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MATEMATIK ANALIZ ASOSLARI

$$f: R^m \to R$$

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2-qism

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Ko'p o'zgaruvchili funksiyalar, ularning limiti, uzluksizligi

«Matematik analiz asoslari» kursining I-qismida bir o'zgaruvchili funksiyalar batafsil o'rganildi.

Matematika, fizika, texnika va fanning boshqa tarmoqlarida shunday funksiyalar uchraydiki, ular ko'p o'zgaruvchilarga bog'liq bo'ladi. Masalan, doiraviy silindrning hajmi

$$V = \pi r^2 h \qquad (12.1)$$

ikki o'zgaruvchi: *r*–radius hamda *h*–balandlikka bog'liq.

Tok kuchi

$$J = \frac{E}{R} \qquad (12.2)$$

ham ikki o'zgaruvchi: *E*–elektr yurituvchi kuch va *R*-qarshilikning funksiyasi bo'ladi. Bunda silindrning hajmi (12.1) formula yordamida bir-biriga bog'liq bo'lmagan *r* va *h* o'zgaruvchilarning qiymatlariga ko'ra, tok kuchi (12.2) formula yordamida bir-biriga bog'liq bo'lmagan *E* va *R* o'zgaruvchilarning qiymatlariga ko'ra topiladi. Shunga o'xshash misollarni juda ko'plab keltirish mumkin. Binobarin, ko'p o'zgaruvchili funksiyalarni yuqoridagidek chuqurroq o'rganish vazifasi tug'ildi.

Ko'p o'zgaruvchili funksiyalar nazariyasida ham bir o'zgaruvchili funksiyalar nazariyasidagidek, funksiya va uning limiti, funksiyaning uzluksizligi va xakazo kabi tushunchalar o'rganiladi. Bunda bir o'zgaruvchili funksiyalar haqidagi ma'lumotlardan muttasil foydalana boriladi.

Ma'lumki, bir o'zgaruvchili funksiyalarni o'rganishni ularning aniqlanish to'plamlarini (sohalarini) o'rganishdan boshlagan edik. Ko'p o'zgaruvchili funksiyalarni o'rganishni ham ularning aniqlanish to'plamlarini (sohalarini) bayon etishdan boshlaymiz.

1-§. R^m fazo va uning muhim to'plamlari

1^0. R^m fazo. *m* ta A_1, A_2, … A_m ($m \geq 1$ butun son) to'plamlarning Dekart ko'paytmasi ikkita *A* va *V* to'plamlarning Dekart ko'paytmasiga o'xshash ta'riflanadi. Agar $A_1 = A_2 = ... = A_m = R$ bo'lsa, u holda

$$A_1 \times A_2 \times ... \times A_m = R \times R \times ... \times R = \{(x_1, x_2, ..., x_m) : x_1 \in R, x_2 \in R, ..., x_m \in R\}$$

bo'ladi. Ushbu

$$\{(x_1, x_2, ..., x_m) : x_1 \in R, x_2 \in R, ..., x_m \in R\}$$

to'plam R^m to'plam deb ataladi. R^m to'plamning elementi $(x_1, x_2, ..., x_m)$ shu to'plam nuqtasi deyiladi va u odatda bitta harf bilan belgilanadi: $x = (x_1, x_2, ..., x_m)$. Bunda $x_1, x_2, ..., x_m$ sonlar *x* nuqtaning mos ravishda birinchi, ikkinchi, … *m*-koordinatalari deyiladi.

Agar $x=(x_1,x_2,...,x_m)\in R^m$, $y=(y_1,y_2,...,y_m)\in R^m$ nuqtalar uchun $x_1=y_1, x_2=y_2,...,x_m=y_m$ bo'lsa, u holda $x=y$ deb ataladi.

R^m to'plamda ixtiyoriy $x=(x_1,x_2,...,x_m)$, $y=(y_1,y_2,...,y_m)$ nuqtalarni olaylik.

1-ta'rif. Ushbu

$$\rho(x,y)=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+...+(y_m-x_m)^2}=\sqrt{\sum_{i=1}^{m}(y_i-x_i)^2} \qquad (12.3)$$

miqdor x va y nuqtalar orasidagi masofa (Evklid masofasi) deb ataladi. Bunday aniqlangan masofa quyidagi xossalarga ega (bunda $\forall x,y,z\in \mathrm{R}^m$)

1) $\rho(x,y)\ge 0$ va $\rho(x,y)=0 \Leftrightarrow \mathrm{x}=\mathrm{y}$,

2) $\rho(x,y)=\rho(y,x)$,

3) $\rho(x,z)\le \rho(x,y)+\rho(y,z)$

Bu xossalarni isbotlaylik. (12.3) munosabatdan $\rho(x,y)$ miqdorning har doim manfiy emasligini ko'ramiz. Agar $\rho(x,y)=0$ bo'lsa, unda $y_1-x_1=0$, $y_2-x_2=0,...y_m-x_m=0$ bo'lib, $x_1=y_1$, $\mathrm{x}_2=y_2,...,x_m=y_m$ ya'ni $x=y$ bo'ladi. Aksincha $x=y$, ya'ni $x_1=y_1$, $\mathrm{x}_2=y_2,...,x_m=y_m$ bo'lsa, u holda (12.3) dan $\rho(x,y)=0$ bo'lishi kelib chiqadi. Bu esa 1)-xossani isbotlaydi.

(12.3) munosabatdan

$$\rho(x,y)=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+...+(y_m-x_m)^2}=$$
$$=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+...+(x_m-y_m)^2}=\rho(y,x)$$

bo'ladi.

Masofaning 3)-xossasi ushbu

$$\sqrt{\sum_{i=1}^{m}(a_i+b_i)^2}\le\sqrt{\sum_{i=1}^{m}a_i^2}+\sqrt{\sum_{i=1}^{m}b_i^2} \qquad (12.4)$$

tengsizlikka asoslanib isbotlanadi, $a_1,a_2,...,a_m;\ b_1,b_2,...,b_m$ ixtiyoriy haqiqiy sonlar. Avvalo shu tengsizlikning to'g'riligini ko'rsataylik. Ravshanki, $\forall x\in R$ uchun

$$\sum_{i=1}^{m}(a_ix+b_i)^2\ge 0.$$

Bundan, x ga nisbatan kvadrat uchxadning manfiy emasligi

$$\left(\sum_{i=1}^{m}a_i^2\right)x^2+\left(2\sum_{i=1}^{m}a_ib_i\right)x+\sum_{i=1}^{m}b_i^2\ge 0$$

kelib chiqadi. Demak, bu kvadrat uchxad ikkita turli haqiqiy ildizga ega bo'lmaydi. Binobarin, uning diskriminanti

$$-\sum_{i=1}^{m}a_i^2\ \sum_{i=1}^{m}b_i^2+\left[\sum_{i=1}^{m}a_ib_i\right]^2\le 0$$

bo'lishi kerak. Bundan esa

$$\sum_{i=1}^{m}a_ib_i\le\sqrt{\sum_{i=1}^{m}a_i^2}\ \sqrt{\sum_{i=1}^{m}b_i^2}$$

bo'lib,

$$\sum_{i=1}^{m} a_i^2 + \sum_{i=1}^{m} b_i^2 + 2\sum_{i=1}^{m} a_i b_i \le \left[\sqrt{\sum_{i=1}^{m} a_i^2}\right]^2 + \left[\sqrt{\sum_{i=1}^{m} b_i^2}\right]^2 + 2\sqrt{\sum_{i=1}^{m} a_i^2}\cdot\sqrt{\sum_{i=1}^{m} b_i^2}$$

bo'ladi. Keyingi tengsizlikdan esa

$$\sqrt{\sum_{i=1}^{m}(a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{m} a_i^2} + \sqrt{\sum_{i=1}^{m} b_i^2}$$

bo'lishi kelib chiqadi. Odatda (12.4) tengsizlik Koshi-Bunyakovskiy tengsizlik deb ataladi.

Ixtiyoriy $x=(x_1,x_2,...,x_m)\in R^m$, $y=(y_1,y_2,...,y_m)\in R^m$, $z=(z_1,z_2,...,z_m)\in R^m$ nuqtalarni olib, ular orasidagi masofani (12.3) formuladan foydalanib topamiz:

$$\rho(x,y)=\sqrt{\sum_{i=1}^{m}(y_i - x_i)^2},$$

$$\rho(y,z)=\sqrt{\sum_{i=1}^{m}(z_i - y_i)^2}, \qquad (12.5)$$

$$\rho(x,z)=\sqrt{\sum_{i=1}^{m}(z_i - x_i)^2}.$$

Endi Koshi-Bunyakovskiy tengsizligi (12.4) da

$$a_i = y_i - x_i, \quad b_i = z_i - y_i \qquad (i=1,2,...,m)$$

deb olsak, unda

$$a_i + b_i = z_i - x_i \qquad (i=1,2,...,m)$$

bo'lib,

$$\sqrt{\sum_{i=1}^{m}(z_i - x_i)^2} \le \sqrt{\sum_{i=1}^{m}(y_i - x_i)^2} + \sqrt{\sum_{i=1}^{m}(z_i - x_i)^2}$$

bo'ladi. Yuqoridagi (12.5) munosabatlarni e'tiborga olib, topamiz:

$$\rho(x,z)\le \rho(x,y)+\rho(y,z).$$

Bu esa 3)-xossani isbotlaydi.

R^m to'plam R^m fazo (m o'lchovli Evklid fazosi) deb ataladi. Endi R^m fazoning ba'zi bir muhim to'plamlarini keltiramiz.

Biror $a=(a_1,a_2,...,a_m)\in R^m$ nuqta va $r>0$ sonni olaylik. Quyidagi

$$\left\{x=(x_1,x_2,...,x_m)\in R^m : (x_1-a_1)^2+(x_2-a_2)^2+...+(x_m-a_m)^2 \le r^2\right\}, \qquad (12.6)$$

$$\left\{x=(x_1,x_2,...,x_m)\in R^m : (x_1-a_1)^2+(x_2-a_2)^2+...+(x_m-a_m)^2 < r^2\right\}, \qquad (12.7)$$

ya'ni

$$\left\{x\in R^m : \rho(x,a)\le r\right\},$$

$$\left\{x\in R^m : \rho(x,a)< r\right\}$$

to'plamlar mos ravishda shar hamda ochiq shar deb ataladi. Bunda a nuqta shar markazi, r esa shar radiusi deyiladi.

Ushbu

$$\left\{x=(x_1,x_2,...,x_m)\in R^m : (x_1-a_1)^2+(x_2-a_2)^2+...+(x_m-a_m)^2 = r^2\right\}$$

ya'ni

$$\left\{x\in R^m : \rho(x,a)= r\right\}$$

to'plam sfera deb ataladi. Bu sfera (12.6) va (12.7) to'plamlarning chegarasi bo'ladi.

Ushbu

$$\{x=(x_1,x_2,...,x_m)\in R^m : a_1 \le x_1 \le b_1,\ a_2 \le x_2 \le b_2,..,a_m \le x_m \le b_m,\},$$
$$\{x=(x_1,x_2,...,x_m)\in R^m : a_1 < x_1 < b_1,\ a_2 < x_2 < b_2,..,a_m < x_m < b_m,\}$$

to'plamlar (bunda $a_1,a_2,...,a_m;\ b_1,b_2,...,b_m$ haqiqiy sonlar) mos ravishda parallelepiped hamda ochiq parallelepiped deb ataladi.

Ushbu

$$\{x=(x_1,x_2,...,x_m)\in R^m:\ x_1 \ge 0,\ x_2 \ge 0,...,x_m \ge 0,\ \ x_1+x_2+...+x_m \le h\}$$

to'plam (m o'lchovli) simpleks deb ataladi, bunda h- musbat son.

Yuqorida keltirilgan to'plamlar tez-tez ishlatilib turiladi. Ular yordamida muhim tushunchalar, jumladan atrof tushunchasi ta'riflanadi.

2^0. R^m fazoda ochiq va yopiq to'plamlar. Biror $x^0=(x_1^0,x_2^0,...,x_m^0)\in R^m$ nuqta hamda $\varepsilon > 0$ sonni olaylik.

2-ta'rif. Markazi x^0 nuqtada, radiusi $\varepsilon > 0$ga teng bo'lgan ochiq shar x^0 nuqtaning sferik atrofi (ε atrofi) deyiladi va $U_\varepsilon(x^0)$ kabi belgilanadi:

$$U_\varepsilon(x^0)=\{x\in R^m : \rho(x,x^0)<\varepsilon\}.$$

Nuqtaning boshqacha atrofi tushunchasini ham kiritishimiz mumkin.

3-ta'rif. Ushbu

$$\{x=(x_1,x_2,...,x_m)\in R^m:\ x_1^0-\delta_1 < x_1 < x_1^0+\delta_1,\ ,x_2^0-\delta_2 < x_2 < x_2^0+\delta_2,\ ....,\ x_m^0-\delta_m < x_m < x_m^0+\delta_m\} \qquad (12.8)$$

ochiq parallelepiped x^0 nuqtaning parallelepipedial atrofi deb ataladi va $\tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)$ kabi belgilanadi.

Xususan $\delta_1=\delta_2=...=\delta_m=\delta$ bo'lsa, (12.8) ochiq parallelepiped kubga aylanadi va uni $\tilde{U}_\delta(x^0)$ kabi belgilanadi.

1-lemma. $x^0\in R^m$ nuqtaning har qanday $U_\varepsilon(x^0)$ sferik atrofi olinganda ham har doim x^0 nuqtaning shunday $\tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)$ parallelepipedial atrofi mavjudki, bunda

$$\tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)\subset U_\varepsilon(x^0)$$

bo'ladi.

Shuningdek, x^0 nuqtaning har qanday $\tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)$ parallelepipedial atrofi olinganda ham har doim shu nuqtaning shunday $U_\varepsilon(x^0)$ sferik atrofi mavjudki, bunda

$$U_\varepsilon(x^0)\subset \tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)$$

bo'ladi.

◄ $x^0\in R^m$ nuqtaning sferik atrofi

$$U_\varepsilon(x^0)=\{x\in R^m : \rho(x,x^0)<\varepsilon\}$$

berilgan bo'lsin. Bundagi $\varepsilon>0$ songa ko'ra $\delta<\frac{\varepsilon}{\sqrt{m}}$ tengsizlikni qanoatlantiruvchi $\delta>0$ sonni olamiz. So'ng x^0 nuqtaning ushbu

$$\tilde{U}_\varepsilon(x^0)=\{\, x=(x_1,x_2,...,x_m)\in R^m : x_1^0-\delta<x_1<x_1^0+\delta,$$
$$,x_2^0-\delta<x_2<x_2^0+\delta,...,x_m^0-\delta<x_m<x_m^0+\delta\,\}$$

parallelepipedial atrofni tuzamiz.

Aytaylik, $x\in\tilde{U}_\varepsilon(x^0)$ bo'lsin. Unda $|x_i-x_i^0|<\delta$ $(\mathrm{i}=1,2,...,\mathrm{m})$ bo'lib,

$$\sqrt{\sum_{i=1}^{m}(x_i-x_i^0)^2}<\sqrt{\sum_{i=1}^{m}\delta^2}=\delta\sqrt{m}$$

bo'ladi. Yuqoridagi $\delta<\frac{\varepsilon}{\sqrt{m}}$ tengsizlikni e'tiborga olib topamiz:

$$\rho(x,x^0)=\sqrt{\sum_{i=1}^{m}(x_i-x_i^0)^2}<\varepsilon$$

Demak, $\rho(x,x^0)<\varepsilon$. Bu esa $x\in U_\varepsilon(x^0)$ ekanini bildiradi. Shunday qilib,

$$\forall x\in\tilde{U}_\delta(x^0)\ \Rightarrow\ x\in U_\varepsilon(x^0)$$

ya'ni

$$\tilde{U}_\delta(x^0)\subset U_\varepsilon(x^0)$$

bo'ladi.

$x^0\in R^m$ nuqtaning

$$\tilde{U}_{\delta_1\delta_2...\delta_m}(x^0)=\{\, x=(x_1,x_2,...,x_m)\in R^m : x_1-\delta_1<x_1<x_1^0+\delta_1,$$
$$,x_2^0-\delta_2<x_2<x_2^0+\delta_2,...,x_m^0-\delta_m<x_m<x_m^0+\delta_m\,\}$$

parallelepiped atrofi berilgan bo'lsin. Unda

$$\varepsilon=min\{\,\delta_1,\delta_2,...,\delta_m\,\}$$

ni olib x^0 nuqtaning sferik atrofi

$$U_\varepsilon(x^0)=\{x\in R^m : \rho(x,x^0)<\varepsilon\}$$

ni tuzamiz.

Aytaylik $x\in U_\varepsilon(x^0)$ bo'lsin. U holda

$$\rho(x,x^0)=\sqrt{\sum_{i=1}^{m}(x_i-x_i^0)^2}<\varepsilon\le\delta_i \qquad (i=1,2,...,m)$$

bo'ladi. Demak,

$$|x_i-x_i^0|\le\sqrt{\sum_{i=1}^{m}(x_i-x_i^0)^2}<\delta_i \qquad (i=1,2,...,m).$$

Bundan esa $x \in \tilde{U}_{\delta_1 \delta_2 \dots \delta_m}(x^0)$ bo'lishi kelib chiqadi. Shunday qilib,

$$\forall x \in U_\varepsilon(x^0) \Rightarrow x \in \tilde{U}_{\delta_1 \delta_2 \dots \delta_m}(x^0)$$

ya'ni

$$U_\varepsilon(x^0) \subset \tilde{U}_{\delta_1 \delta_2 \dots \delta_m}(x^0)$$

bo'ladi. ►

Aytaylik, R^m fazo va G to'plam berilgan bo'lsin: $G \subset R^m$

4-ta'rif. Agar $x^0 \in G$ nuqtaning shunday $U_\delta(x^0)$ atrofi $(\delta > 0)$ topilsaki,

$$U_\delta(x^0) \subset G$$

bo'lsa, x^0 nuqta G to'plamning ichki nuqtasi deyiladi.

5-ta'rif. To'plamning har bir nuqtasi uning ichki nuqtasi bo'lsa, u ochiq to'plam deb ataladi.

12.1-misol. R^m fazodagi ochiq shar ochiq to'plam bo'lishi isbotlansin.

◄ Aytaylik,

$$A = \{ x \in R^m : \ \rho(x, x^0) < r \}$$

R^m fazodagi biror ochiq shar bo'lsin. Ravshanki, $z \in A \Rightarrow \rho(z, x) < r$.

Ushbu $r_1 = r - \rho(z, x^0)$ sonni olib, quyidagi

$$B = \{ x \in R^m : \ \rho(x, x^0) < r_1 \}$$

sharni qaraymiz. $\forall y \in B$ uchun,

$$\rho(y, x^0) \le \rho(y, z) + \rho(z, x^0) < r_1 + \rho(z, x^0)$$

munosabatga ko'ra

$$B \subset A$$

bo'ladi. ►

R^m fazoda biror F to'plam va biror x^0 nuqta berilgan bo'lsin:

6-ta'rif. Agar

$$\forall r > 0, \ \exists x \in F, x \ne x^0 : \ x \in \{x \in R^m : \rho(x, x^0) < r\}$$

bo'lsa, x^0 nuqta F to'plamning limit nuqtasi deyiladi.

Masalan, ushbu

$$A = \{ x \in R^m : \ \rho(x, x^0) \le r \}$$

sharning barcha nuqtalari uning limit nuqtalari bo'ladi.

7-ta'rif. $F \subset R^m$ to'plamning barcha limit nuqtalari shu to'plamga tegishli bo'lsa, F yopiq to'plam deb ataladi.

Masalan,

$$E = \{ x \in R^m : \ \rho(x, x^0) \le r \}$$

yopiq to'plam bo'ladi.

Shuni ta'kidlash lozimki, ochiq va yopiq to'plamlar ta'riflarini qanoatlantirmaydigan to'plamlar ham ko'pdir.

Biror $M \subset R^m$ to'plamni qaraylik. Ravshanki, $R^m \setminus M$ ayirma M to'plamni R^m to'plamga to'ldiruvchi to'plam bo'ladi. (qaralsin I-qism, I-bob, I-§).

