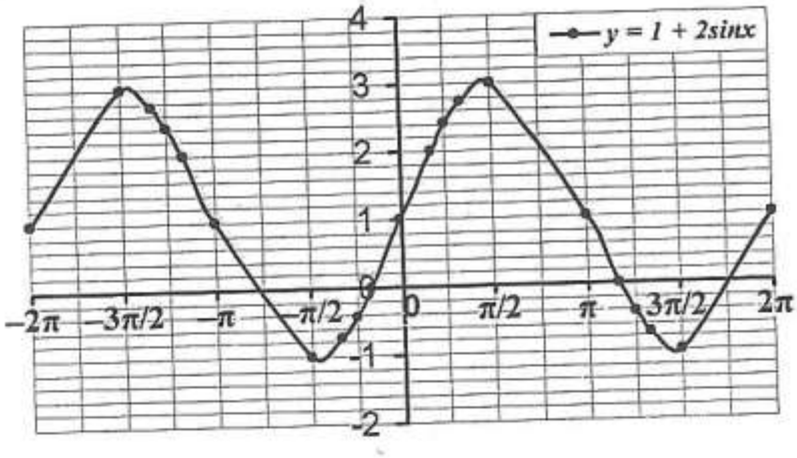
VII.2. 1. a) (−∞, +∞); [−4, 2];



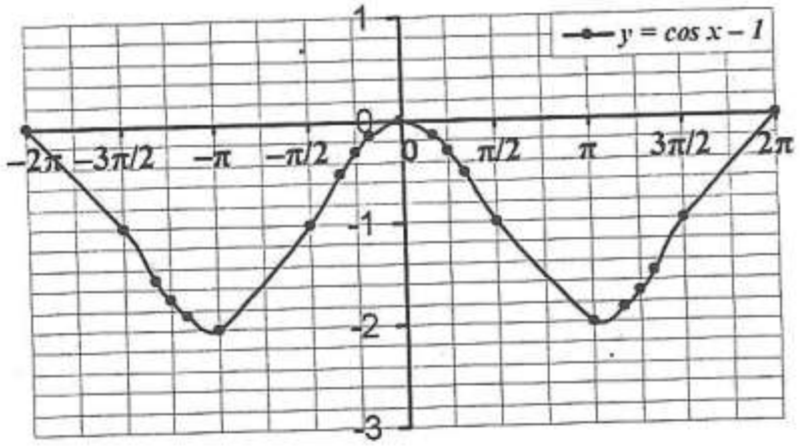
b) (−∞, +∞); [1, 3];



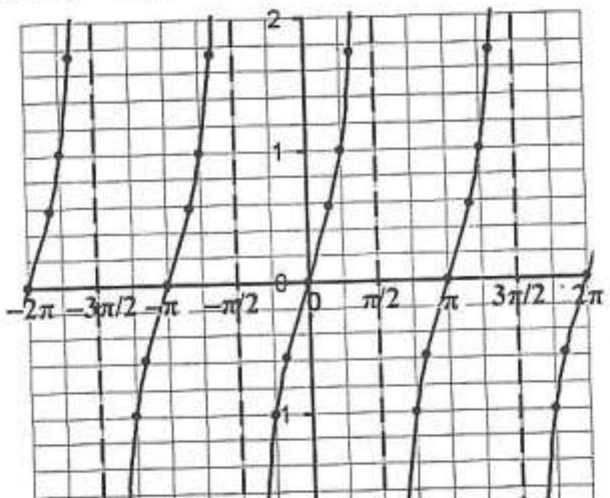
c) $(-\infty, +\infty)$; $[-1, 3]$;