8-ta'rif. Agar $x^0\left(x^0 \in R^m\right)$ nuqtaning istalgan $U_\varepsilon\left(x^0\right)$ atrofida ham M to'plamning, ham $R^m \setminus M$ to'plamning nuqtalari bo'lsa, x^0 nuqta M to'plamning chegaraviy nuqtasi deb ataladi. M to'plamning barcha chegaraviy nuqtalaridan iborat to'plam M to'plamning chegarasi deyiladi va uni odatda $\partial(M)$ kabi belgilanadi.

Bu tushuncha yordamida yopiq to'plamni quyidagicha ham ta'riflash mumkin.

9-ta'rif. Agar $F\left(F \subset R^m\right)$ to'plamning chegarasi shu to'plamga tegishli, ya'ni $\partial(F) \subset F$ bo'lsa, F yopiq to'plam deb ataladi.

Yopiq to'plamning yuqorida keltirilgan 12.7- va 12.9- ta'riflari ekvivalent ta'riflardir.

Biror $M \subset R^m$ to'plam berilgan bo'lsin.

10-ta'rif. Agar R^m fazoda shunday shar

$$U^0 = \left\{x \in R^m : \rho(x, \mathbf{0}) < r\right\} \quad (\mathbf{0} = (0, 0, ..., 0))$$

topilsaki, $M \subset U^0$ bo'lsa, M chegaralangan to'plam deb ataladi.

Faraz qilaylik, $x_1(t), x_2(t), ..., x_m(t)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz bo'lsin.

Ushbu

$$\{x_1(t), x_2(t), ..., x_m(t)\} \qquad (a \le t \le b) \tag{12.9}$$

sistema yoki nuqtalar to'plami R^m fazoda egri chiziq deb ataladi. Xususan,

$$x_1 = \alpha_1 t + \beta_1,\ x_2 = \alpha_2 t + \beta_2, ...,\ x_m = \alpha_m t + \beta_m$$

$(-\infty < t < +\infty,\ \alpha_1, \alpha_2, ..., \alpha_m;\ \beta_1, \beta_2, ..., \beta_m$ haqiqiy sonlar va $\alpha_1, \alpha_2, ... \alpha_m$ larning xech bo'lmaganda bittasi nolga teng emas) bo'lganda (12.9) sistema R^m fazoda to'g'ri chiziq deyiladi.

R^m fazoda ixtiyoriy ikkita $x' = (x_1', x_2', ..., x_m')$ va $x'' = (x_1'', x_2'', ..., x_m'')$ nuqtani olaylik. Bu nuqtalar orqali o'tuvchi to'g'ri chiziq quyidagi

$$\{(x_1' + t(x_1'' - x_1'),\ x_2' + t(x_2'' - x_2'),\ ..., x_m' + t(x_m'' - x_m'))\} \quad (-\infty < t < +\infty) \tag{12.10}$$

sistemasi bilan ifodalanadi. Bunda $t = 0$ va $t = 1$ bo'lganda R^m fazoning mos ravishda x' va x'' nuqtalari hosil bo'lib, $0 \le t \le 1$ bo'lganda (12.10) sistema R^m fazoda x' va x'' nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi bo'ladi.

R^m fazoda chekli sondagi to'g'ri chiziq kesmalarni birin-ketin birlashtirishdan tashkil topgan chiziq, siniq chiziq deb ataladi.

$M \subset R^m$ to'plam berilgan bo'lsin.

11-ta'rif. Agar M to'plamning ixtiyoriy ikki nuqtasini birlashtiruvchi shunday siniq chiziq topilsaki, u M to'plamga tegishli bo'lsa, M bog'lamli to'plam deb ataladi.

12-ta'rif. Agar $M \subset R^m$ to'plam ochiq hamda bog'lamli to'plam bo'lsa, u soha deb ataladi.

R^m fazoda ochiq parallelepiped, ochiq shar, ochiq simplekslar fazodagi sohalar bo'ladi.

2-§. R^m fazoda ketma-ketlik va uning limiti

Natural sonlar to'plami N va R^m fazo berilgan bo'lib, f har bir $n(n \in N)$ ga R^m fazoning biror muayyan $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}) \in R^m$ nuqtasini mos qo'yuvchi akslantirish bo'lsin:

$$f : N \to R^m \text{ yoki } n \to x^n = \left(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\right).$$

Bu akslantirishni quyidagicha tasvirlash mumkin:

$$1 \to x^{(1)} = (x_1^{(1)}, x_2^{(1)}, ..., x_m^{(1)}),$$
$$2 \to x^{(2)} = (x_1^{(2)}, x_2^{(2)}, ..., x_m^{(2)}),$$
$$\dots\dots\dots\dots\dots\dots$$
$$n \to x^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}),$$
$$\dots\dots\dots\dots\dots\dots$$

$f : N \to R^m$ akslantirishning tasvirlari (obrazlari) dan tuzilgan

$$x^{(1)}, x^{(2)}, ..., x^{(n)}, ...$$

to'plam ketma-ketlik deb ataladi va u $\{x^{(n)}\}$ kabi beligilanadi. Har bir $x^{(n)} \in R^m \quad (n = 1,2,...)$ ni ketma-ketlikning hadi deyiladi.

Shuni ta'kidlash kerakki, $\{x^{(n)}\}$ ketma-ketlikning mos koordinatalardan tuzilgan $\{x_1^{(n)}\}$, $\{x_2^{(n)}\}$, ..., $\{x_m^{(n)}\}$ lar sonli ketma-ketliklar bo'lib, $\{x^{(n)}\}$ ketma-ketlikni shu m ta ketma-ketlikning (ma'lum tartibdagi) birlikda qaralishi deb hisoblash mumkin.

1^0. Ketma-ketlikning limiti.

R^m fazoda biror

$$x^{(1)}, x^{(2)}, ..., x^{(n)}, ... \qquad (12.11)$$

ketma-ketlik hamda $a = (a_1, a_2, ..., a_m) \in R^m$ nuqta berilgan bo'lsin.

13-ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $n_0 \in N$ topilsaki, barcha $n > n_0$ uchun

$$\rho(x^n, a) < \varepsilon$$

tengsizlik bajarilsa, a nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deb ataladi va $\lim\limits_{n \to \infty} x^n = a$ yoki $n \to \infty$ da $x^n \to a$ kabi belgilanadi.

1-§da keltirilgan a nuqtaning ε-atrofi ta'rifini e'tiborga olib, $\{x^{(n)}\}$ ketma-ketlikning limitini quyidagicha ham ta'riflasa bo'ladi.

14-ta'rif. Agar a nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olinganda ham, $\{x^{(n)}\}$ ketma-ketlikning biror hadidan boshlab, keyingi barcha hadlari shu atrofga tegishli bo'lsa, a nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deb ataladi.

Agar (12.11) ketma-ketlik limitga ega bo'lsa, u yaqinlashtiruvchi ketma-ketlik deb ataladi.

Limit ta'rifidagi shartni qanoatlantiruvchi a mavjud bo'lmasa, $\{x^{(n)}\}$ ketma-ketlikning limitga ega emas deyiladi, ketma-ketlikning o'zi esa uzoqlashtiruvchi deb ataladi.

Shunga e'tibor berish kerakki, ketma-ketlikning limiti ta'rifidagi ε ixtiyoriy musbat son bo'lib, $n_0 (n_0 \in N)$ esa ε ga (va, tabiiyki, qaralayotgan ketma-ketlikka) bog'liq ravishda topiladi.

12.2-misol. R^m fazoda ushbu $\{x^n\} = \left\{\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)\right\}$ ketma-ketlikning limiti $a = (0,0,...,0)$ bo'lishi ko'rsatilsin.

◀ $\forall \varepsilon > 0$ sonni olaylik. Shu ε ga ko'ra $n_0 = \left[\frac{\sqrt{m}}{\varepsilon}\right] + 1$ ni topamiz. Natijada barcha $n > n_0$ uchun

$$\rho(x^n, a) = \rho\left(\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right), (0,0,...,0)\right) = \frac{\sqrt{m}}{n} < \frac{\sqrt{m}}{n_0} = \frac{\sqrt{m}}{\left[\frac{\sqrt{m}}{\varepsilon}\right] + 1} < \varepsilon$$

bo'ladi. Demak, ta'rifga ko'ra,

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty}\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right) = (0,0,...,0) = a$$

bo'ladi.▶

R^m fazoda $\{x^{(n)}\} = \{x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\}$ ketma-ketlik berilgan bo'lib, u $a = (a_1, a_2, ..., a_m)$ limitga ega bo'lsin. U holda limit ta'rifiga ko'ra, $\forall \varepsilon > 0$ berilganda ham, $\{x^{(n)}\}$ ketma-ketlikning biror n_0 hadidan boshlab keyingi hadlari a nuqtaning

$$U_\varepsilon(a) = \{x \in R^m : \rho(x, a) < \varepsilon\}$$

sferik atrofiga tegishli bo'ladi. Bu sferik atrof ushbu bobning 1-§dagi 1-lemmaga muvofiq shu a nuqtaning $\tilde{U}_\varepsilon(a)$ parallelepipedial atrofining qismi bo'ladi:

$$U_\varepsilon(a) \subset \tilde{U}_\varepsilon(a)$$

Demak, $\{x^{(n)}\}$ ketma-ketlikning o'sha n_0 hadidan boshlab, keyingi barcha hadlari a nuqtaning $\tilde{U}_\varepsilon(a)$ atrofida yotadi, ya'ni barcha $n > n_0$ uchun

$$x^{(n)} \in \tilde{U}_\varepsilon(a) = \{(x_1, x_2, ..., x_m) \in R^m : a_1 - \varepsilon < x_1 < a_1 + \varepsilon,$$
$$, a_2 - \varepsilon < x_2 < a_2 + \varepsilon, ..., a_m - \varepsilon < x_m < a_m + \varepsilon\}$$

bo'ladi. Bundan esa, barcha $n > n_0$ uchun

$$a_1 - \varepsilon < x_1^{(n)} < a_1 + \varepsilon,$$
$$a_2 - \varepsilon < x_2^{(n)} < a_2 + \varepsilon,$$
$$.................................,$$
$$a_m - \varepsilon < x_m^{(n)} < a_m + \varepsilon$$

bo'lishi kelib chiqadi. Demak, $\forall \varepsilon > 0$ olinganda ham, shunday $n > n_0$ topiladiki, barcha $n > n_0$ uchun

$$\left|x_1^{(n)} - a_1\right| < \varepsilon,\ \left|x_2^{(n)} - a_2\right| < \varepsilon,\ ...,\left|x_m^{(n)} - a_m\right| < \varepsilon$$

bo'ladi. Bu esa

$$\lim_{n\to\infty} x_1^{(n)} = a_1$$
$$\lim_{n\to\infty} x_2^{(n)} = a_2$$
$$....................$$
$$\lim_{n\to\infty} x_m^{(n)} = a_m$$

ekanligini bildiradi.

Shunday qilib, R^m fazoda $\left\{x^{(n)}\right\} = \left\{x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\right\}$ ketma-ketlikning limiti $a = (a_1, a_2, ..., a_m)$ bo'lsa, u holda bu ketma-ketlikning koordinatalaridan tashkil topgan sonlar ketma-ketliklari $\left\{x_1^{(n)}\right\}, \left\{x_2^{(n)}\right\}, ..., \left\{x_m^{(n)}\right\}$ ham limitga ega bo'lib, ular mos ravishda a nuqtaning $a_1, a_2, ..., a_m$ koordinatalariga teng bo'ladi.

Demak,

$$\lim_{n\to\infty} x^{(n)} = a \Rightarrow \begin{cases} \lim\limits_{n\to\infty} x_1^{(n)} = a_1 \\ \lim\limits_{n\to\infty} x_2^{(n)} = a_2 \\ .... \\ \lim\limits_{n\to\infty} x_m^{(n)} = a_m \end{cases} \qquad (12.12)$$

Endi R^m fazoda ketma-ketlikning koordinatalaridan tashkil topgan $\left\{x_1^{(n)}\right\}, \left\{x_2^{(n)}\right\}, ..., \left\{x_m^{(n)}\right\}$ sonlar ketma-ketliklari limitga ega bo'lib, ularning limitlari mos ravishda $a = (a_1, a_2, ..., a_m) \in R^m$ nuqta koordinatalari $a_1, a_2, ..., a_m$ larga teng bo'lsin:

$$\lim_{n\to\infty} x_1^{(n)} = a_1$$
$$\lim_{n\to\infty} x_2^{(n)} = a_2$$
$$....................$$
$$\lim_{n\to\infty} x_m^{(n)} = a_m.$$

Limit ta'rifiga asosan, $\forall\varepsilon > 0$ olinganda ham, $\frac{\varepsilon}{\sqrt{m}}$ ga ko'ra shunday $n_0^{(1)} \in N$ topiladiki, barcha $n > n_0^{(1)}$ uchun

$$\left|x_1^{(n)} - a_1\right| < \frac{\varepsilon}{\sqrt{m}},$$

shunday $n_0^{(2)} \in N$ topiladiki, barcha $n > n_0^{(2)}$ uchun

$$\left|x_2^{(n)} - a_2\right| < \frac{\varepsilon}{\sqrt{m}}$$

va xokazo, shunday $n_0^{(m)} \in N$ topiladiki, barcha $n > n_0^{(m)}$ uchun

$$\left|x_m^{(n)} - a_m\right| < \frac{\varepsilon}{\sqrt{m}}$$

bo'ladi. Agar $n_0 = max\left\{n_0^{(1)}, n_0^{(2)}, ..., n_0^{(m)}\right\}$ deb olsak, unda barcha $n > n_0$ uchun bir yo'la

$$\left|x_i^{(n)} - a_i\right| < \frac{\varepsilon}{\sqrt{m}} \qquad (i = 1,\ 2,\ ...,\ m)$$

tengsizliklar bajariladi. U holda

$$\sqrt{\sum_{i=1}^{m}\left(x_i^{(n)} - a_i\right)^2} < \sqrt{\sum_{i=1}^{m}\left(\frac{\varepsilon}{\sqrt{m}}\right)^2} = \varepsilon$$

bo'lib, undan

$$\rho\left(x^{(n)}, a\right) < \varepsilon$$

bo'lishi kelib chiqadi. Bu esa

$$\lim_{n\to\infty} x^{(n)} = a$$

ekanini bildiradi.

Demak, $\left\{x^{(n)}\right\} = \left\{x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}\right\}$ ketma-ketlik koordinatalaridan tashkil topgan $\left\{x_1^{(n)}\right\}, \left\{x_2^{(n)}\right\}, ..., \left\{x_m^{(n)}\right\}$ sonlar ketma-ketliklarining limitlari mos ravishda $a = (a_1, a_2, ..., a_m)$ nuqta koordinatalari $a_1, a_2, ..., a_m$ larga teng bo'lsa, $\left\{x^{(n)}\right\}$ ketma-ketlikning limiti yuqoridagi ta'rif ma'nosida shu a nuqta bo'ladi:

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} x_1^{(n)} = a_1 \\ \lim\limits_{n\to\infty} x_2^{(n)} = a_2 \\ \dots\dots\dots\dots \\ \lim\limits_{n\to\infty} x_m^{(n)} = a_m \end{array}\right\} \Rightarrow \lim_{n\to\infty} x^{(n)} = a. \qquad (12.13)$$

Yuqoridagi *(12.12)* va *(12.13)* munosabatlardan

$$\lim_{n\to\infty} x^{(n)} = a \Leftrightarrow \begin{cases} \lim\limits_{n\to\infty} x_1^{(n)} = a_1 \\ \lim\limits_{n\to\infty} x_2^{(n)} = a_2 \\ \dots\dots\dots\dots\dots\dots \\ \lim\limits_{n\to\infty} x_m^{(n)} = a_m \end{cases}$$

ekanligi kelib chiqadi.

Shunday qilib, quyidagi muhim teoremaga kelamiz:

1-teorema. R^m fazoda $\{x^{(n)}\} = \{x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}\}$ ketma-ketlikning $a = (a_1, a_2, \dots, a_m) \in R^m$ ga intilishi

$$x^{(n)} \to a \ \ (n \to \infty \text{ da})$$

uchun $n \to \infty$ bir yo'la

$$x_1^{(n)} \to a_1,$$
$$x_2^{(n)} \to a_2,$$
$$\dots\dots\dots\dots$$
$$x_m^{(n)} \to a_m$$

bo'lishi zarur va etarli.

Bu teoremada R^m fazoda ketma-ketlikning limitini o'rganishni sonli ketma-ketliklarning limitini o'rganishga keltirilishini ifodalaydi. Ma'lumki, «Matematik analiz asoslari» kursining 1-qism, 3-bobida ketma-ketligi va uning limiti batafsil o'rganilgan.

2^0. Ketma-ketlik limitiga doir ba'zi tasdiqlar

Sonlar ketma-ketligi limiti haqidagi ma'lumotlar (tushuncha va tasdiqlar) R^m fazo nuqtalaridan iborat ketma-ketliklarda ham o'rinli bo'ladi. Quyida biz ularni keltirish bilangina kifoyalanamiz (keltirilgan tasdiqlarni isbotlashni o'quvchiga havola etamiz).

1) R^m fazoda $\{x^{(n)}\} = \{(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})\}$ ketma-ketlikning chegaralangan bo'lishi uchun bu ketma-ketlik koordinatalardan iborat $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ sonlar ketma-ketliklarining har birining chegaralangan bo'lishi zarur va etarli.

2) R^m fazoda $\{x^{(n)}\}$ ketma-ketlik uchun $\forall \varepsilon > 0$ olinganda ham shunday $n_0 \in N$ topilsaki, barcha $n > n_0$, $\rho > n_0$ da

$$\rho\left(x^{(p)}, x^{(n)}\right) < \varepsilon$$

tengsizlik bajarilsa, $\{x^{(n)}\}$ fundamental ketma-ketlik deyiladi.

R^m fazoda $\{x^{(n)}\} = \{(x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})\}$ ketma-ketlik fundamental bo'lishi uchun bu ketma-ketlik koordinatalaridan iborat $\{x_1^{(n)}\}, \{x_2^{(n)}\}, \dots, \{x_m^{(n)}\}$ ketma-ketliklarining har birining fundamental bo'lishi zarur va etarli.

R^m fazoda $\{x^{(n)}\}$ ketma-ketlikning yaqinlashuvchi bo'lishi uchun u fundamental bo'lishi zarur va etarli (Koshi teoremasi).

3) Markazlari $a^{(n)}=\left(a_1^{(n)},a_2^{(n)},...,a_m^{(n)}\right)\in R^m$ nuqtalarda, radiuslari r_n $(r_n>0,\ n=1,\ 2,\ ...)$ bo'lgan

$$S_n=S_n\left(a^{(n)},r_n\right)=\left\{x\in R^m:\rho\left(x,a^{(n)}\right)\le r_n\right\}\qquad (n=1,\ 2,\ 3,\ ...)$$

sharlar berilgan bo'lsin. Agar

$$S_1\supset S_2\supset...\supset S_n\supset...$$

munosabat o'rinli bo'lsa, $\{S_n\}$ ichma-ich joylashgan sharlar ketma-ketligi deyiladi.

Agar R^m fazoda ichma-ich joylashgan sharlar ketma-ketligi $\{S_n\}$ uchun

$$\lim_{n\to\infty} r_n=0$$

bo'lsa, u holda barcha sharlarga tegishli bo'lgan a $\left(a\in R^m\right)$ nuqta mavjud va yagonadir. (Ichma-ich joylashgan sharlar prinsipi).

4) R^m fazoda $\left\{x^{(n)}\right\}$:

$$x^{(1)},\ x^{(2)},\ ...,\ x^{(n)},...\quad \left(x^{(n)}\in R^m,\qquad n=1,\ 2,\ ...\right)$$

ketma-ketlik berilgan bo'lsin. Bu ketma-ketlikning

$$n_1,n_2,...,n_k,...\quad (n_1<n_2<...<n_k<...,\ n_k\in N,\ k=1,\ 2,\ ...)$$

nomerli hadlardan tashkil topgan ushbu

$$x^{(n_1)},x^{(n_2)},...,x^{(n_k)},...\quad \left(x^{(n_k)}\in R^m\right)$$

ketma-ketlik $\left\{x^{(n)}\right\}$ ketma-ketlikning qismiy ketma-ketligi deyiladi va $\left\{x^{(n_k)}\right\}$ kabi belgilanadi.

Agar $\left\{x^{(n)}\right\}$ ketma-ketlik yaqinlashuvchi bo'lib, uning limiti a $\left(a\in R^m\right)$ bo'lsa, bu ketma-ketlikning har qanday qismiy $\left\{x^{(n_k)}\right\}$ ketma-ketligi ham yaqinlashuvchi bo'lib, uning limiti ham a ga teng bo'ladi.

Har qanday chegaralangan $\left\{x^{(n)}\right\}$ ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin. (Baltsano-Veyershtrass teoremasi).

3-§. Ko'p o'zgaruvchili funksiya va uning limiti

1⁰. Funksiya. Biror $M\left(M\subset R^m\right)$ to'plam berilgan bo'lsin.

15-ta'rif. Agar M to'plamdagi har bir $x=(x_1,\ x_2,\ ...,\ x_m)$ nuqtaga biror qoida yoki qonunga ko'ra bitta haqiqiy son $y\,(y\in R)$ mos qo'yilgan bo'lsa, M to'plamda ko'p o'zgaruvchili (m ta o'zgaruvchili) funksiya berilgan (aniqlangan) deb ataladi va uni

$$f:(x_1,x_2,...,x_m)\to y \text{ yoki } y=f(x_1,x_2,...,x_m) \qquad \textit{(12.14)}$$

kabi belgilanadi. Bunda M-funksiyaning berilishi (aniqlanish) to'plami, $x_1,x_2,...,x_m$ erkli o'zgaruvchilar – funksiya argumentlari, y erksiz o'zgaruvchi - $x_1,x_2,...,x_m$ o'zgaruvchilarning funksiyasi deyiladi.

$(x_1,x_2,...,x_m)$ nuqta bitta x bilan belgilanishini e'tiborga olib, bundan keyin deyarli hamma vaqt $(x_1,x_2,...,x_m)$ o'rniga x ni ishlataveramiz. Unda yuqoridagi *(12.14)* belgilashlar quyidagicha yoziladi.

$$f:x\to y \text{ yoki } y=f(x)\qquad \left(x\in R^m,y\in R\right).$$

Funksiyaning berilish to’plamidan olingan $x^0 \in M$ nuqtaga mos keluvchi y_0 son $y = f(x)$ funksiyaning $x = x^0$ nuqtadagi xususiy qiymati deb ataladi.

Masalan, $f - R^m$ fazodagi har bir x nuqtaga shu nuqta koordinatalari kvadratlarining yig’indisini mos qo’yuvchi qoida, ushbu

$$f{:}x \to x_1^2 + x_2^2 + \ldots + x_m^2, \qquad y = x_1^2 + x_2^2 + \ldots + x_m^2$$

funksiyani hosil qiladi. Bu funksiya $M = R^m$ to’plamda berilgan.

$f(x)$ funksiya $M \subset R^m$ to’plamda berilgan bo’lsin. Ushbu $\{f(x) : x \in M\}$ to’plam funksiya qiymatlari to’plami (funksiyaning o’zgarish sohasi) deb ataladi.

R^{m+1} fazoning (x, y) $(x \in R^m, y = f(x) \in R)$ nuqtalardan iborat ushbu

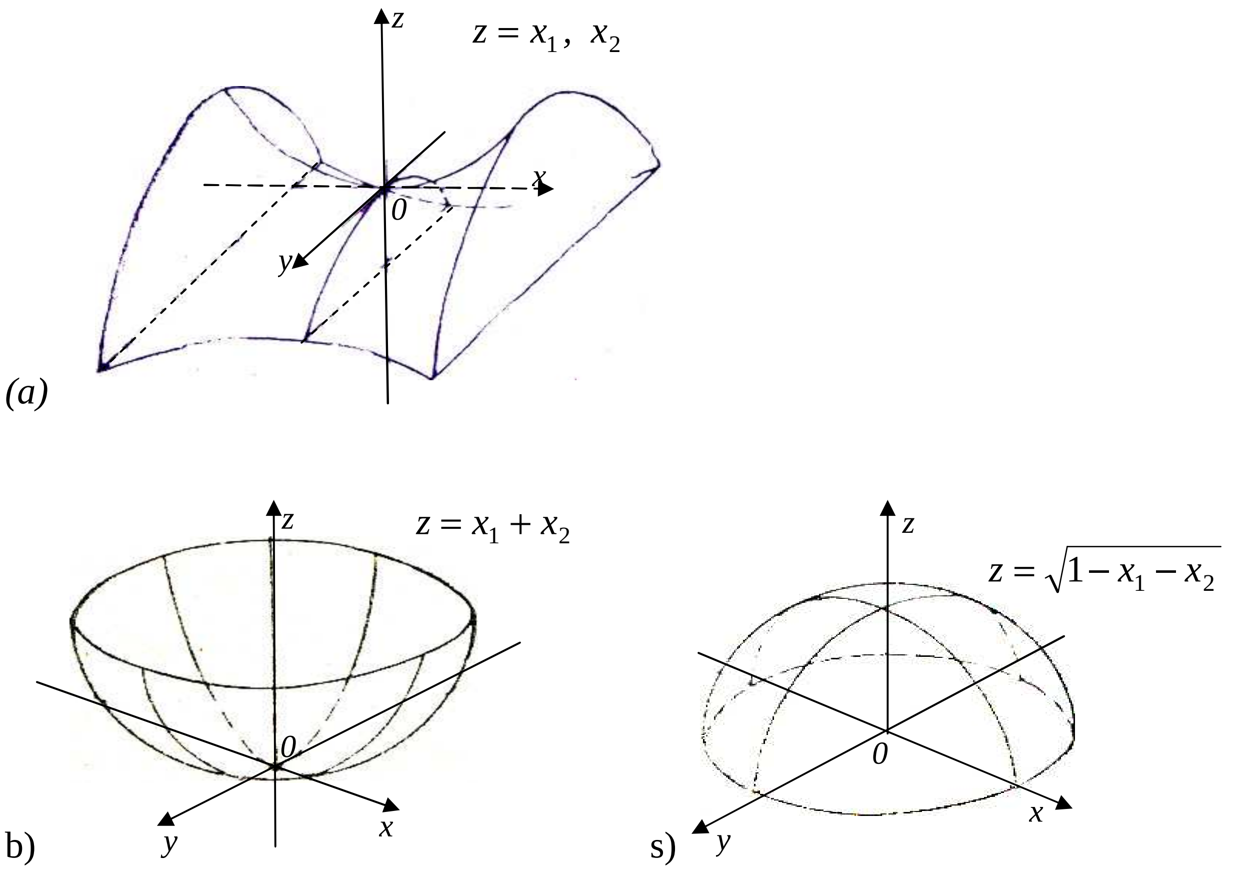
$$\{(x, f(x))\} = \{(x, f(x)) : x \in R^m, f(x) \in R\}$$

to’plam $y = f(x)$ funksiya grafigi deb ataladi.

Masalan, $m = 2$ bo’lganda (R^2 fazoda)

$$y = x_1, x_2, \quad y = x_1^2, x_2^2, \ \mathrm{y} = \sqrt{1 - \mathrm{x}_1^2 - x_2^2}$$

funksiyalar grafigi (47-chizma) mos ravishda R^3 fazoda giperbolik paraboloid (a), aylanma paraboloid (b) hamda yuqori yarim sferalar (s) dan iboratdir.



47-chizma

$M \subset R^m$ to'plamda $y = f(x) = f(x_1, x_2, ..., x_m)$ funksiya berilgan bo'lib, $x_1, x_2, ..., x_m$ larning har biri $T \subset R^k$ $(k \in N)$ to'plamda berilgan funksiyalar bo'lsin:

$$x_1 = \varphi_1(t) = \varphi_1(t_1, t_2, ..., t_k),$$
$$x_2 = \varphi_2(t) = \varphi_2(t_1, t_2, ..., t_k),$$
$$.......................................,$$
$$x_m = \varphi_m(t) = \varphi_m(t_1, t_2, ..., t_k)$$

Bunda $t = (t_1, t_2, ..., t_k)$ o'zgaruvchi $T \subset R^k$ to'plamda o'zgarganda ularga mos $x = (x_1, x_2, ..., x_m)$ nuqta $M \subset R^m$ to'plamga tegishli bo'lsin. Natijada y o'zgaruvchi $x = (x_1, x_2, ..., x_m)$ o'zgaruvchi orqali $t = (t_1, t_2, ..., t_k)$ o'zgaruvchilarning funksiyasi bo'ladi:

$$t \to x \to y,$$
$$((t_1, t_2 ... t_k) \to (x_1, x_2, ..., x_m) \to y),$$
$$y = f(x(t)) = f(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k)).$$

Bu murakkab funksiya yoki $f(x)$ hamda $\varphi_i(t)$ $(i = 1, 2, ..., m)$ funksiyalar superpozitsiyasi deb ataladi.

Elementar funksiyalar ustida qo'shish, ayirish, ko'paytirish va bo'lish amallari hamda funksiyalar superpozitsiyasi yordamida ko'p o'zgaruvchili funksiyalar hosil qilinadi. Ushbu

$$y = e^{x_1 x_2 ... x_m}, \quad y = ln\sqrt{x_1 + x_2 + ... + x_m},$$
$$y = sin(x_1 x_2) + sin(x_2 x_3) + ... + sin(x_{m-1} x_m)$$

funksiyalar shular jumlasidandir.

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya $M \subset R^m$ to'plam berilgan bo'lsin. Agar bu funksiya qiymatlari to'plami

$$Y = \{f(x_1, x_2, ..., x_m) : (x_1, x_2, ..., x_m) \in M\}$$

yuqoridan (quyidan) chegaralangan bo'lsa, ya'ni shunday o'zgarmas C (o'zgarmas P) son topilsaki, $\forall (x_1, x_2, ..., x_m) \in M$ uchun

$$f(x_1, x_2, ..., x_m) \le C \qquad (f(x_1, x_2, ..., x_m) \ge P)$$

tengsizlik o'rinli bo'lsa, $f(x) = f(x_1, x_2, ..., x_m)$ funksiya M to'plamda yuqoridan (quyidan) chegaralangan deb ataladi, aks holda, ya'ni har qanday katta musbat S son olinganda ham M to'plamda shunday $(x_1^0, x_2^0, ..., x_m^0)$ nuqta topilsaki,

$$f(x_1^0, x_2^0, ..., x_m^0) > S \qquad (f(x_1^0, x_2^0, ..., x_m^0) < -S)$$

tengsizlik o'rinli bo'lsa, $f(x) = f(x_1, x_2, ..., x_m)$ funksiya M to'plamda yuqoridan (quyidan) chegaralanmagan deb ataladi.

Agar $f(x) = f(x_1, x_2, .. x_m)$ funksiya M to'plamda ham yuqoridan, ham quyidan chegaralangan bo'lsa, funksiya shu to'plamda chegaralangan deyiladi.

12.3-misol. Ushbu

$$y=\frac{\sqrt{1-x_1^2-x_2^2-...-x_m^2}}{ln\left(x_1^2+x_2^2+...+x_m^2-1/4\right)}$$

funksiyaning aniqlanish to'plami topilsin.

◀ Qaralayotgan munosabat ma'noga ega bo'lishi uchun

$$1-x_1^2-x_2^2-...-x_m^2\geq 0,$$

$$x_1^2+x_2^2+...+x_m^2-\frac{1}{4}>0$$

bo'lishi kerak. Ravshanki,

$1-x_1^2-x_2^2-...-x_m^2\geq 0 \;\Rightarrow\; x_1^2+x_2^2+...+x_m^2\leq 1$ bo'lib, u R^m fazoda

$$\{x\in R^m : \rho(x,\mathbf{0})\leq 1\}$$

sharni $(\mathbf{0}=(0,0,\ ...,0))$;

$x_1^2+x_2^2+...+x_m^2-\frac{1}{4}>0 \;\Rightarrow\; x_1^2+x_2^2+...+x_m^2>\frac{1}{4}$ bo'lib, u R^m fazoda

$$\left\{x\in R^m : \rho(x,\mathbf{0})>\frac{1}{2}\right\}$$

to'plamni (markazi $\mathbf{0}=(0,0,...,0)$ nuqtada, radiusi $\frac{1}{2}$ bo'lgan shar tashqarini) ifodalaydi.

Demak, berilgan funksiyaning aniqlanish to'plami

$$\{x\in R^m : \rho(x,\mathbf{0})\leq 1\}\cap\left\{x\in R^m : \rho(x,\mathbf{0})>\frac{1}{2}\right\}$$

bo'ladi. ▶

2^0. Funksiyaning limiti. R^m fazoda biror M to'plam olaylik. a nuqta $(a=(a_1,a_2,...,a_m))$ shu to'plamning limit nuqtasi bo'lsin. U holda M to'plamning nuqtalaridan a ga intiluvchi turli $\{x^{(n)}\}$ $(x^{(n)}\in M, x^{(n)}\neq a, n=1,2,...)$ ketma-ketliklar tuzish mumkin:

$$\lim_{n\to\infty} x^{(n)}=a\,.$$

Endi shu M to'plamda biror $y=f(x)$ funksiya berilgan bo'lsin.

16.-ta'rif. (Geyne ta'rifi). Agar M to'plamning nuqtalaridan tuzilgan a ga intiluvchi har qanday $\{x^{(n)}\}$ $(x^{(n)}\neq a, n=1,2,...)$ ketma-ketlik olinganda ham mos $\{f(x^{(n)})\}$ ketma-ketlik hamma vaqt yagona $в$ (chekli yoki cheksiz) limitga intilsa, $в$ $f(x)$ funksiyaning a nuqtadagi (yoki $x\to a$ dagi) limiti deb ataladi va u

$$\lim_{x\to a} f(x)=в \text{ yoki } x\to a \text{ da } f(x)\to в$$

kabi belgilanadi.

Funksiya limitini boshqacha ham ta'riflash mumkin.

17-ta'rif. (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun shundan $\delta>0$ topilsaki, ushbu $0<\rho(x,a)<\delta$ tengsizlikni qanoatlantiruvchi barcha $x\in M$ nuqtalarda

$$|f(x)-b|<\varepsilon$$

tengsizlik bajarilsa, $в$ son $f(x)$ funksiyaning a nuqtadagi ($x\to a$ dagi) limiti deb ataladi.

18-ta'rif. (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun shunday $\delta>0$ topilsaki, ushbu $0<\rho(x,a)<\delta$ tengsizlikni qanoatlantiruvchi barcha $x\in M$ nuqtalarda

$$|f(x)|>\varepsilon \quad (f(x)>\varepsilon, \quad f(x)<-\varepsilon)$$

bo'lsa, $f(x)$ funksiyaning a nuqtadagi ($x\to a$ dagi) limiti $\infty(+\infty;-\infty)$ deyiladi.

Yuqoridagi $\lim\limits_{x\to a} f(x)=в$ yoki $x\to a$ da $f(x)\to в$ belgilashlarni, $x=(x_1,x_2,...,x_m)$, $a=(a_1,a_2,...,a_m)$ hamda

$$x\to a \Leftrightarrow \begin{cases} x_1\to a_1, \\ x_2\to a_2 \\ .... \\ x_m\to a_m \end{cases}$$

ekanligi e'tiborga olib quyidagicha

$$\lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ ....\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=в$$

yoki

$$\begin{matrix} x_1\to a_1 \\ x_2\to a_2 \\ .... \\ x_m\to a_m \end{matrix} \quad \text{da} \quad f(x_1,x_2,...,x_m)\to в$$

yozsa ham bo'ladi.

R^m fazoda biror M to'plam berilgan bo'lib, ∞ esa shu to'plamning limit nuqtasi bo'lsin. Bu M to'plamda $y=f(x)$ funksiya berilgan.

19-ta'rif. (Geyne ta'rifi). Agar M to'plamning nuqtalaridan tuzilgan har qanday $\{x^{(n)}\}$ ketma-ketlik uchun $x^{(n)}\to\infty$ da mos $\{f(x^{(n)})\}$ ketma-ketlik hamma vaqt yagona $в$ ga intilsa, $в$ $f(x)$ funksiyaning $x\to\infty$ dagi limiti deb ataladi va

$$\lim_{x\to\infty} f(x)=в$$

kabi belgilanadi.

20-ta'rif. (Koshi ta'rifi). Agar $\forall\varepsilon>0$ son uchun shunday $\delta>0$ topilsaki, ushbu $\rho(x,0)>\delta$ tengsizlikni qanoatlantiruvchi barcha $x\in M$ nuqtalarda

$$|f(x)-в|<\varepsilon$$

tengsizlik bajarilsa, $в$ $f(x)$ funksiyaning $x\to\infty$ dagi limiti deb ataladi va

$$\lim_{x\to\infty} f(x)=в$$

kabi belgilanadi.

Shuni ta'kidlash lozimki, funksiya limiti tushunchasi kiritilishida limiti qaralayotgan nuqtada funksiyaning berilishi (aniqlanishi) shart emas.

1-eslatma. Yuqoridagi funksiya limitiga berilgan Geyne ta'rifining mohiyati, har qanday $\{x^{(n)}\}$ $\left(x^{(n)} \neq a,\ n = 1, 2, ...;\ x^{(n)} \to a\right)$ ketma-ketlik uchun mos $\{f(x^{(n)})\}$ ketma-ketlikning limiti olingan $\{x^{(n)}\}$ ketma-ketlikka bog'liq bo'lmasligidadir.

12.4-misol. Ushbu

$$f(x) = f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}, \text{agar } x_1^2 + x_2^2 > 0 \text{ bo'lsa} \\ 0 \quad, \text{agar } x_1^2 + x_2^2 = 0 \text{ bo'lsa} \end{cases}$$

funksiyaning $x = (x_1, x_2) \to (0,0)$ (ya'ni $x_1 \to 0,\ x_2 \to 0$) dagi limiti nol ekanligi ko'rsatilsin.

◀Bu funksiya R^2 to'plamda berilgan bo'lib, $(0,0)$ nuqta shu to'plamning limit nuqtasi.

a) Geyne ta'rifi bo'yicha: $(0,0)$ nuqtaga intiluvchi ixtiyoriy $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}\right) \to (0,0)$ (ya'ni $x_1^{(n)} \to 0,\ x_2^{(n)} \to 0,\ x^{(n)} \neq (0, 0)$) ketma-ketlik olamiz. Unga mos $\{f(x^{(n)})\}$ ketma-ketlik uchun quyidagicha

$$f\left(x^{(n)}\right) = f\left(x_1^{(n)}, x_2^{n}\right) = \frac{x_1^{(n)} x_2^{(n)}}{\sqrt{\left(x_1^{(n)}\right)^2 + \left(x_2^{(n)}\right)^2}} = \sqrt{\frac{x_1^{(n)} x_2^{(n)}}{\sqrt{\left(x_1^{(n)}\right)^2 + \left(x_2^{(n)}\right)^2}}} \cdot \sqrt{x_1^{(n)} x_2^{(n)}} \leq$$

$$\leq \frac{1}{\sqrt{2}} \sqrt{x_1^{(n)} x_2^{(n)}}$$

bo'lib, $x_1^{(n)} \to 0,\ x_2^{(n)} \to 0$ da

$$\lim_{(x_1, x_2) \to (0,0)} f\left(x^{(n)}\right) = 0$$

bo'ladi. Demak,

$$\lim_{(x_1, x_2) \to (0,0)} f(x) = \lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} = 0.$$

b) Koshi ta'rifi bo'yicha: $\forall \varepsilon > 0$ songa ko'ra $\delta = 2\varepsilon$ deb olinsa, u holda $0 < \rho(x, 0) < \delta$ tengsizlikni qanoatlantiruvchi barcha x nuqtalarda

$$|f(x) - 0| = \left|\frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}\right| = \frac{|x_1| \cdot |x_2|}{\sqrt{x_1^2 + x_2^2}} \leq \frac{1}{2}\sqrt{x_1^2 + x_2^2} = \frac{1}{2}\rho(x, \mathbf{0}) < \frac{1}{2}\delta = \varepsilon$$

tengsizlik o'rinli bo'ladi. Bu esa

$$\lim_{x \to (0,0)} f(x) = \lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} = 0$$

ekanligini bildiradi.▶

12.5-misol. Quyidagi

$$f(x) = f(x_1, x_2) = \frac{x_1^2 \cdot x_2^2}{x_1^2 x_2^2 + (x_1 - x_2)^2}$$

funksiyaning $x=(x_1,x_2)\to(0,0)$ ya'ni $x_1\to 0,\ x_2\to 0$ dagi limitining mavjud emasligi ko'rsatilsin.