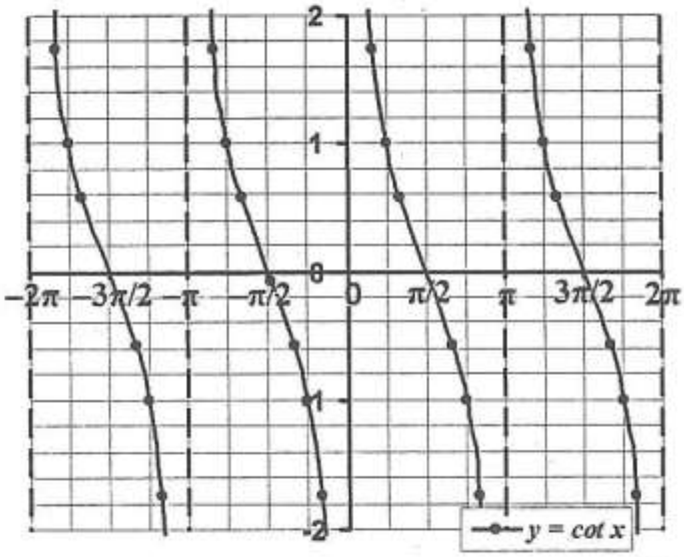
d) $(-\infty, +\infty)$; $[-2,0]$;



e) $x \neq \pi/2+\pi n$, $n \in Z$; $(-\infty,+\infty)$;



f) $x \neq \pi n,\ n \in Z$; $(-\infty, +\infty)$.



2. a) even; b) odd; c) odd; d) odd. **3.** a) $\frac{\sqrt{3}}{2}$; b) $-\frac{1}{2}$; c) $-\sqrt{3}$; d) $\frac{\sqrt{6}-\sqrt{2}}{4}$; e) $\frac{\sqrt{2+\sqrt{2}}}{2}$; f) $2\sqrt{2}+1$. **4.** a) 1; b) 2; c) 1. **5.** a) $2\pi/5$; b) π; c) No period.

Homework: 1. a) $(-\infty, +\infty)$; $[-4, 2]$; b) $(-\infty, +\infty)$; $[-1, 3]$.

c) $x \neq \frac{\pi}{2} + \pi n, n \in Z$; $y \in (-\infty, +\infty)$. **2.** a) $\frac{4\pi}{3}$; b) 70π. c) No period. **3.** 3/8. **4.** $8\cos^4\theta$. **5.** a) 2; b) 0. **6.** $2\sqrt{3}$.

VII.3. 1. $\frac{5\pi}{3}$. **2.** a) $(-1)^n\frac{\pi}{6} + \pi k, k \in Z$;

b) $\pm\frac{\pi}{2} + \pi k, k \in Z, \frac{\pi}{4} + \pi k, k \in Z$; c) $\pi k, k \in Z, \frac{\pi}{4} + \frac{\pi k}{2}, k \in Z$. **3.**
$\frac{\pi}{2} + 2\pi k, k \in Z$. **4.** a) $\frac{\pi k}{3}, k \in Z, \frac{\pi k}{2}, k \in Z$; b) $\frac{\pi k}{2}, k \in Z$. **5.** $\cos\theta$. **6.** a)
$\left(\frac{\pi}{6} + 2\pi k, k \in Z; \frac{5\pi}{6} + 2\pi k, k \in Z\right)$;

b) $\left(\frac{\pi}{3} + 2\pi k, k \in Z; \frac{2\pi}{3} + 2\pi k, k \in Z\right)$;

c) $\left(\frac{\pi}{4} + 2\pi k, k \in Z; \frac{3\pi}{4} + 2\pi k, k \in Z\right)$;

d) $\left(-\frac{\pi}{2}+\pi k, k\in Z; \frac{\pi}{4}+\pi k, k\in Z\right)$.