◀Bu funksiya $R^2\setminus(0,0)$ to'plamda berilgan bo'lib, $(0,0)$ shu to'plamning limit nuqtasi.

$(0,0)$ nuqtaga intiluvchi ikkita

$$x^{(n)}=\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0),$$

$$\overline{x}^{(n)}=\left(\frac{1}{n},-\frac{1}{n}\right)\to(0,0)$$

ketma-ketliklar olinsa, ular uchun mos ravishda

$$f\left(x^{(n)}\right)=\frac{\frac{1}{n^4}}{\frac{1}{n^4}}=1\to 1$$

$$f\left(\overline{x}^{(n)}\right)=\frac{\frac{1}{n^4}}{\frac{1}{n^4}+\frac{4}{n^2}}=\frac{1}{1+4n^2}\to 0$$

bo'ladi. Bu esa $x\to(0,0)$ da berilgan funksiyaning limiti mavjud emasligini bildiradi.▶

3^0. Limitga ega bo'lgan funksiyalarning xossalari. Chekli limitga ega bo'lgan ko'p o'zgaruvchili funksiyalar ham chekli limitga ega bo'lgan bir o'zgaruvchili funksiyalarning xossalariga (qaralsin 1-qism, 4-bob, 4-§) o'xshash xossalarga ega. Ularning isboti xuddi bir o'zgaruvchili funksiyalar xossalarining isboti kabidir.

Biror $M\subset R^m$ to'plamda $f(x)$ funksiya berilgan bo'lib, $a\ (a\in R^m)$ nuqta shu M to'plamning limit nuqtasi bo'lsin.

1) Agar

$$\lim_{x\to a} f(x)=в$$

mavjud bo'lib, $в>p\ (в<q)$ bo'lsa, a nuqtaning etarli kichik atrofidagi $x\in M\ (x\neq a)$ nuqtalarda $f(x)>p\ (f(x)<q)$ bo'ladi. Xususan, $в\neq 0$ bo'lsa, u holda a nuqtaning etarlicha kichik atrofida $f(x)\neq 0$ bo'ladi.

2) Agar

$$\lim_{x\to a} f(x)=в$$

mavjud bo'lsa, a nuqtaning etarlicha kichik $U_\delta(a)$ atrofida ($x\in M\ (x\neq a)$ nuqtalarda) $f(x)$ funksiya chegaralangan bo'ladi.

Endi $M\subset R^m$ da ikkita $f_1(x)$ va $f_2(x)$ funksiyalar berilgan bo'lsin.

3) Agar

$$\lim_{x\to a} f_1(x)=в_1,\quad \lim_{x\to a} f_2(x)=в_2$$

bo'lib, a nuqtaning $U_\delta(a)$ atrofidagi barcha x nuqtalarda $(x \in M \cap U_\delta(a))$, $f_1(x) \le f_2(x)$ bo'lsa, u holda $в_1 \le в_2$ bo'ladi.

4) Agar a nuqtaning $U_\delta(a)$ atrofidagi $x \in M \cap U_\delta(a)$ nuqtalarda

$$f_1(x) \le f(x) \le f_2(x)$$

bo'lib, $x \to a$ da $f_1(x)$ va $f_2(x)$ funksiyalar limitga ega hamda

$$\lim_{x\to a} f_1(x) = \lim_{x\to a} f_2(x) = в$$

bo'lsa, u holda $f(x)$ funksiya ham limitga ega va

$$\lim_{x\to a} f(x) = в$$

bo'ladi.

5) Agar $x \to a$ da $f_1(x)$ va $f_2(x)$ funksiyalar limitga ega bo'lsa, $f_1(x) \pm f_2(x)$ funksiyalar limitga ega bo'ladi va

$$\lim_{x\to a}[f_1(x) \pm f_2(x)] = \lim_{x\to a} f_1(x) \pm \lim_{x\to a} f_2(x).$$

6) Agar $x \to a$ da $f_1(x)$ va $f_2(x)$ funksiyalar limitga ega bo'lsa, $f_1(x) \cdot f_2(x)$ funksiya ham limitga ega bo'ladi va

$$\lim_{x\to a}[f_1(x) \cdot f_2(x)] = \lim_{x\to a} f_1(x) \cdot \lim_{x\to a} f_2(x).$$

7) Agar $x \to a$ da $f_1(x)$ va $f_2(x)$ funksiyalar limitga ega bo'lib, $\lim_{x\to a} f_2(x) \ne 0$ bo'lsa, $\dfrac{f_1(x)}{f_2(x)}$ funksiya ham limitga ega bo'ladi va

$$\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)}.$$

2-eslatma. Bir o'zgaruvchili funksiyalardagidek $x \to a$ da $f_1(x)$ va $f_2(x)$ funksiyalar yig'indisi, ko'paytmasi va nisbatan iborat bo'lgan funksiyalarning limitga ega bo'lishidan bu funksiyalarning har birining limitga ega bo'lishi kelib chiqavermaydi.

3-eslatma. Agar $x \to a$ da 1) $f_1(x)$ va $f_2(x)$ funksiyalarning har birining limiti nol (yoki cheksiz) bo'lsa, $\dfrac{f_1(x)}{f_2(x)}$ ifoda: 2) $f_1(x) \to 0$, $f_2(x) \to \infty$ bo'lganda $f_1(x) \cdot f_2(x)$ ifoda va nixoyat 3) $f_1(x)$ va $f_2(x)$ turli ishorali cheksiz limitga ega bo'lganda $f_1(x) + f_2(x)$ yig'indi mos ravishda $\dfrac{0}{0}$, $\left(\dfrac{\infty}{\infty}\right)$, $0 \cdot \infty$, ∞-∞ ko'rinishdagi aniqmasliklarni ifodalaydi.

Agar $x \to a$ da 1) $f_1(x) \to 0$, $f_2(x) \to 0$ bo'lsa, 2) $f_1(x) \to 1$, $f_2(x) \to \infty$ bo'lsa, 3) $f_1(x) \to \infty$, $f_2(x) \to 0$ bo'lsa, u holda $[f_1(x)]^{f_2(x)}$ mos ravishda $0^0, 1^\infty, \infty^0$ ko'rinishdagi aniqmasliklarni ifodalaydi. Bunday aniqmasliklar bir o'zgaruvchili funksiyalarda qaralganidek, $f_1(x)$ va $f_2(x)$ funksiyaning o'z limitlariga intilish xarakteriga qarab ochiladi.

4^0. Takroriy limitlar. Biz yuqorida $f(x)=f(x_1,x_2,...,x_m)$ funksiyaning $a=(a_1,a_2,...,a_m)$ nuqtadagi limiti

$$\lim_{x\to a} f(x)=в \quad \left(\lim_{\substack{x_1\to a_1\\ x_2\to a_2\\ \dots\dots\dots\\ x_m\to a_m}} f(x_1,x_2,...,x_m)=в\right)$$

bilan tanishdik. Demak, funksiyaning limiti, uning argumentlari $x_1,x_2,...,x_m$ larning bir yo'la, mos ravishda $a_1,a_2,...,a_m$ sonlarga intilgandagi limitidan iboratdir.

Ko'p o'zgaruvchili funksiyalar uchun (ulargagina xos bo'lgan) boshqa formadagi limit tushunchasini ham kiritish mumkin.

$f(x_1,x_2,...,x_m)$ funksiya $M\subset R^m$ to'plamda berilgan bo'lib, $a=(a_1,a_2,...,a_m)$ nuqta M to'plamning limit nuqtasi bo'lsin. Bu funksiyaning $x_1\to a_1$ dagi (boshqa barcha o'zgaruvchilarni tayinlab) limiti

$$\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)$$

ni qaraylik. Ravshanki, bu limit, birinchidan bir o'zgaruvchili funksiya limitining o'zginasi, ikkinchidan u $x_2,x_3,...,x_m$ o'zgaruvchilarga bog'liq:

$$\lim_{x_1\to a_1} f(x_1,x_2,...,x_m)=\varphi_1(x_2,x_3,...,x_m).$$

So'ng $\varphi_1(x_2,x_3,...,x_m)$ funksiyaning $x_2\to a_2$ dagi (boshqa barcha o'zgaruvchilarini tayinlab) limiti

$$\lim_{x_2\to a_2} \varphi_1(x_2,x_3,...,x_m)=\varphi_2(x_3,x_4,...,x_m)$$

ni qaraylik.

Yuqoridagidek birin-ketin $x_3\to a_3,\ x_4\to a_4,...,x_m\to a_m$ da limitga o'tib

$$\lim_{x_m\to a_m}\ \lim_{x_{m-1}\to a_{m-1}} \dots \lim_{x_1\to a_1} f(x_1,x_2,...,x_m)$$

ni hosil qilamiz. Bu limit $f(x_1,x_2,...,x_m)$ funksiyaning takroriy limiti deb ataladi.

Demak, funksiyaning takroriy limiti, uning argumentlari $x_1,x_2,...,x_m$ larning har birining birin-ketin mos ravishda $a_1,a_2,...,a_m$ sonlarga intilgandagi limitidan iborat.

Shuni ham aytish kerakki, $f(x_1,x_2,...,x_m)$ funksiya argumentlari $x_1,x_2,...,x_m$ lar mos ravishda $a_1,a_2,...,a_m$ sonlarga turli tartibda intilganda funksiyaning turli takroriy limitlari hosil bo'ladi.

12.6-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{x_1x_2}{\sqrt{x_1^2+x_2^2}}, & \text{agar } x_1^2+x_2^2>0 \text{ bo'lsa}\\ 0, & \text{agar } x_1^2+x_2^2=0 \text{ bo'lsa}\end{cases}$$

funksiyaning takroriy limitlari topilsin.

◀ Bu funksiyaning takroriy limitlari mavjud va ular ham 0 ga teng. Haqiqatdan ham,

$$\lim_{x_1\to 0} f(x_1,x_2)=\lim_{x_1\to 0}\frac{x_1 x_2}{\sqrt{x_1^2+x_2^2}}=0, \quad \lim_{x_2\to 0}\lim_{x_1\to 0} f(x_1,x_2)=0.$$

Shuningdek,

$$\lim_{x_2\to 0} f(x_1,x_2)=\lim_{x_2\to 0}\frac{x_1 x_2}{\sqrt{x_1^2+x_2^2}}=0, \quad \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1,x_2)=0.$$

Demak, berilgan funksiyaning takroriy limitlari mavjud va ular bir-biriga teng. ▶

Bu funksiyaning (karrali) limiti 0 ga teng bo'lishini ko'rgan edi.

12.7-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{2x_1-x_2}{x_1+3x_2}, \text{agar } x_1+3x_2\neq 0 \text{ bo'lsa}\\ \quad 0 \quad\ , \text{agar } x_1+3x_2=0 \text{ bo'lsa}\end{cases}$$

funksiyaning karrali va takroriy limitlari topilsin.

◀ Bu funksiyaning takroriy limitlari quyidagicha:

$$\lim_{x_1\to 0}\frac{2x_1-x_2}{x_1+3x_2}=-\frac{1}{3}, \quad \lim_{x_2\to 0}\lim_{x_1\to 0}\frac{2x_1-x_2}{x_1+3x_2}=-\frac{1}{3},$$

$$\lim_{x_2\to 0}\frac{2x_1-x_2}{x_1+3x_2}=2, \quad \lim_{x_1\to 0}\lim_{x_2\to 0}\frac{2x_1-x_2}{x_1+3x_2}=2.$$

Demak, berilgan funksiyaning takroriy limitlari mavjud bo'lib, ularning biri $-\frac{1}{3}$ ga, ikkinchisi esa 2 ga teng.

Biroq $x=(x_1,x_2)\to(0,0)$ da $f(x_1,x_2)$ funksiyaning (karrali) limiti mavjud emas. Chunki $(0,0)$ nuqtaga intiluvchi ikkita $x^{(n)}=\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)$, $x^{-(n)}=\left(\frac{5}{n},\frac{4}{n}\right)\to 0$ ketma-ketliklar olinsa, ular uchun mos ravishda $f\left(\frac{1}{n},\frac{1}{n}\right)=\frac{1}{4}\to\frac{1}{4}$, $f\left(\frac{5}{n},\frac{4}{n}\right)=\frac{6}{17}\to\frac{6}{17}$ bo'ladi. Bu esa $(x_1,x_2)\to(0,0)$ da berilgan funksiyaning (karrali) limiti mavjud emasligini bildiradi. ▶

12.8-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}x_1+x_2\sin\dfrac{1}{x_1}, \text{agar } x_1\neq 0 \text{ bo'lsa}\\ \quad 0 \quad\ , \text{agar } x_1=0 \text{ bo'lsa}\end{cases}$$

funksiyaning karrali va takroriy limitlari topilsin.

◀ Bu funksiya uchun

$$\lim_{x_2\to 0} f(x_1,x_2)=x_1, \quad \lim_{x_1\to 0}\lim_{x_2\to 0} f(x_1,x_2)=0$$

bo'lib, $\lim\limits_{x_2 \to 0} \lim\limits_{x_1 \to 0} f(x_1, x_2)$ mavjud emas. Demak, berilgan funksiyaning bitta takroriy limiti mavjud bo'lib, ikkinchi takroriy limiti esa mavjud emas. Ammo

$$|f(x_1,x_2)-0| = \left| x_1 + x_2 \sin\frac{1}{x_1} \right| \leq |x_1| + |x_2| \qquad (x_1 \neq 0)$$

munosabatdan $(x_1, x_2) \to (0,0)$ da $f(x_1, x_2)$ funksiyaning (karrali) limiti mavjud va

$$\lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} f(x_1, x_2) = 0$$

bo'lishi kelib chiqadi.▶

2-teorema. Agar 1) $(x_1, x_2) \to (x_1^0, x_2^0)$ da $f(x_1, x_2)$ funksiyaning (karrali) limiti mavjud;

$$\lim_{\substack{x_1 \to x_1^0 \\ x_2 \to x_2^0}} f(x_1, x_2) = в,$$

2) har bir tayinlangan x_1 da quyidagi

$$\lim_{x_2 \to x_2^0} f(x_1, x_2) = \varphi(x_1),$$

limit mavjud bo'lsa, u holda

$$\lim_{x_1 \to x_1^0} \lim_{x_2 \to x_2^0} f(x_1, x_2)$$

takroriy limit ham mavjud bo'lib,

$$\lim_{x_1 \to x_1^0} \lim_{x_2 \to x_2^0} f(x_1, x_2) = в$$

bo'ladi.

◀ $f(x_1, x_2)$ funksiya $(x_1, x_2) \to (x_1^0, x_2^0)$ da karrali

$$\lim_{\substack{x_1 \to x_1^0 \\ x_2 \to x_2^0}} f(x_1, x_2) = в$$

limitga ega bo'lsin. Limitning ta'rifiga ko'ra, $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ topiladiki, ushbu

$$\left\{(x_1, x_2) \in R^2 : \left|x_1 - x_1^0\right| < \delta, \left|x_2 - x_2^0\right| < \delta\right\} \subset M$$

to'plamning barcha (x_1, x_2) nuqtalari uchun

$$|f(x_1,x_2) - в| < \varepsilon \qquad (12.15)$$

bo'ladi. Endi teoremaning 2) shartini e'tiborga olib x_1 o'zgaruvchining $\left|x_1 - x_1^0\right| < \delta$ tengsizlikni qanoatlantiradigan qiymatini tayinlab, $x_2 \to x_2^0$ da (12.15) tengsizlikda limitga o'tib

$$|\varphi(x_1) - в| \leq \varepsilon$$

ni topamiz. Demak, $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ topiladiki, $\left|x_1 - x_1^0\right| < \delta$ bo'lganda $|\varphi(x_1) - в| \leq \varepsilon$ bo'ladi. Bu esa

$$\lim_{x_1 \to x_1^0} \varphi(x_1) = в$$

bo'lishini bildiradi. Keyingi munosabatdan

$$\lim_{x_1\to x_1^0}\lim_{x_2\to x_2^0} f(x_1,x_2)=в$$

bo'lishi kelib chiqadi.▶

Qo'yidagi teorema xuddi shunga o'xshash isbotlanadi.

3-teorema. Agar 1) $(x_1,x_2)\to(x_1^0,x_2^0)$ da $f(x_1,x_2)$ funksiyaning karrali limiti mavjud:

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} f(x_1,x_2)=в$$

2) har bir tayinlangan x_2 da quyidagi

$$\lim_{x_1\to x_1^0} f(x_1,x_2)=\varphi(x_2)$$

limit mavjud bo'lsa, u holda

$$\lim_{x_2\to x_2^0}\lim_{x_1\to x_1^0} f(x_1,x_2)$$

takroriy limit ham mavjud bo'lib,

$$\lim_{x_2\to x_2^0}\lim_{x_1\to x_1^0} f(x_1,x_2)=в$$

bo'ladi.

1-natija. Agar bir vaqtda yuqoridagi 2- va 3- teoremalarning shartlari bajarilsa, u holda

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} f(x_1,x_2)=\lim_{x_1\to x_1^0}\lim_{x_2\to x_2^0} f(x_1,x_2)=\lim_{x_2\to x_2^0}\lim_{x_1\to x_1^0} f(x_1,x_2)$$

bo'ladi.

5^0. Koshi teoremasi (yaqinlashish prinsipi). Endi ko'p o'zgaruvchili funksiya limitining mavjudligi haqida umumiy teorema keltiramiz.

R^m fazoda M to'plam berilgan bo'lib, $a\ (a\in R^m)$ uning limit nuqtasi bo'lsin. Bu to'plamda $f(x)$ funksiya berilgan.

19-ta'rif. Agar $\forall\varepsilon>0$ son uchun shunday $\delta>0$ son topilsaki, ushbu $0<\rho(\bar{x},a)<\delta,\ 0<\rho(x,a)<\delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy x va $\bar{x}$ $(x\in M,\bar{x}\in M)$ nuqtalarda

$$\left|f(\bar{x})-f(x)\right|<\varepsilon$$

tengsizlik o'rinli bo'lsa, $f(x)$ funksiya uchun a nuqtada Koshi sharti bajariladi deyiladi.

4-teorema (Koshi teoremasi). $f(x)$ funksiya a nuqtada chekli limitga ega bo'lishi uchun a nuqtada Koshi shartining bajarilishi zarur va etarli.

◀***Zarurligi.*** $x\to a$ da $f(x)$ funksiya chekli limit

$$\lim_{x\to a} f(x)=в$$

ga ega bo'lsin. Ta'rifga binoan $\forall\varepsilon>0$ son olinganda ham $\frac{\varepsilon}{2}$ ga ko'ra shunday $\delta>0$ topiladiki, ushbu $0<\rho(x,a)<\delta$ tengsizlikni qanoatlantiruvchi barcha $x\ (x\in M)$ nuqtalarda

$$|f(x)-в|<\frac{\varepsilon}{2},$$

jumladan $0<\rho(\overline{x},a)<\delta \Rightarrow \left|f(\overline{x})-в\right|<\frac{\varepsilon}{2}$ bo'ladi. Bu tengsizliklardan

$$\left|f(\overline{x})-f(x)\right| \le \left|f(\overline{x})-в\right|+|f(x)-в|<\varepsilon$$

bo'lishi kelib chiqadi.

Etarliligi. $f(x)$ funksiya uchun a nuqtada Koshi sharti bajarilsin, ya'ni $\forall\varepsilon>0$ son olinganda ham, shunday $\delta>0$ topiladiki, ushbu $0<\rho(x,a)<\delta,\ 0<\rho(\overline{x},a)<\delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy x va $\overline{x}$ $(x,\overline{x}\in M)$ nuqtalarda

$$\left|f(\overline{x})-f(x)\right|<\varepsilon$$

bo'lsin. Bu holda $f(x)$ funksiya $x\to a$ da chekli limitga ega bo'lishini ko'rsatamiz.

a nuqta M to'plamning limit nuqtasi. Shuning uchun M to'plamning nuqtalaridan $\{x^{(n)}\}$ $(x^n\ne a,\ n=1,\ 2,\ ...,)$ ketma-ketlik tuzish mumkinki, bunda

$$\lim_{n\to\infty} x^{(n)}=a$$

bo'ladi. Limitning ta'rifiga binoan, yuqorida keltirilgan $\delta>0$ ga ko'ra, shunday $n_0\in N$ topiladiki, barcha $n>n_0,\ \rho>n_0$ uchun $0<\rho\left(x^{(n)},a\right)<\delta,\ 0<\rho\left(x^{(p)},a\right)<\delta$ bo'ladi. Bu tengsizliklarning bajarilishidan esa, shartga ko'ra:

$$\left|f\left(x^{(p)}\right)-f\left(x^{(n)}\right)\right|<\varepsilon$$

bo'ladi. Demak, $\left\{f\left(x^{(n)}\right)\right\}$ fundamental ketma-ketlik. Binobarin, $\left\{f\left(x^{(n)}\right)\right\}$ ketma-ketlik yaqinlashuvchi.

Bu ketma-ketlikning limitini $в$ bilan belgilaylik:

$$\lim_{n\to\infty} f\left(x^{(n)}\right)=в.$$

Endi M to'plamning nuqtalaridan tuzilgan va a nuqtaga intiluvchi ixtiyoriy $\left\{\overline{x}^{(n)}\right\}$ ketma-ketlik

$$\overline{x}^{(n)}\to a \quad \left(\overline{x}^{(n)}\ne a,\quad n=1,\ 2,\ ...\right)$$

olinganda ham mos $\left\{f\left(\overline{x}^{(n)}\right)\right\}$ ketma-ketlik (u yuqorida ko'rsatganimizga binoan yaqinlashuvchi bo'ladi) ham o'sha $в$ ga intilishini ko'rsatamiz.

Faraz qilaylik $\overline{x}^{(n)}\to a \quad \left(\overline{x}^{(n)}\ne a,\quad n=1,\ 2,\ ...\right)$ bo'lganda

$$f\left(\overline{x}^{(n)}\right)=в^1$$

bo'lsin. $\{x^{(n)}\}$, $\left\{\overline{x}^{(n)}\right\}$ ketma-ketlik hadlaridan ushbu

$$x^{(1)},\overline{x}^{(1)},x^{(2)},\overline{x}^{(2)},...,x^{(n)},\overline{x}^{(n)},...$$

ketma-ketlik tuzaylik. Ravshanki, bu ketma-ketlik a $(a\in R^m)$ ga intiladi. U holda

$$f\left(x^{(1)}\right),\ f\left(\overline{x}^{(1)}\right),\ f\left(x^{(2)}\right),\ f\left(\overline{x}^{(2)}\right),...,\ f\left(x^{(n)}\right),\ f\left(\overline{x}^{(n)}\right),... \qquad (12.16)$$

ketma-ketlik chekli limitga ega. Uni $в^*$ orqali belgilaylik. Agar $\left\{f\left(x^{(n)}\right)\right\}$ va $\left\{f\left(\overline{x}^{(n)}\right)\right\}$ ketma-ketliklarning har biri (12.16) ketma-ketlikning qismiy ketma-ketliklari ekanligini e'tiborga olsak, u holda

$$f\left(x^{(n)}\right) \to в^* \quad f\left(\overline{x}^{(n)}\right) \to в^*$$

bo'lishini topamiz. Demak,

$$в^* = в = в'$$

Shunday qilib, $f(x)$ funksiya uchun a nuqtada Koshi shartining bajarilishidan M to'plam nuqtalaridan tuzilgan va a ga intiluvchi har qanday $\left(x^{(n)}\right)$ $\left(x^{(n)} \neq a, \quad n = 1, 2, ...\right)$ ketma-ketlik olinganda mos ketma-ketlik bitta songa intilishini topdik. Bu esa funksiya limitining Geyne ta'rifiga ko'ra $f(x)$ funksiya a nuqtada chekli limitga ega bo'lishini bildiradi. ▶

3-eslatma. Koshi sharti va Koshi teoremasi $x \to \infty$ da ham yuqoridagiga o'xshash ifodalanadi va isbot etiladi.

4-§. Ko'p o'zgaruvchili funksiyaning uzluksizligi

1^0. Funksiya uzluksizligi ta'rifi. $M \subset R^m$ to'plamda $f(x) = f(x_1, x_2, ..., x_m)$ funksiya berilgan bo'lib, $a \in M$ $(a = (a_1, a_2, ..., a_m))$ nuqta esa M to'plamning limit nuqtasi bo'lsin.

19-ta'rif. Agar $x \to a$ da $f(x)$ funksiyaning limiti mavjud bo'lib,

$$\lim_{x \to a} f(x) = f(a)$$

$$\left(\lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2 \\ \cdots\cdots\cdots \\ x_m \to a_m}} f(x_1, x_2, ..., x_m) = f(a_1, a_2, ..., a_m) \right) \tag{12.17}$$

bo'lsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

20-ta'rif. (Geyne ta'rifi). Agar $M \subset R^m$ to'plamning nuqtalaridan tuzilgan, a $(a \in M)$ ga intiluvchi har qanday $\left\{x^{(n)}\right\}$ ketma-ketlik olinganda ham, mos $\left\{f\left(x^{(n)}\right)\right\}$ ketma-ketlik hamma vaqt $f(a)$ ga intilsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

21-ta'rif. (Koshi ta'rifi). Agar $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ topilsaki, ushbu $\rho(x, a) < \delta$ tengsizlikni qanoatlantiruvchi barcha $x \in M$ nuqtalarda

$$|f(x) - f(a)| < \varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

Atrof tushunchasi yordamida funksiyaning uzluksizligini quyidagicha ham ta'riflash mumkin.

22-ta'rif. Agar $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ topilsaki, barcha $x \in U_\delta(a) \cap M$ nuqtalarda $f(x)$ funksiyaning qiymatlari $f(x) \in U_\varepsilon(f(a))$, ya'ni

$$x \in U_\delta(a) \cap M \Rightarrow f(x) \in U_\varepsilon(f(a))$$

bo'lsa, $f(x)$ funksiya a nuqtada uzluksiz deb ataladi.

$f(x)=f(x_1,x_2,...,x_m)$ funksiyaning $a=(a_1,a_2,...,a_m)$ nuqtada uzluksizligini funksiya orttirmasi yordamida ham ta'riflash mumkin.

Funksiya argumentlarining orttirmalari

$$\Delta x_1 = x_1 - a_1,\ \Delta x_2 = x_2 - a_2,...,\Delta x_m = x_m - a_m$$

ga mos ushbu

$$f(x)-f(a)=f(x_1,x_2,...,x_m)-f(a_1,a_2,...,a_m)=$$
$$=f(a_1+\Delta x_1,\ a_2+\Delta x_2,...,\ a_m+\Delta x_m)-f(a_1,a_2,...,a_m)$$

ayirma $f(x)$ funksiyaning a nuqtadagi to'liq orttirmasi deb ataladi va Δf yoki $\Delta f(a)$ kabi belgilanadi:

$$\Delta f(a)=f(a_1+\Delta x_1,\ a_2+\Delta x_2,...,\ a_m+\Delta x_m)-f(a_1,a_2,...,a_m).$$

Quyidagi

$$f(a_1+\Delta x_1,\ a_2,...,\ a_m)-f(a_1,a_2,...,a_m),$$
$$f(a_1,\ a_2+\Delta x_2,\ a_3,...,\ a_m)-f(a_1,a_2,...,a_m),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$f(a_1,\ a_2,\ ...,a_{m-1}\ a_m+\Delta x_m)-f(a_1,a_2,...,a_m)$$

ayirmalar $f(x)$ funksiyaning a nuqtadagi xususiy orttirmalari deyiladi va ular mos ravishda $\Delta_{x_1} f, \Delta_{x_2} f, ..., \Delta_{x_m} f$ kabi belgilanadi.

Yuqoridagi (12.17) limit munosabatdan topamiz:

$$\lim_{x\to a} f(x)=f(a) \quad \Rightarrow \quad \lim_{x\to a}[f(x)-f(a)]=0.$$

Natijada (12.17) tenglik quyidagi

$$\lim_{x\to a}\Delta f(a)=0 \text{ ya'ni } \lim_{\substack{\Delta x_1\to 0\\ \dots\dots\\ \Delta x_m\to 0}}\Delta f(a)=0$$

ko'rinishga keladi. Demak, $f(x)$ funksiyaning a nuqtadagi uzluksizligi

$$\lim_{x\to a}\Delta f(a)=0 \quad \left(\lim_{\substack{\Delta x_1\to 0\\ \dots\dots\\ \Delta x_m\to 0}}\Delta f(a)=0\right)$$

kabi ham ta'riflanishi mumkin ekan.

23-ta'rif. Agar $f(x)$ funksiya $M(M\subset R^m)$ to'plamning har bir nuqtasida uzluksiz bo'lsa, funksiya shu M to'plamda uzluksiz deb ataladi.

12.9-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{x_1\cdot x_2}{\sqrt{x_1^2+x_2^2}}, & agar\ x_1^2+x_2^2\neq 0\\ 0 & ,\ agar\ x_1^2+x_2^2=0\end{cases}$$

funksiya uzluksizlikka tekshirilsin.

◄Ravshanki, bu funksiya R^2 da aniqlangan. Aytaylik $(x_1^0,x_2^0)\neq(0,0)$ bo'lsin. Limit xossalaridan foydalanib topamiz:

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} f(x_1,x_2)=\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \frac{x_1\cdot x_2}{\sqrt{x_1^2+x_2^2}}=\frac{\lim\limits_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}(x_1,x_2)}{\lim\limits_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\sqrt{x_1^2+x_2^2}}=\frac{x_1^0\cdot x_2^0}{\sqrt{x_1^{0^2}+x_2^{0^2}}}=f\left(x_1^0,x_2^0\right).$$

$\left(x_1^0,x_2^0\right)=(0,0)$ bo'lgan holda

$$\lim_{\substack{x_1\to 0\\ x_2\to 0}} f(x_1,x_2)=0=f(0,0)$$

bo'ladi (qaralsin, 12.4-misol).

Demak, berilgan funksiya R^2 da uzluksiz.

24-ta'rif. Agar $x\to a$ da $f(x)$ funksiyaning limiti mavjud bo'lmasa, yoki

$$\lim_{x\to a} f(x)=\infty,$$

yoki funksiyaning limiti mavjud, chekli bo'lib,

$$\lim_{x\to a} f(x)=в\neq f(a)$$

bo'lsa, funksiya a nuqtada uzilishga ega deb ataladi.

12.10-misol. Ushbu

$$f(x_1,x_2)=\begin{cases} x_1^2+x_2^2, \text{agar } (x_1,x_2)\neq(0,0) \text{ bo'lsa,}\\ \quad 1 \quad, \text{ agar } (x_1,x_2)=(0,0) \text{ bo'lsa}\end{cases}$$

funksiya uzluksizlikka tekshirilsin.

◀ Bu funksiya R^2 to'plamda berilgan bo'lib, uning $(0,\ 0)$ nuqtadagi limiti

$$\lim_{\substack{x_1\to 0\\ x_2\to 0}} f(x_1,x_2)=0\neq f(0,0)=1$$

bo'ladi. Demak, berilgan funksiya $(0,\ 0)$ nuqtada uzilishga ega, qolgan barcha nuqtalarda uzluksiz.▶

12.11-misol. Quyidagi

$$f(x_1,x_2)=\begin{cases} \dfrac{1}{x_1^2+x_2^2-1}, \text{agar } x_1^2+x_2^2\neq 1 \text{ bo'lsa,}\\ \quad 0 \qquad, \text{agar } x_1^2+x_2^2=1 \text{ bo'lsa}\end{cases}$$

funksiya uzluksizlikka tekshirilsin.

◀ Bu funksiya $\left\{(x_1,x_2)\in R^2 : x_1^2+x_2^2=1\right\}$ to'plamning har bir nuqtasida uzilishga ega bo'ladi, chunki $(x_1,x_2)\to\left(x_1^0,x_2^0\right)$ $\left(x_1^{0^2}+x_2^{0^2}=1\right)$ da $f(x_1,x_2)$ funksiyaning chekli limiti mavjud emas.▶

2⁰. Uzluksiz funksiyalar ustida arifmetik amallar. Murakkab funksiyaning uzluksizligi.

5-teorema. Agar $f_1(x)$ va $f_2(x)$ funksiyalarning har biri $M\subset R^m$ to'plamda berilgan bo'lib, ular $a\in M$ nuqtada uzluksiz bo'lsa,

$$f_1(x)\pm f_2(x),\quad f_1(x)\cdot f_2(x) \text{ hamda } \frac{f_1(x)}{f_2(x)} \quad (f_2(x)\neq 0)$$

funksiyalar ham shu nuqtada uzluksiz bo'ladi.

◀Bu teoremaning isboti, limitga ega bo'lgan funksiyalar ustida arifmetik amallar haqidagi ma'lumotlardan (ushbu bobning 3-§ dagi 5, 6 va 7- xossalar) bevosita kelib chiqadi.▶

Faraz qilaylik, $M \subset R^m$ to'plamda $y = f(x) = f(x_1, x_2, ..., x_m)$ funksiya berilgan bo'lib, $x_1, x_2, ..., x_m$ larning har bir $T \subset R^k$ $(k \in N)$ to'plamda berilgan funksiyalar bo'lsin:

$$x_1 = \varphi_1(t) = \varphi_1(t_1, t_2, ..., t_k),$$
$$x_2 = \varphi_2(t) = \varphi_2(t_1, t_2, ..., t_k),$$
$$.......................................,$$
$$x_m = \varphi_m(t) = \varphi_m(t_1, t_2, ..., t_k)$$

Biz $t = (t_1, t_2, ..., t_k) \in T$ bo'lganda unga mos $x = (x_1, x_2, ..., x_m) \in M$ deb qaraymiz. Bu funksiyalar yordamida

$$y = f(\varphi_1(t_1, t_2, ..., t_k),\ \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k)) = \Phi(t_1, t_2, ..., t_k) = \Phi(t)$$

murakkab funksiyani tuzamiz.

6-teorema. Agar $\varphi_i(t) = \varphi_i(t_1, t_2, ..., t_k)$ $(i = 1,\ 2,\ ..., m)$ funksiyalarning har bir $t^0 = (t_1^0, t_2^0, ..., t_m^0)$ nuqtada uzluksiz bo'lib, $f(x) = f(x_1, x_2, ..., x_m)$ funksiya esa $t^0 = (t_1^0, t_2^0, ..., t_k^0)$ nuqtaga mos

$$x^0 = (x_1^0, x_2^0, ..., x_m^0)\ \left(x_1^0 = \varphi_1(t_1^0, t_2^0, ..., t_k^0),\ x_2^0 = \varphi_2(t_1^0, t_2^0, ..., t_k^0), ..., x_m^0 = \varphi_m(t_1^0, t_2^0, ..., t_k^0)\right)$$

nuqtada uzluksiz bo'lsa, $y = \Phi(t) = \Phi(t_1, t_2, ..., t_k)$ murakkab funksiya $t^0 = (t_1^0, t_2^0, ..., t_k^0)$ nuqtada uzluksiz bo'ladi.

◀ $x_i = \varphi_i(t) = \varphi_i(t_1, t_2, ..., t_k)$ $(i = 1,\ 2,\ ...,\ m)$ funksiya $t^0 = (t_1^0, t_2^0, ..., t_k^0)$ nuqtada uzluksiz bo'lsin.

$T \subset R^k$ to'plamda $t^0 = (t_1^0, t_2^0, ..., t_k^0)$ nuqtaga intiluvchi ixtiyoriy

$$\{t^{(n)}\} = \{(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)})\} \quad (n = 1,\ 2,\ ...)$$

ketma-ketlikni olaylik. U holda uzluksizlikning Geyne ta'rifiga ko'ra

$$\left.\begin{array}{l} t_1^{(n)} \to t_1^0 \\ t_2^{(n)} \to t_2^0 \\ .... \\ t_k^{(n)} \to t_k^0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1^{(n)} = \varphi_1(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}) \to \varphi_1(t_1^0, t_2^0, ..., t_k^0) = x_1^0, \\ x_2^{(n)} = \varphi_2(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}) \to \varphi_2(t_1^0, t_2^0, ..., t_k^0) = x_2^0, \\ ...., \\ x_m^{(n)} = \varphi_m(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}) \to \varphi_m(t_1^0, t_2^0, ..., t_k^0) = x_m^0 \end{array}$$

bo'ladi.

$y = f(x_1, x_2, ..., x_m)$ funksiya $(x_1^0, x_2^0, ..., x_m^0)$ nuqtada uzluksiz. U holda yana Geyne ta'rifiga ko'ra

$$\left.\begin{array}{l} x_1^{(n)} \to x_1^0 \\ x_2^{(n)} \to x_2^0 \\ .... \\ x_k^{(n)} \to x_k^0 \end{array}\right\} \Rightarrow f(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)}) \to f(x_1^0, x_2^0, ..., x_m^0)$$

bo'ladi. Demak, $t_1^{(n)} \to t_1^0,\ t_2^{(n)} \to t_2^0, ..., t_k^{(n)} \to t_k^0$ da

$$f\left(\varphi_1\left(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}\right), \varphi_2\left(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}\right), ..., \varphi_m\left(t_1^{(n)}, t_2^{(n)}, ..., t_k^{(n)}\right)\right) \to$$
$$\to f\left(\varphi_1\left(t_1^0, t_2^0, ..., t_k^0\right), \varphi_2\left(t_1^0, t_2^0, ..., t_k^0\right), ..., \varphi_m\left(t_1^0, t_2^0, ..., t_k^0\right)\right).$$

Bu esa $y = f(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k)) = \Phi(t_1, t_2, ..., t_k)$ funksiyaning $t^0 = \left(t_1^0, t_2^0, ..., t_k^0\right)$ nuqtada uzluksiz ekanligini bildiradi.►

5-§. Uzluksiz funksiyalarning xossalari

Biz quyida ko'p o'zgaruvchili uzluksiz funksiyalarning xossalarini keltiramiz. Bunda bir o'zgaruvchili uzluksiz funksiyalarning xossalari to'g'risida ma'lumotlardan to'la foydalana boramiz.

Ko'p o'zgaruvchili uzluksiz funksiyalar ham bir o'zgaruvchili uzluksiz funksiyalarning xossalari kabi xossalarga ega.

1^0. Nuqtada uzluksiz bo'lgan funksiyalarning xossalari (lokal xossalari). $f(x)$ funksiya $M\left(M \subset R^m\right)$ to'plamda berilgan bo'lib, $x^0 \in M$ nuqtada uzluksiz bo'lsin. Bunday $f(x)$ funksiyaning x^0 nuqtaning etarli kichik atrofi $U_\delta\left(x^0\right) \subset M$ dagi xossalarini (lokal xossalarini) o'rganimiz.

1) Agar $f(x)$ funksiya $x^0 \in M$ nuqtada uzluksiz bo'lsa, u holda x^0 nuqtaning etarli kichik atrofida funksiya chegaralangan bo'ladi.

◄Funksiya uzluksizligi ta'rifiga ko'ra

$$\lim_{x \to x^0} f(x) = f\left(x^0\right)$$

bo'lib, undan $f(x)$ funksiyani x^0 nuqtada chekli limitga ega ekanligi kelib chiqadi. Chekli limitga ega bo'lgan funksiyaning xossalaridan esa, $f(x)$ funksiyani x^0 nuqtaning etarli kichik atrofida chegaralanganligini topamiz.►

2) Agar $f(x)$ funksiya x^0 nuqtada uzluksiz bo'lib, $f\left(x^0\right) > 0$ ($f\left(x^0\right) < 0$) bo'lsa, x^0 nuqtaning etarli kichik atrofida x nuqtalarda $f(x) > 0$ ($f(x) < 0$) bo'ladi.

◄Funksiya x^0 nuqtada uzluksizligi ta'rifiga ko'ra, $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ topiladiki, barcha $x \in U_\delta\left(x^0\right) \cap M$ nuqtalar uchun

$$f\left(x^0\right) - \varepsilon < f(x) < f\left(x^0\right) + \varepsilon$$

bo'ladi.