Homework: 1. a) 0; b) $-\frac{17}{8}$. **2.** a) $\pm\frac{\pi}{6}+\pi k, k\in Z$; b)

$\pi k, k\in Z; \pm\arccos\left(-\frac{1}{3}\right)+2\pi k, k\in Z$;

c) $\pi k, k\in Z$; d) $\pm\frac{\pi}{3}+\pi k, k\in Z$; e) Ø. **3.** a) $\left(\frac{\pi k}{2}, k\in Z; \frac{\pi}{2}+\frac{\pi k}{2}, k\in Z\right)$;

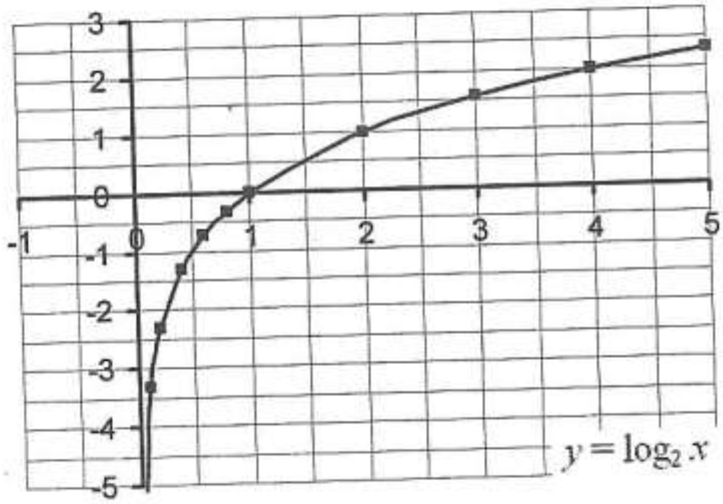
b) $-\frac{\pi}{2}+2\pi k, k\in Z$; c) $\left(\frac{\pi k}{2}, k\in Z; \frac{\pi}{4}+\frac{\pi k}{2}, k\in Z\right)$;

d) $\left(\frac{2\pi}{3}+\pi k, k\in Z; \frac{4\pi}{3}+\pi k, k\in Z\right)$;

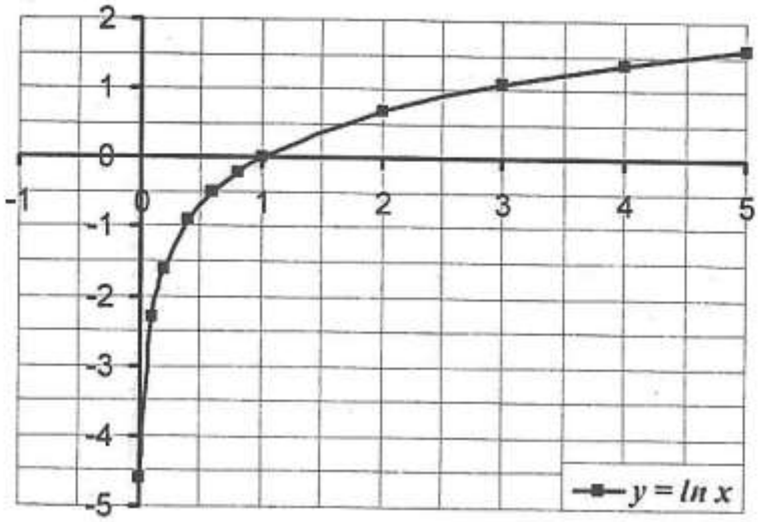
e) $\left(\frac{\pi}{8}+\frac{\pi k}{2}, k\in Z; \frac{3\pi}{8}+\frac{\pi k}{2}, k\in Z\right)$. **4.** $\frac{\pi}{3}$; $\frac{\pi}{2}$. **5.** $\frac{\pi}{10}$; $\frac{3\pi}{10}$; $\frac{7\pi}{10}$; $\frac{9\pi}{10}$.

VII.4. 1. a) –2; b) 1. **2.** a) $(-\infty, 0)\cup(2, +\infty)$; b) (0, 1)U(1, 6).

3. a)



b)



4. a) 18; b) 2; c) 3; d) 16; e) $\log_3 7+5$; f) 5. **5.** 3. **6.** a) (0.5, 4.5); b) (1, +∞); c) $(-\infty,-2.5)\cup(0,+\infty)$; d) (1, 2).

Homework: 1. a) 1.5; b) 7.5; c) 1/14; d) 27. **2.** a) – 0.9; b) 4; c) 2; d) 0.01; 10. **3.** a) (–3, 7); b) (0, 4).