Bu erda $\varepsilon = f\left(x^0\right) > 0$ (agar $f\left(x^0\right) < 0$ bo'lsa, $\varepsilon = -f\left(x^0\right)$) deb olsak, fikrimizning tasdig'iga ega bo'lamiz.►

Demak, $f(x)$ funksiya x^0 nuqtada uzluksiz va $f\left(x^0\right) \neq 0$ bo'lsa, x^0 nuqtaning etarli kichik atrofidagi x nuqtalarda funksiya qiymatlarining ishorasi $f\left(x^0\right)$ ning ishorasi bilan bir xil bo'lar ekan:

$$sign f(x) = sign f\left(x^0\right).$$

3) Agar $f(x)$ funksiya x^0 nuqtada uzluksiz bo'lsa, x^0 nuqtaning etarli kichik atrofidagi $x' \in M$, $x'' \in M$ nuqtalar uchun

$$\left|f(x')-f(x'')\right|<\varepsilon$$

tengsizlik o'rinli bo'ladi.

◀ $f(x)$ funksiyaning x^0 nuqtada uzluksizligiga asosan, $\forall\varepsilon>0$ olinganda ham $\frac{\varepsilon}{2}$ ga ko'ra shunday $\delta>0$ topiladiki, barcha $x\in U_\delta(x^0)$ nuqtalar uchun

$$\left|f(x)-f(x^0)\right|<\frac{\varepsilon}{2}$$

bo'ladi. Jumladan, $x'\in U_\delta(x^0)$, $x''\in U_\delta(x^0)$ nuqtalar uchun ham

$$\left|f(x')-f(x^0)\right|<\frac{\varepsilon}{2},\quad \left|f(x'')-f(x^0)\right|<\frac{\varepsilon}{2}$$

tengsizliklar o'rinli bo'ladi. Keyingi tengsizliklardan esa $\left|f(x')-f(x'')\right|<\varepsilon$ bo'lishi kelib chiqadi.▶

2^0. To'plamda uzluksiz bo'lgan funksiyalarning xossalari (global xossalari). Endi $M\subset R^m$ to'plamda uzluksiz bo'lgan funksiyalarning xossalarini (global xossalari), aniqrog'i $f(x)$ funksiya qiymatlaridan iborat $\{f(x);\ x\in M\}$ to'plamning xossalarini o'rganamiz.

17-teorema (Boltsano-Koshining birinchi teoremasi).

$f(x)=f(x_1,x_2,...,x_m)$ funksiya bog'lamli $M\subset R^m$ to'plamda uzluksiz bo'lsin. Agar bu funksiya to'plamning ikkita $a=(a_1,a_2,...,a_m)$ va $в=(в_1,в_2,...,в_m)$ nuqtasida har xil ishorali qiymatlarga ega bo'lsa, u holda shunday $c=(c_1,c_2,...,c_m)\in M$ nuqta topiladiki, bu nuqtada funksiya nolga aylanadi:

$$f(c)=f(c_1,c_2,...,c_m)=0.$$

◀Aniqlik uchun $f(a)=f(a_1,a_2,...,a_m)<0$, $f(в)=f(в_1,в_2,...,в_m)>0$ bo'lsin. $M\subset R^m$ bog'lamli to'plam bo'lgani uchun bu a va $в$ nuqtalarni birlashtiruvchi va M to'plamda yotuvchi siniq chiziq topiladi. Bu siniq chiziq uchlari bo'lgan nuqtalarda $f(x)$ funksiyaning qiymatlarini hisoblab boramiz. Bunda ikki xol yuz beradi:

1) Siniq chiziq uchlarining birida $f(x)$ funksiya nolga aylanadi. Bu holda siniq chiziqning shu uchini teoremadagi c nuqta deb olinsa, $f(c)=0$ bo'lib, teorema isbotlanadi.

2) Siniq chiziq uchlarida $f(x)$ funksiya nolga aylanadi. Bu holda siniq chiziqning shunday kesmasi topiladiki, uning uchlarida $f(x)$ funksiyaning qiymatlari har xil ishorali bo'ladi. Siniq chiziqning xuddi shu uchlarining birini $a'=(a'_1,a'_2,...,a'_m)$ bilan, ikkinchi uchini esa $в'=(в'_1,в'_2,...,в'_m)$ bilan belgilasak, unda

$$f(a')=f(a'_1,a'_2,...,a'_m)<0,$$
$$f(в')=f(в'_1,в'_2,...,в'_m)>0$$

bo'ladi. Siniq chiziqning bu kesmasining tenglamasi ushbu

$$x_1 = a_1' + t(в_1' - a_1'),$$
$$x_2 = a_2' + t(в_2' - a_2'),$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$x_m = a_m' + t(в_m' - a_m')$$

$(0 \le t \le 1)$ ko'rinishda yoziladi.

Agar o'zgaruvchi $x = (x_1, x_2, \ldots, x_m) \in M$ nuqtani siniq chiziqning shu kesmasi bo'yichagina o'zgaradi deb olinadigan bo'lsa, u holda $f(x) = f(x_1, x_2, \ldots, x_m)$ ko'p o'zgaruvchili funksiya quyidagicha

$$F(t) = f\left(a_1' + t(в_1' - a_1'),\ a_2' + t(в_2' - a_2'), \ldots,\ a_m' + t(в_m' - a_m')\right)$$

bitta t o'zgaruvchining murakkab funksiyasi bo'lib qoladi. Murakkab funksiyaning uzluksizligi haqidagi teoremaga ko'ra $F(t)$ funksiya $[0,1]$ segmentda uzluksizdir. Ikkinchi tomondan $t = 0$ va $t = 1$ da bu funksiya turli ishorali qiymatlarga ega:

$$F(0) = f(a_1', a_2', \ldots, a_m') < 0,$$
$$F(1) = f(в_1', в_2', \ldots, в_m') > 0.$$

Shunday qilib, $F(t)$ funksiya $[0,1]$segmentda uzluksiz va shu segmentning chetki nuqtalarida har xil ishorali qiymatlarga ega. U holda 1-qism, 5-bob, 6-§ dagi 5-teoremaga ko'ra, $(0,1)$ intervalda shunday t_0 nuqta topiladiki,

$$F(t_0) = 0$$

bo'ladi. Demak,

$$F(t_0) = f\left(a_1' + t_0(в_1' - a_1'),\ a_2' + t_0(в_2' - a_2'), \ldots,\ a_m' + t_0(в_m' - a_m')\right) = 0.$$

Agar

$$c_1 = a_1' + t_0(в_1' - a_1'),$$
$$c_2 = a_2' + t_0(в_2' - a_2'),$$
$$\ldots\ldots\ldots\ldots\ldots,$$
$$c_m = a_m' + t_0(в_m' - a_m')$$

deb olsak, ravshanki, $c = (c_1, c_2, \ldots, c_m) \in M$ va $f(c) = f(c_1, c_2, \ldots, c_m) = 0$ bo'ladi.▶

Quyidagi teorema ham shunga o'xshash isbotlanadi.

18-teorema (Boltsano-Koshining ikkinchi teoremasi). $f(x) = f(x_1, x_2, \ldots, x_m)$ funksiya bog'lamli $M \subset R^m$ to'plamda uzluksiz bo'lib, M to'plamning ikkita $a = (a_1, a_2, \ldots, a_m)$ va $в = (в_1, в_2, \ldots, в_m)$ nuqtasida $f(a) = A,\ f(в) = B,\ (\mathrm{A} \ne \mathrm{B})$ qiymatlarga ega bo'lsin. A va V orasida har qanday C son olinsa ham M to'plamda shunday $c = (c_1, c_2, \ldots, c_m)$ nuqta topiladiki,

$$f(c) = f(c_1, c_2, \ldots, c_m) = C$$

bo'ladi.

19-teorema (Veyershtrassning birinchi teoremasi). Agar $f(x)$ funksiya chegaralangan yopiq $M \subset R^m$ to'plamda uzluksiz bo'lsa, funksiya shu M to'plamda chegaralangan bo'ladi.

◀Teskarisini faraz qilaylik, ya'ni $f(x)$ funksiya chegaralangan yopiq M to'plamda uzluksiz bo'lsa ham, u shu to'plamda chegaralanmagan bo'lsin. U holda $\forall n \in N$ uchun shunday $x^{(n)} \in M$ nuqta topiladiki,

$$\left|f\left(x^{(n)}\right)\right| > n \qquad (12.18)$$

bo'ladi. Bunday nuqtalardan $\left\{x^{(n)}\right\}$, $\left(x^{(n)} \in M,\ n = 1,2,...\right)$ ketma-ketlik tuzamiz. Modomiki, M to'plam chegaralangan ekan, unda $\left\{x^{(n)}\right\}$ ketma-ketlik ham chegaralangandir. Boltsano-Veyershtrass teoremasiga (ushbu bobning 2-§ iga) ko'ra $\left\{x^{(n)}\right\}$ ketma-ketlikdan yaqinlashuvchi bo'lgan $\left\{x^{(n_k)}\right\}$ qismiy ketma-ketlik ajratish mumkin: $\left\{x^{(n_k)}\right\} \to x^0$ $(k \to \infty)$. M yopiq to'plam bo'lgani uchun $x^0 \in M$ bo'ladi. $f(x)$ funksiyaning M to'plamda uzluksiz ekanligidan esa

$$f\left(x^{(n_k)}\right) \to f\left(x^0\right)$$

bo'lishi kelib chiqadi. Natijada bir tomondan (12.18) munosabatga ko'ra

$$\left|f\left(x^{(n_k)}\right)\right| > n_k$$

ya'ni $f\left(x^{(n_k)}\right) \to \infty$ $(k \to \infty)$ bo'lsa, ikkinchi tomondan $f\left(x^{(n_k)}\right) \to f\left(x^0\right)$ bo'lib qoldi. Bunday ziddiyat $f(x)$ funksiyani M to'plamda chegaralanmagan deb olinishi oqibatida kelib chiqadi. Demak, $f(x)$ funksiya M to'plamda chegaralangan.▶

20-teorema (Veyershtrassning ikkinchi teoremasi). Agar $f(x)$ funksiya chegaralangan yopiq $M \subset R^m$ to'plamda uzluksiz bo'lsa, u shu to'plamda o'zining aniq yuqori hamda aniq quyi chegaralariga erishadi.

Bu teoremaning isboti 1-qism, 5-bob, 6-§ dagi 8-teoremaning isboti kabidir. Uni isbotlashni o'quvchiga havola etamiz.

6-§. Ko'p o'zgaruvchili funksiyaning tekis uzluksizligi. Kantor teoremasi

$f(x)$ funksiya $M \subset R^m$ to'plamda berilgan bo'lsin.

24-ta'rif. Agar $\forall \varepsilon > 0$ son uchun $\delta > 0$ topilsaki, M to'plamning $\rho(x', x'') < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy x' va x'' $(x' \in M,\ x'' \in M)$ nuqtalarida

$$\left|f(x') - f(x'')\right| < \varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya M to'plamda tekis uzluksiz funksiya deb ataladi.

Funksiyaning tekis uzluksizligi ta'rifidagi $\delta > 0$ son $\varepsilon > 0$ gagina bog'liq bo'ladi. Ravshanki, agar $f(x)$ funksiya $M \subset R^m$ to'plamda tekis uzluksiz bo'lsa, u shu to'plamda uzluksiz bo'ladi.

12.12-misol. Ushbu

$$f(x_1, x_2) = x_1^2 + x_2^2$$

funksiyaning $D = \left\{(x_1, x_2) \in R^2 :\ x_1^2 + x_2^2 \le 1\right\}$ to'plamda tekis uzluksiz bo'lishi ko'rsatilsin.

◀ $\forall\varepsilon>0$ sonni olib, unga ko'ra topiladigan $\delta>0$ sonni $\delta<\frac{\varepsilon}{4}$ deb olsak, u holda

$$\rho(x',x'')=\rho((x_1',x_2'),(x_1'',x_2''))=\sqrt{(x_1''-x_1')+(x_2''-x_2')}<\delta$$

tengsizlikni qanoatlantiruvchi $\forall(x_1',x_2')\in D,\ \forall(x_1'',x_2'')\in D$ nuqtalar uchun

$$|f(x_1',x_2')-f(x_1'',x_2'')|=\left|(x_1')^2+(x_2')^2-\left((x_1'')^2+(x_2'')^2\right)\right|=$$

$$=|(x_1'-x_1'')(x_1'+x_1'')+(x_2'-x_2'')(x_2'+x_2'')|\le 2\sqrt{(x_1'-x_1'')^2+(x_2'-x_2'')^2}+$$

$$+2\sqrt{(x_1'-x_1'')^2+(x_2'-x_2'')^2}=4\delta<\varepsilon$$

bo'ladi.

Demak, berilgan funksiya $D\subset R^2$ to'plamda tekis uzluksiz.▶

11-teorema. (Kantor teoremasi). Agar $f(x)$ funksiya chegaralangan yopiq $M\left(M\subset R^m\right)$ to'plamda uzluksiz bo'lsa, funksiya shu to'plamda tekis uzluksiz bo'ladi.

◀Teskarisini faraz qilaylik, ya'ni $f(x)$ funksiya chegaralangan yopiq M to'plamda uzluksiz bo'lsinu, ammo tekis uzluksizlik ta'rifidagi shart bajarilmasin. Bu holda biror $\varepsilon>0$ son va ixtiyoriy $\delta>0$ son uchun M to'plamda $\rho(x',x'')<\delta$ tengsizlikni qanoatlantiruvchi shunday x' va x'' $(x'\in M,\ x''\in M)$ nuqtalari topiladiki,

$$|f(x')-f(x'')|\ge\varepsilon$$

bo'ladi.

Nolga intiluvchi musbat sonlar ketma-ketligi $\delta_1,\delta_2,...,\delta_n,...$ ni olaylik:

$$\delta_n\to 0\quad(\delta_n>0,\ \ n=1,2,...)\ . \qquad (12.19)$$

Farazimizga ko'ra, yuqoridagi $\varepsilon>0$ son va ixtiyoriy $\delta_n>0,(n=1,2,...)$ uchun M to'plamda shunday $a^{(n)}$ va $в^{(n)}$ $(n=1,2,...)$ nuqtalar topiladiki,

$$\rho\left(a^{(1)},в^{(1)}\right)<\delta_1 \text{ va } \left|f\left(a^{(1)}\right)-f\left(в^{(1)}\right)\right|\ge\varepsilon$$

$$\rho\left(a^{(2)},в^{(2)}\right)<\delta_2 \text{ va } \left|f\left(a^{(2)}\right)-f\left(в^{(2)}\right)\right|\ge\varepsilon$$

....

$$\rho\left(a^{(n)},в^{(n)}\right)<\delta_n \text{ va } \left|f\left(a^{(n)}\right)-f\left(в^{(n)}\right)\right|\ge\varepsilon$$

....

bo'ladi.

Modomiki, M - chegaralangan to'plam va $a^{(n)}\in M$ $(n=1,2,...)$ ekan, unda Boltsano-Veyershtrass teoremasiga ko'ra $\left\{a^{(n)}\right\}$ ketma-ketlikdan yaqinlashuvchi qismiy $\left\{a^{(n_k)}\right\}$ ketma-ketlik ajratish mumkin:

$$\lim_{k\to+\infty}a^{(n_k)}=a^0\ . \qquad (12.20)$$

M yopiq to’plam bo’lganligi sababli $a^0 \in M$ bo’ladi. Yuqoridagi $\{в^{(n)}\}$ ketma-ketlikdan ajratilgan $\{в^{(n_k)}\}$ qismiy ketma-ketlikning limiti ham a^0 ga teng bo’ladi. Haqiqatdan ham, ushbu

$$\rho\left(в^{(n_k)},a^0\right) \leq \rho\left(в^{(n_k)},a^{(n_k)}\right) + \rho\left(a^{(n_k)},a^0\right) < \delta_{n_k} + \rho\left(a^{(n_k)},a^0\right)$$

tengsizlikdagi δ_{n_k} va $\rho\left(a^{(n_k)},a^0\right)$ lar uchun (12.19) va (12.20) munosabatlarga ko’ra $k \to \infty$ da

$$\delta_{n_k} \to 0, \quad \rho\left(a^{(n_k)},a^0\right) \to 0$$

bo’lishini e’tiborga olib, $k \to \infty$ da $\rho\left(в^{(n_k)},a^0\right) \to 0$ ekanini topamiz.

Shunday qilib,

$$a^{(n_k)} \to a^0, \quad в^{(n_k)} \to a^0.$$

Qaralayotgan $f(x)$ funksiyaning, shartga ko’ra M to’plamda uzluksiz ekanligidan

$$f\left(a^{(n_k)}\right) \to f\left(a^0\right), \quad f\left(в^{(n_k)}\right) \to f\left(a^0\right)$$

bo’lib, ulardan esa

$$f\left(в^{(n_k)}\right) - f\left(a^{(n_k)}\right) \to 0$$

bo’lishi kelib chiqadi. Bu esa $\forall n_k$ lar uchun

$$\left| f\left(в^{(n_k)}\right) - f\left(a^{(n_k)}\right) \right| \geq \varepsilon$$

deb qilingan farazga ziddir. Bunday ziddiyatning kelib chiqishga sabab $f(x)$ funksiyaning M to’plamda tekis uzluksizlik shartini qanoatlantirmaydi deb olinishidir. Demak, funksiya M to’plamda tekis uzluksiz.▶

Biror $M \subset R^m$ to’plam berilgan bo’lsin. Bu to’plamda ixtiyoriy ikkita x' va x'' nuqtalarni olib, ular orasidagi $\rho(x',x'')$ masofani topamiz. Agar x' va x'' nuqtalarni M to’plamda o’zgartira borsak, unda $\{\rho(x',x'')\}$ to’plam hosil bo’ladi. Odatda, bu to’plamning aniq yuqori chegarasi $sup\{\rho(x',x'')\}$ $(x' \in M,\ x'' \in M)$ to’plamning diametri deb ataladi va u $d(M)$ kabi belgilanadi:

$$d(M) = sup\{\rho(x',x'')\} \quad (x' \in M,\ x'' \in M).$$

25-ta’rif. Ushbu

$$sup\{|f(x'') - f(x')|\} \quad (x' \in M,\ x'' \in M)$$

miqdor $f(x)$ funksiyaning M to’plamdagi tebranishi deb ataladi va u $\omega(f,M)$ kabi belgilanadi:

$$\omega(f;M) = sup\{|f(x'') - f(x')|\} \quad (x' \in M,\ x'' \in M)$$

Yuqorida keltirilgan Kantor teoremasidan muhim natija kelib chiqadi.

2-natija. $f(x)$ funksiya chegaralangan yopiq to’plamda uzluksiz bo’lsin. U holda $\forall \varepsilon > 0$ son olinganda ham M to’plamni chekli sondagi M_k to’plamlarga shunday ajratish mumkinki,

$$\bigcup_k M_k = M,$$
$$M_k \cap M_i = \phi \qquad (k \neq i)$$
$$\omega(f, M_k) < \varepsilon$$
bo'ladi.

◀ $f(x)$ funksiya chegaralngan yopiq M to'plamda uzluksiz bo'lsin. Kantor teoremasiga ko'ra tekis uzluksiz bo'ladi. Binobarin, $\forall\varepsilon > 0$ son uchun shunday $\delta > 0$ topiladiki, $\rho(x', x'') < \delta$ bo'lgan $\forall x', x'' \in M$ uchun
$$|f(x') - f(x'')| < \varepsilon$$
bo'ladi.

M to'plamni diametrlari shu δ bo'lgan M_k to'plamlarga ajratamiz. Ravshanki, bu holda
$$\rho(x', x'') < \delta \qquad (x', x'' \in M_k)$$
bo'ladi va demak,
$$|f(x'') - f(x')| < \varepsilon$$
tengsizlik bajariladi. Bundan
$$sup\{|f(x'') - f(x')|\} \leq \varepsilon$$
ya'ni $\omega(f, M_k) \leq \varepsilon$ bo'lishi kelib chiqadi. ▶

Mashqlar

12.13. R^2 va R^3 to'plamlarda ikki nuqta orasidagi masofa yozilsin va masofaning 3 ta xossasi isbotlansin.

12.14. R^2 va R^3 fazolarda ochiq shar va sharlarning geometrik tasvirlari keltirilsin.

12.15. R^m fazodagi $\{x^{(n)}\}$ ketma-ketlik yaqinlashuchi bo'lsa, uning chegaralanganligi isbotlansin.

12.16. Ikki va uch o'zgaruvchili funksiyalarning ta'riflari keltirilsin.

12.17. Ushbu
$$f(x_1, x_2) = arcsin(x_1 + x_2)$$
funksiyaning aniqlanish to'plami topilsin.

12.18. Ushbu
$$f(x_1, x_2) = \begin{cases} x_1 + x_2\, sin\dfrac{1}{x_1}, & \text{agar } x_1 \neq 0 \text{ bo'lsa,} \\ 0, & \text{agar } x_1 = 0 \text{ bo'lsa} \end{cases}$$
funksiya uchun
$$\lim_{\substack{x_1 \to 0 \\ x_2 \to 0}} f(x_1, x_2) = 0$$
bo'lishi isbotlansin.

12.19. Ushbu
$$f(x_1, x_2) = \begin{cases} (x_1 + x_2) sin\dfrac{1}{x_1}\, sin\dfrac{1}{x_2}, & \text{agar } (x_1, x_2) \neq (0, 0) \text{ bo'lsa,} \\ 0, & \text{agar } (x_1, x_2) = (0, 0) \text{ bo'lsa} \end{cases}$$

funksiyaning $(0,\,0)$ nuqtada takroriy limitlarining mavjud emasligi isbotlansin.

12.20. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{1}{sin^2\pi x_1+sin^2\pi x_2}, & \text{agar } sin^2\pi x_1+sin^2\pi x_2\neq 0 \text{ bo'lsa,}\\ 0, & \text{agar } sin^2\pi x_1+sin^2\pi x_2=0 \text{ bo'lsa}\end{cases}$$

funksiya uzluksizlikka tekshirilsin.

13-BOB

Ko'p o'zgaruvchili funksiyaning hosila va differensiallari

Ushbu bobda biz ko'p o'zgaruvchili funksiyalar differensial hisobi bilan shug'ullanamiz. Kiritiladigan va o'rganiladigan hosilalar va differensiallar tushunchalari bir o'zgaruvchining funksiyalari uchun kiritilgan mos tushunchalarning tegishlicha umumlashtirilishidan iborat bo'ladi. Ayni paytda, biz ko'ramizki, ko'p o'zgaruvchili funksiyalar uchun xos bo'lgan bir qancha yangi tushunchalar ham (yo'nalish bo'yicha hosila, to'la differensial va xokazo) o'rganiladi.

1-§. Ko'p o'zgaruvchili funksiyaning hosilalari

1^0. Funksiya xususiy hosilasining ta'rifi. $f(x)=f(x_1,x_2,...,x_m)$ funksiya ochiq $M\left(M\subset R^m\right)$ to'plamda berilgan bo'lsin. Bu to'plamda $x^0=\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqta olib, uning birinchi koordinatasi x_1^0 ga shunday Δx_1 $(\Delta x_1 \gtrless 0)$ orttirma beraylikki, $\left(x_1^0+\Delta x_1,x_2^0,...,x_m^0\right)\in M$ bo'lsin. Natijada $f(x_1,x_2,...,x_m)$ funksiya ham $\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqtada x_1 o'zgaruvchisi bo'yicha

$$\Delta_{x_1} f = f\left(x_1^0+\Delta x_1,x_2^0,...,x_m^0\right)-f\left(x_1^0,x_2^0,...,x_m^0\right)$$

xususiy orttirmaga ega bo'ladi.

1-ta'rif. Agar $\Delta x_1\to 0$ da ushbu limit

$$\lim_{\Delta x_1\to 0}\frac{\Delta x_1 f}{\Delta x_1}=\lim_{\Delta x_1\to 0}\frac{f\left(x_1^0+\Delta x_1,x_2^0,...,x_m^0\right)-f\left(x_1^0,x_2^0,...,x_m^0\right)}{\Delta x_1}$$

mavjud va chekli bo'lsa, bu limit $f(x_1,x_2,...,x_m)$ funksiyaning $\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqtada x_1 o'zgaruvchisi bo'yicha xususiy hosilasi deb ataladi va

$$\frac{\partial f\left(x_1^0,x_2^0,...,x_m^0\right)}{\partial x_1},\quad \frac{\partial f}{\partial x_1},\quad f'_{x_1}\left(x_1^0,x_2^0,...,x_m^0\right),\quad f'_{x_1}$$

belgilarning biri bilan belgilanadi. Demak,

$$f'_{x_1}\left(x^0\right)=\frac{\partial f\left(x^0\right)}{\partial x_1}=\lim_{\Delta x_1\to 0}\frac{\Delta x_1 f}{\Delta x_1}.$$

$f(x_1,x_2,...,x_m)$ funksiyaning $(x_1^0,x_2^0,...,x_m^0)$ nuqtada x_1 o'zgaruvchisi bo'yicha xususiy hosilasini quyidagi

$$f'_{x_1}(x^0)=\lim_{x_1\to x_1^0}\frac{f(x_1,x_2^0,...,x_m^0)-f(x_1^0,x_2^0,...,x_m^0)}{x_1-x_1^0}$$

ham ta'riflash mumkin.

Xuddi shunga o'xshash $f(x_1,x_2,...,x_m)$ funksiyaning boshqa o'zgaruvchilari buyicha xususiy hosilalari ta'riflanadi:

$$\frac{\partial f}{\partial x_2}=\lim_{\Delta x_2\to 0}\frac{\Delta x_2 f}{\Delta x_2}=\lim_{\Delta x_2\to 0}\frac{f(x_1^0,x_2^0+\Delta x_2,x_3^0,...,x_m^0)-f(x_1^0,x_2^0,...,x_m^0)}{\Delta x_2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$\frac{\partial f}{\partial x_m}=\lim_{\Delta x_m\to 0}\frac{\Delta x_m f}{\Delta x_m}=\lim_{\Delta x_m\to 0}\frac{f(x_1^0,x_2^0,...,x_{m-1}^0,x_m^0+\Delta x_m)-f(x_1^0,x_2^0,...,x_m^0)}{\Delta x_m}.$$

Demak, ko'p o'zgaruvchili $f(x_1,x_2,...,x_m)$ funksiyaning biror $(x_1^0,x_2^0,...,x_m^0)$ nuqtada x_k $(k=1,2,...,m)$ o'zgaruvchisi bo'yicha xususiy hosilasini ta'riflashda bu funksiyaning x_k $(k=1,2,...,m)$ o'zgaruvchidan boshqa barcha o'zgaruvchilari o'zgarmas deb hisoblanar ekan. Shunday qilib, $f(x_1,x_2,...,x_m)$ funksiyaning xususiy hosilalari $f'x_1, f'x_2,..., f'x_m$ 1-qism, 6-bob, 1-§ da o'rganilgan hosila – bir o'zgaruvchili funksiya hosilasi kabi ekanligini ko'ramiz. Demak, ko'p o'zgaruvchili funksiyalarning xususiy hosilalarini hisoblashda bir o'zgaruvchili funksiyaning hosilasini hisoblashdagi ma'lum bo'lgan qoida va jadvallardan to'liq foydalanish mumkin.

13.1-misol. Ushbu

$$f(x_1,x_2)=\frac{1}{\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}}$$

funksiyaning $(x_1,x_2)\in R^2$ $(x_2>0)$ nuqtadagi xususiy hosilalari topilsin.

◀ $$\frac{\partial f}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}}\right)=-\frac{1}{2\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}},$$

$$\frac{\partial f}{\partial x_2}=\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}}\right)=-\frac{1}{2\sqrt{x_2^3}}e^{-\frac{x_1+x_2}{2}}-\frac{1}{2\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}}=$$

$$=-\frac{1}{2\sqrt{x_2}}e^{-\frac{x_1+x_2}{2}}\left(1+\frac{1}{x_2}\right).$$ ▶

13.2-misol. Ushbu

$$f(x_1,x_2)=\begin{cases}\dfrac{2x_1x_2}{x_1^2+x_2^2}, \text{agar } (x_1,x_2)\neq(0,0) \text{ bo'lsa,}\\ 0 \quad, \text{agar } (x_1,x_2)=(0,0) \text{ bo'lsa}\end{cases}$$

funksiyaning xususiy hosilalari topilsin.

◀ Aytaylik, $(x_1,x_2)\neq(0,0)$ bo'lsin. U holda

$$\frac{\partial f}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{2x_1x_2}{x_1^2+x_2^2}\right)=\frac{2x_2\left(x_1^2+x_2^2\right)-2x_1x_2\,2x_1}{\left(x_1^2+x_2^2\right)^2}=\frac{2x_2\left(x_2^2-x_1^2\right)}{\left(x_1^2+x_2^2\right)^2},$$

$$\frac{\partial f}{\partial x_2}=\frac{\partial}{\partial x_2}\left(\frac{2x_1x_2}{x_1^2+x_2^2}\right)=\frac{2x_1\left(x_1^2+x_2^2\right)-2x_1x_2\,2x_2}{\left(x_1^2+x_2^2\right)^2}=\frac{2x_1\left(x_1^2-x_2^2\right)}{\left(x_1^2+x_2^2\right)^2}$$

bo'ladi.

Endi $(x_1,x_2)=(0,0)$ bo'lsin. U holda ta'rifga binoan

$$\frac{\partial f(0,0)}{\partial x_1}=\lim_{\Delta x_1\to 0}\frac{f(\Delta x_1,0)-f(0,0)}{\Delta x_1}=0,$$

$$\frac{\partial f(0,0)}{\partial x_2}=\lim_{\Delta x_2\to 0}\frac{f(0,\Delta x_2)-f(0,0)}{\Delta x_2}=0$$

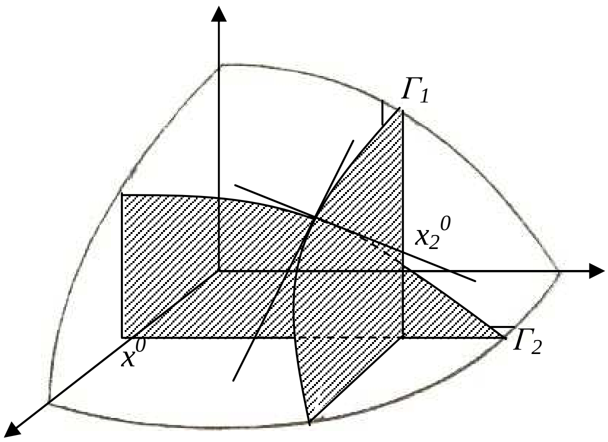
bo'ladi.

Demak, berilgan $f(x_1,x_2)$ funksiya $\forall(x_1,x_2)\in \mathrm{R}^2$ da xususiy hosilalarga ega.▶

2⁰. Xususiy hosilaning geometrik ma'nosi. Soddalik uchun ikki o'zgaruvchili funksiya xususiy hosilalarining geometrik ma'nosini keltiramiz.

$f(x_1,x_2)$ funksiya ochiq $M\left(M\subset R^2\right)$ to'plamda berilgan bo'lib, $\left(x_1^0,x_2^0\right)\in M$ bo'lsin. Bu funksiya $\left(x_1^0,x_2^0\right)$ nuqtada $f'_{x_1}\left(x_1^0,x_2^0\right)$, $f'_{x_2}\left(x_1^0,x_2^0\right)$ xususiy hosilalarga ega deylik. Ta'rifga ko'ra $f'_{x_1}\left(x_1^0,x_2^0\right)$ va $f'_{x_2}\left(x_1^0,x_2^0\right)$ xususiy hosilalar mos ravishda ushbu $y_1=f\left(x_1,x_2^0\right)$ va $y_2=f\left(x_1^0,x_2\right)$ bir o'zgaruvchili funksiyalarning x_1^0 va x_2^0 dagi hosilalaridan iborat.

Faraz qilaylik, $y=f\left(x_1,x_2\right)$ funksiyaning grafigi 48-chizmada ko'rsatilgan sirtni tasvirlasin.



48-chizma

Unda $y_1=f\left(x_1,x_2^0\right)$ va $y_2=f\left(x_1^0,x_2\right)$ funksiyalarning grafiklari mos ravishda $y=f\left(x_1,x_2\right)$ sirt bilan $x_2=x_2^0$ tekislikning hamda shu sirt bilan $x_1=x_1^0$ tekislikning kesishidan hosil bo'lgan Γ_1 va Γ_2 chiziqlardan iborat.

Ma’lumki, bir o’zgaruvchili $u=\varphi(x)$ funksiyaning biror $x_0\,(x_0\in R)$ nuqtadagi hosilasining geometrik ma’nosi (1-qism, 6-bob, 1-§) bu funksiya tasvirlangan egri chiziqqa $(x_0,\varphi(x_0))$ nuqtada o’tkazilgan urinmaning burchak koeffitsientidan iborat edi. $f'_{x_1}\left(x_1^0,x_2^0\right)$ va $f'_{x_2}\left(x_1^0,x_2^0\right)$ xususiy hosilalar mos ravishda Γ_1 va Γ_2 egri chiziqlarga $\left(x_1^0,x_2^0\right)$ nuqtada o’tkazilgan urinmalarning ox_1 va ox_2 o’qlari bilan tashkil etgan burchakning tangensini bildiradi. Demak, $f'_{x_1}\left(x_1^0,x_2^0\right)$ va $f'_{x_2}\left(x_1^0,x_2^0\right)$ xususiy hosilalar $y=f\left(x_1,x_2\right)$ sirtning mos ravishda ox_1 va ox_2 o’qlar yo’nalishi bo’yicha o’zgarish darajasini ko’rsatadi.

2-§. Ko’p o’zgaruvchili funksiyalarning differensiallanuvchiligi

1⁰. Funksiyaning differensiallanuvchiligi tushunchasi. Differensiallanuvchilikning zaruriy sharti. $f(x)=f\left(x_1,x_2,\ldots,\ ,x_m\right)$ funksiya ochiq $M\left(M\subset R^m\right)$ to’plamda berilgan bo’lsin. Bu to’plamda $x^0=\left(x_1^0,x_2^0,\ldots,x_m^0\right)$ nuqta bilan birga $\left(x_1^0+\Delta x_1,x_2^0+\Delta x_m,\ldots,x_m^0+\Delta x_m\right)$ nuqtani olib, berilgan funksiyaning to’la orttirmasi

$$\Delta f\left(x^0\right)=f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,\ldots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,\ldots,x_m^0\right)$$

ni qaraymiz.

Ravshanki, funksiyaning $\Delta f\left(x_0\right)$ orttirmasi argumentlar orttirmalari $\Delta x_1,\Delta x_2,\ldots,\Delta x_m$ larga bog’liq.

2-ta’rif. Agar $f(x)$ funksiyaning x^0 nuqtadagi $\Delta f\left(x_0\right)$ orttirmasini

$$\Delta f\left(x^0\right)=A_1\Delta x_1+A_2\Delta x_2+\ldots+A_m\Delta x_m+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\ldots+\alpha_m\Delta x_m \qquad (13.1)$$

ko’rinishda ifodalash mumkin bo’lsa, $f(x)$ funksiya x^0 nuqtada differensiallanuvchi deb ataladi, bunda $A_1,A_2,\ldots,A_m$ lar $\Delta x_1,\Delta x_2,\ldots,\Delta x_m$ larga bog’liq bo’lmagan o’zgarmaslar, $\alpha_1,\alpha_2,\ldots,\alpha_m$ lari esa $\Delta x_1,\Delta x_2,\ldots,\Delta x_m$ larga bog’liq va $\Delta x_1\to 0,\Delta x_2\to 0,\ldots,\Delta x_m\to 0$ da $\alpha_1\to 0,\alpha_2\to 0,\ldots,\alpha_m\to 0$ ($\Delta x_1=\Delta x_2=,\ldots,\Delta x_m=0$ bo’lganda $\alpha_1=\alpha_2=\ldots=\alpha_m=0$ deb olinadi.)

Agar $f(x)$ funksiya M to’plamning har bir nuqtasida differensiallanuvchi bo’lsa, $f(x)$ funksiya M to’plamda differensiallanuvchi deb ataladi.

13.3-misol. Ushbu $f\left(x_1,x_2\right)=x_1^2+x_2^2$ funksiyani $\forall\left(x_1^0,x_2^0\right)\in R^2$ nuqtada differensiallanuvchi bo’lishi ko’rsatilsin.

◄Haqiqatdan ham, $\left(x_1^0,x_2^0\right)$ nuqtada funksiyaning orttirmasi

$$\Delta f=f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2\right)-f\left(x_1^0,x_2^0\right)=\left(x_1^0+\Delta x_1\right)^2+\left(x_2^0+\Delta x_2\right)^2-$$

$$-\left(x_1^0\right)^2-\left(x_2^0\right)^2=2x_1^0\Delta x_1+2x_2^0\Delta x_2+\left(\Delta x_1\right)^2+\left(\Delta x_2\right)^2$$

bo’lib, unda $A_1=2x_1^0$, $A_2=2x_2^0$, $\alpha_1=\Delta x_1$, $\alpha_2=\Delta x_2$ deyilsa, natijada

$$\Delta f=A_1\Delta x_1+A_2\Delta x_2+\alpha_1\Delta x_1+\alpha_2\Delta x_2$$

bo'ladi. Bu esa berilgan funksiyaning $\forall(x_1, x_2) \in R^2$ nuqtada differensiallanuvchi ekanligini bildiradi.▶

$f(x)$ funksiyaning x^0 nuqtada differensiallanuvchilik sharti (13.1) ni quyidagi

$$\Delta f(x^0) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + o(\rho) \qquad (13.2)$$

ko'rinishda ham yozish mumkin, bunda

$$\rho = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + ... + (\Delta x_m)^2}\,.$$

Endi differensiallanuvchi funksiyalar haqida ikkita teorema keltiramiz.

1-teorema. Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsa, u holda bu funksiya shu nuqtada uzluksiz bo'ladi.

◀ $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsin. U holda ta'rifga ko'ra funksiya orttirmasi uchun

$$\Delta f(x^0) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m$$

bo'ladi, bunda $A_1, A_2, ..., A_m$ o'zgarmas, $\Delta x_1 \to 0, \Delta x_2 \to 0, ..., \Delta x_m \to 0$ da $\alpha_1 \to 0$, $,\alpha_2 \to 0, ..., \alpha_m \to 0$.

Yuqoridagi tenglikdan

$$\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0 \\ .... \\ \Delta x_m \to 0}} \Delta f(x^0) = 0$$

bo'lishi kelib chiqadi. Bu esa $f(x)$ funksiya x^0 nuqtada uzluksizligini bildiradi.▶

2-teorema. Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsa, u holda bu funksiyaning shu nuqtada barcha xususiy hosilalari $f'_{x_1}(x^0), f'_{x_2}(x^0), ..., f'_{x_m}(x^0)$ mavjud va ular mos ravishda (13.1) munosabatdagi $A_1, A_2, ..., A_m$ larga teng bo'ladi:

$$f'_{x_1}(x^0) = A_1, \quad f'_{x_2}(x^0) = A_2, ..., f'_{x_m}(x^0) = A_m\,.$$

◀ $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'lsin.
U holda ta'rifga ko'ra

$$\Delta f(x^0) = A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m$$

bo'ladi. Bu tenglikda

$$\Delta x_1 \neq 0, \quad \Delta x_2 = \Delta x_3 = ... = \Delta x_m = 0$$

deb olsak, unda (13.1) ushbu

$$\Delta x_1 f(x^0) = A_1\Delta x_1 + \alpha_1\Delta x_1$$

ko'rinishni oladi. Bu tenglikdan quyidagini topamiz:

$$\lim_{\Delta x_1 \to 0} \frac{\Delta x_1 f(x_0)}{\Delta x_1} = \lim_{\Delta x_1 \to 0} (A_1 + \alpha_1) = A_1\,.$$

Demak,

$$f'_{x_1}(x_0) = A_1\,.$$

Xuddi shunga o’xshash $f(x)$ funksiya x^0 nuqtada $f'_{x_2}(x^0), f'_{x_3}(x^0), ..., f'_{x_m}(x^0)$ xususiy hosilalarining mavjudligi hamda

$$f'_{x_2}(x^0) = A_2, \quad f'_{x_3}(x^0) = A_3, ..., f'_{x_m}(x^0) = A_m .$$

ekanligi ko’rsatiladi.▶

1-natija. Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo’lsa, u holda

$$\Delta f(x_0) = f'_{x_1}(x^0)\Delta x_1 + f'_{x_2}(x^0)\Delta x_2 + ... + f'_{x_m}(x^0)\Delta x_m + o(\rho)$$

bo’ladi.

1-eslatma. $f(x)$ funksiyaning biror x^0 nuqtada barcha xususiy hosilalari $f'_{x_1}(x^0), f'_{x_2}(x^0), ..., f'_{x_m}(x^0)$ ning mavjud bo’lishidan funksiyaning shu nuqtada differensiallanuvchi bo’lishi har doim kelib chiqavermaydi.

13.4-misol. Ushbu

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}, \text{agar } (x_1, x_2) \neq (0,\ 0) \\ 0 \quad , \text{agar } (x_1, x_2) = (0,\ 0) \end{cases}$$

funksiyaning $(0,0)$ nuqtada differensiallanuvchi emasligi ko’rsatilsin.

◀Bu funksiya $(0,0)$ nuqtada xususiy hosilalarga ega:

$$f'_{x_1}(0,0) = \lim_{\Delta x_1 \to 0} \frac{f(\Delta x_1, 0) - f(0,0)}{\Delta x_1} = 0,$$

$$f'_{x_2}(0,0) = \lim_{\Delta x_2 \to 0} \frac{f(0, \Delta x_1) - f(0,0)}{\Delta x_2} = 0.$$

Berilgan funksiyaning $(0,0)$ nuqtada orttirmasi

$$\Delta f(0,0) = f(\Delta x_1, \Delta x_2) - f(0,0) = \frac{\Delta x_1 \Delta x_2}{\sqrt{\Delta x_1^2 + \Delta x_2^2}}$$

bo’lib, uni (13.1) yoki (13.2) ko’rinishida ifodalab bo’lmaydi. Buni isbotlash maqsadida, teskarisini, ya’ni $f(x_1, x_2)$ funksiya $(0,0)$ nuqtada differensiallanuvchi bo’lsin deb faraz qilaylik. Unda

$$\Delta f(0,0) = f'_{x_1}(0,0)\Delta x_1 + f'_{x_2}(0,0)\Delta x_2 + \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 = \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2,$$

bo’lib, bu munosabatda $\Delta x_1 \to 0, \Delta x_2 \to 0$ da $\alpha_1 \to 0, \alpha_2 \to 0$ bo’lishi lozim. Demak,

$$\frac{\Delta x_1, \Delta x_2}{\sqrt{\Delta x_1^2 + \Delta x_2^2}} = \alpha_1 \Delta x_1 + \alpha_2 \Delta x_2. \qquad (13.3)$$

Ma’lumki, Δx_1 va Δx_2 lar ixtiyoriy orttirmalar. Jumladan, $\Delta x_1 = \Delta x_2$ bo’lganda (13.3) tenglik ushbu

$$\frac{\Delta x_1}{\sqrt{2}} = \Delta x_1(\alpha_1 + \alpha_2)$$

ko’rinishga kelib, undan esa

$$\alpha_1+\alpha_2=\frac{\sqrt{2}}{2}$$

bo'lishi kelib chiqadi. Natijada $\Delta x_1 \to 0, \Delta x_2 \to 0$ da α_1 va α_2 miqdorlarning nolga intilmasligini topamiz. Bu esa $f(x_1,x_2)$ funksiyaning $(0,0)$ nuqtada differensiallanuvchi bo'lsin deb qilingan farazga zid.▶

Shunday qilib, funksiyaning biror nuqtada barcha xususiy hosilalarga ega bo'lishi, funksiyaning shu nuqtada differensiallanuvchi bo'lishining zaruriy shartidan iborat ekan.

2^0. Funksiyaning differensiallanuvchiligining etarli sharti. Endi ko'p o'zgaruvchili funksiya differensiallanuvchi bo'lishining etarli shartini keltiramiz.

$f(x)=f(x_1,x_2,...,x_m)$ funksiya ochiq $M\left(M\subset R^m\right)$ to'plamda berilgan bo'lib, $x^0=\left(x_1^0,x_2^0,...,x_m^0\right)\in M$ bo'lsin.

3-teorema. Agar $f(x)$ funksiya x^0 nuqtaning biror atrofida barcha o'zgaruvchilari bo'yicha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar shu x^0 nuqtada uzluksiz bo'lsa, $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo'ladi.

◀ $x^0\in M$ nuqtani olib, koordinatalariga mos ravishda shunday $\Delta x_1,\Delta x_2,...,\Delta x_m$ orttirmalar beraylikki, $\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m\right)$ nuqta x^0 nuqtaning aytilgan atrofiga tegishli bo'lsin. So'ng funksiya to'la orttirmasi

$$\Delta f\left(x^0\right)=f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,...,x_m^0\right)$$

ni quyidagi yozib olamiz:

$$\begin{aligned}\Delta f\left(x^0\right)=&\left[f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m\right)\right]+\\&+\left[f\left(x_1^0,x_2^0+\Delta x_2,x_3^0+\Delta x_3,...,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,x_3^0+\Delta x_3,...,x_m^0+\Delta x_m\right)\right]+\\&+...+\left[f\left(x_1^0,x_2^0,...,x_{m-1}^0,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,...,x_m^0\right)\right].\end{aligned}$$

Bu tenglikning o'ng tomonidagi har bir ayirma tegishli bitta argumentning funksiyasi orttirmasi sifatida qaralishi mumkin. Uning uchun Lagranj teoremasini tatbiq qila olamiz, chunki teoremamizda keltirilgan shartlar Lagranj teoremasi shartlarining bajarilishni ta'minlaydi:

$$\begin{aligned}\Delta f\left(x^0\right)=&f'_{x_1}\left(x_1^0+\theta_1\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m\right)\cdot\Delta x_1+\\&+f'_{x_2}\left(x_1^0,x_2^0+\theta_2\Delta x_2,x_3^0+\Delta x_3,...,x_m^0+\Delta x_m\right)\cdot\Delta x_2+\\&+...+\\&+f'_{x_m}\left(x_1^0,x_2^0,...,x_{m-1}^0,x_m^0+\theta_m\Delta x_m\right)\cdot\Delta x_m,\end{aligned}\qquad (13.4)$$

bunda

$$0<\theta_i<1 \qquad (i=1,\ 2,\ ...,\ m).$$

Odatda (13.4) funksiya orttirmasining formulasi deb ataladi. Shartga ko'ra x^0 nuqtada $f'_{x_1},f'_{x_2},...,f'_{x_m}$ xususiy hosilalar uzluksiz. Shunga ko'ra

$$f'_{x_1}\left(x_1^0+\theta_1\Delta x_1, x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)=f'_{x_1}\left(x^0\right)+\alpha_1,$$
$$f'_{x_2}\left(x_1^0, x_2^0+\theta_2\Delta x_2, x_3^0+\Delta x_3,\dots,x_m^0+\Delta x_m\right)=f'_{x_2}\left(x^0\right)+\alpha_2, \qquad (13.5)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$f'_{x_m}\left(x_1^0, x_2^0,\dots,x_{m-1}^0, x_m^0+\theta_m\Delta x_m\right)=f'_{x_m}\left(x^0\right)+\alpha_m$$

bo’lib, unda $\Delta x_1 \to 0, \Delta x_2 \to 0,\dots,\Delta x_m \to 0$ da $\alpha_1 \to 0, \alpha_2 \to 0,\dots,\alpha_m \to 0$ bo’ladi.

(13.4) va (13.5) munosabatlardan

$$\Delta f\left(x^0\right)=f'_{x_1}\left(x^0\right)\Delta x_1+f'_{x_2}\left(x^0\right)\Delta x_2+\dots+f'_{x_m}\left(x^0\right)\Delta x_m+$$
$$+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\dots+\alpha_m\Delta x_m$$

bo’lishi kelib chiqadi. Bu esa $f(x)$ funksiyaning x^0 nuqtada differensiallanuvchi ekanligini bildiradi.▶

Bir va ko’p o’zgaruvchili funksiyalarda funksiyaning differensiallanuvchiligi tushunchasi kiritildi. (qaralsin, 1-qism, 6-bob, 4-§ xamda ushbu bobning 2-§.). Ularni solishtirib quyidagi xulosalarga kelamiz.

1) Bir o’zgaruvchili funksiyalarda ham ko’p o’zgaruvchili funksiyalarda ham funksiyaning biror nuqtada differensiallanuvchi bo’lishidan uning shu nuqtada uzluksiz bo’lishi kelib chiqadi. Demak, bir va ko’p o’zgaruvchili funksiyalarda funksiyaning differensiallanuvchi bo’lishi bilan uning uzluksiz bo’lishi orasidagi munosabat bir xil.
2) Ma’lumki, bir o’zgaruvchili funksiyalarda funksiyaning biror nuqtada differensiallanuvchi bo’lishidan uning shu nuqtada chekli hosilaga ega bo’lishi kelib chiqadi va aksincha, funksiyaning biror nuqtada chekli hosilaga ega bo’lishidan uning shu nuqtada differensiallanuvchi bo’lishi kelib chiqadi.

Ko’p o’zgaruvchili funksiyalarda funksiyaning biror nuqtada differensiallanuvchi bo’lishidan uning shu nuqtada barcha chekli xususiy hosilalarga ega bo’lishi kelib chiqadi. Biroq funksiyaning biror nuqtada barcha chekli xususiy hosilalarga ega bo’lishidan uning shu nuqtada differensiallanuvchi bo’lishi har doim kelib chiqavermaydi.

Demak, bir va ko’p o’zgaruvchili funksiyalarda funksiyaning differensiallanuvchi bo’lishi bilan uning hosilaga (xususiy hosilaga) ega bo’lishi orasidagi munosabat bir xil emas ekan.

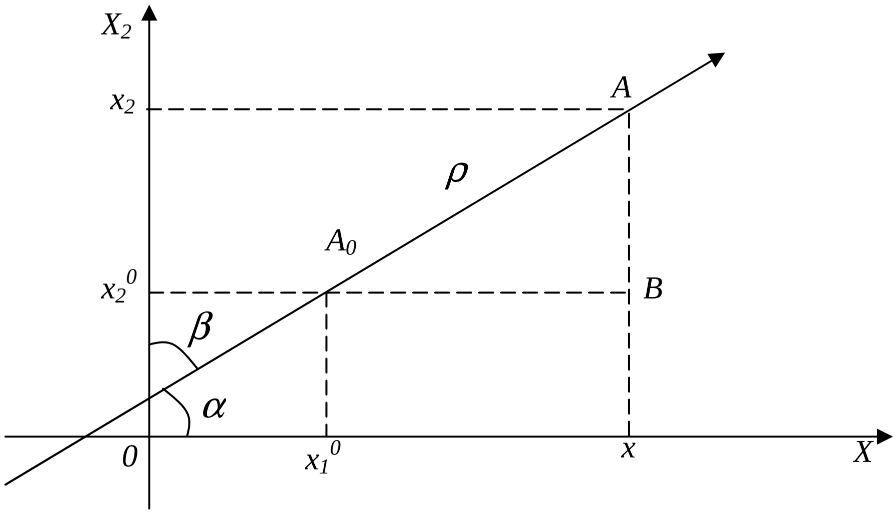
3-§. Yo’nalish bo’yicha hosila

Ma’lumki, bir o’zgaruvchili $y=f(x)$ funksiyaning $\left(x\in R, y\in R\right)$ $\frac{df}{dx}$ hosilasi bu funksiyaning o’zgarish tezligini bildirar edi. Ko’p o’zgaruvchili $y=f\left(x_1,x_2,\dots,x_m\right)$ funksiyaning xususiy hosilalari ham bir o’zgaruvchili funksiyaning hosilasi kabi ekanligini e’tiborga olib, bu $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\dots,\frac{\partial f}{\partial x_m}$ xususiy hosilalar ham $y=f\left(x_1,x_2,\dots,x_m\right)$ funksiyaning mos ravishda $ox_1, ox_2,\dots,ox_m$ o’qlar bo’yicha o’zgarish tezligini ifodalaydi deb aytish mumkin.

Endi funksiyaning ixtiyoriy yo'nalish bo'yicha o'zgarish tezligini ifodalovchi tushuncha bilan tanishaylik. Soddalik uchun ikki o'zgaruvchili funksiyani qaraymiz.

$y = f(x_1, x_2) = f(A)$ funksiya ochiq M to'plamda $(M \subset R^2)$ berilgan bo'lsin. Bo' to'plamda ixtiyoriy $A_0 = (x_1^0, x_2^0)$ nuqtani olib, u orqali biror to'g'ri chiziq o'tkazaylik va undagi ikki yo'nalishdan birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik. Bu yo'nalgan to'g'ri chiziqni l deylik.

α va β deb l yo'nalgan to'g'ri chiziq musbat yo'nalishi bilan mos ravishda ox_1 va ox_2 koordinata o'qlarining musbat yo'nalishi orasidagi burchaklarni olaylik (49-chizma).



49-chizma

l to'g'ri chiziqda A_0 nuqtadan farqli va M to'plamga tegishli bo'lgan A nuqtani $(A = (x_1, x_2))$ olaylikki, A_0A kesma M to'plamga tegishli bo'lsin. Agarda A nuqta A_0 ga nisbatan l to'g'ri chiziqning musbat yo'nalishi tomonidan bo'lsa (shakldagidek), u holda A_0A kesma uzunligi $\rho(A_0, A)$ ni musbat ishora bilan olishga kelishaylik.

ΔA_0AB dan

$$\frac{x_1 - x_1^0}{\rho} = cos\,\alpha, \qquad \frac{x_2 - x_2^0}{\rho} = cos\,\beta \qquad (13.6)$$

bo'lishi kelib chiqadi. Odatda $cos\,\alpha$ va $cos\,\beta$ lar l to'g'ri chiziqning yo'naltiruvchi kosinuslari deyiladi.

3-ta'rif. A nuqta l yo'nalgan to'g'ri chiziq bo'ylab A_0 nuqtaga intilganda $(A \to A_0)$ ushbu nisbat

$$\frac{f(A) - f(A_0)}{\rho(A_0, A)} = \frac{f(x_1, x_2) - f(x_1^0, x_2^0)}{\rho((x_1^0, x_2^0), (x_1, x_2))}$$

ning limiti mavjud bo'lsa, bu limit $f(x_1, x_2) = f(A)$ funksiyaning $A_0 = (x_1^0, x_2^0)$ nuqtadagi l yo'nalish bo'yicha hosilasi deb ataladi va

$$\frac{df(A_0)}{dl} \text{ yoki } \frac{df(x_1^0, x_2^0)}{dl}$$

kabi belgilanadi. Demak,

$$\frac{df}{dl}=\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0,A)}.$$

Endi $f(x_1,x_2)$ funksiyaning l yo'nalish bo'yicha hosilasining mavjudligi hamda uni topish masalasi bilan shug'ullanamiz.

4-teorema. $f(x_1,x_2)$ funksiya ochiq M to'plamda $(M\subset R^2)$ berilgan bo'lib, $A_0=(x_1^0,x_2^0)$ nuqtada $((x_1^0,x_2^0)\in M)$ differensiallanuvchi bo'lsa, funksiya shu nuqtada har qanday yo'nalish bo'yicha hosilaga ega va

$$\frac{df(x_1^0,x_2^0)}{dl}=\frac{\partial f(x_1^0,x_2^0)}{\partial x_1}\cos\alpha+\frac{\partial f(x_1^0,x_2^0)}{\partial x_2}\cos\beta \qquad (13.7)$$

bo'ladi.

◀Shartga ko'ra $f(x_1,x_2)$ funksiya $A_0=(x_1^0,x_2^0)$ nuqtada differensiallanuvchi. Demak, funksiya orttirmasi

$$f(A)-f(A_0)=f(x_1,x_2)-f(x_1^0,x_2^0)$$

uchun

$$f(A)-f(A_0)=\frac{\partial f(x_1^0,x_2^0)}{\partial x_1}(x_1-x_1^0)+\frac{\partial f(x_1^0,x_2^0)}{\partial x_2}(x_2-x_2^0)+o(\rho) \qquad (13.8)$$

bo'ladi, bunda

$$\rho=\rho(A_0,A)=\sqrt{(x_1-x_1^0)^2+(x_2-x_2^0)^2}.$$

(13.8) tenglikning har ikki tomonini $\rho=\rho(A_0,A)$ ga bo'lib, so'ng (13.6) ni e'tiborga olib topamiz.

$$\frac{f(A)-f(A_0)}{\rho(A_0,A)}=\frac{\partial f(x_1^0,x_2^0)}{\partial x_1}\cos\alpha+\frac{\partial f(x_1^0,x_2^0)}{\partial x_2}\cos\beta+\frac{o(\rho)}{\rho}.$$

Bu tenglikda $A\to A_0$ da (ya'ni $\rho\to o$ da) limitga o'tsak, unda

$$\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0,A)}=\lim_{\rho\to o}\frac{f(A)-f(A_0)}{\rho}=\frac{df(x_1^0,x_2^0)}{dx_1}\cos\alpha+\frac{df(x_1^0,x_2^0)}{dx_2}\cos\beta$$

bo'ladi. Demak,

$$\frac{df(x_1^0,x_2^0)}{dl}=\frac{\partial f(x_1^0,x_2^0)}{\partial x_1}\cos\alpha+\frac{\partial f(x_1^0,x_2^0)}{\partial x_2}\cos\beta.$$ ▶

13.5-misol. Ushbu

$$f(x_1,x_2)=arctg\,\frac{x_1}{x_2}$$

funksiyaning l yo'nalish bo'yicha hosilasi topilsin, bunda l birinchi kvadrantning $(1,\ 1)$ nuqtadan o'tuvchi va $(0,\ 0)$ nuqtadan $(1,\ 1)$ nuqtaga qarab yo'nalgan bissektrisasidan iborat.

◀Berilgan funksiyaning $A_0=(1,\ 1)$ nuqtadagi l yo'nalish bo'yicha hosilasini (13.7) formulaga ko'ra topamiz

Ravshanki,

$$f(x_1,x\ )=arctg\frac{x_1}{x_2}$$

funksiya $A_0=(1,1)$ nuqtada differensiallanuvchi. Unda (13.7) formulaga ko'ra

$$\frac{df(1,1)}{dl}=\frac{\partial f(1,1)}{\partial x_1}cos\frac{\pi}{4}+\frac{\partial f(1,1)}{\partial x_2}cos\frac{\pi}{4}=$$

$$=\left(\frac{x_2}{x_1^2+x_2^2}-\frac{x_1}{x_1^2+x_2^2}\right)_{\substack{x_1=1\\x_2=1}}\frac{\sqrt{2}}{2}=0$$

bo'ladi. ▶

4-§. Ko'p o'zgaruvchili murakkab funksiyalarning differensiallanuvchiligi. Murakkab funksiyaning hosilasi

$f(x_1,x_2,...,x_m)$ funksiya M $(M\subset R^m)$ to'plamda berilgan bo'lib, $x_1,x_2,...,x_m$ o'zgaruvchilarning har biri o'z navbatida $t_1,t_2,...,t_k$ o'zgaruvchilarning T $(T\subset R^k)$ to'plamda berilgan funksiya bo'lsin:

$$\begin{aligned}x_1&=\varphi_1(t_1,t_2,...,t_k),\\x_2&=\varphi_2(t_1,t_2,...,t_k),\\&.............................,\\x_m&=\varphi_m(t_1,t_2,...,t_k),\end{aligned}\qquad(13.9)$$

Bunda $(t_1,t_2,...,t_k)\in T$ bo'lganda unga mos $(x_1,x_2,...,x_m)\in M$ bo'lsin. Natijada ushbu

$$f(\varphi_1(t_1,t_2,...,t_k),\varphi_2(t_1,t_2,...,t_k),...,\varphi_m(t_1,t_2,...,t_k))=F(t_1,t_2,...,t_k)$$

murakkab funksiyaga ega bo'lamiz.

1^0. Murakkab funksiyaning differensiallanuvchanligi.

5-teorema. Agar (13.9) funksiyalarning har biri $(t_1^0,t_2^0,...,t_k^0)\in T$ nuqtada differensiallanuvchi bo'lib, $f(x_1,x_2,...,x_m)$ funksiya esa mos $(x_1^0,x_2^0,...,x_k^0)\in M$ nuqtada $(x_1^0=\varphi_1(t_1^0,t_2^0,...,t_k^0),\ x_2^0=\varphi_2(t_1^0,t_2^0,...,t_k^0),...,x_m^0=\varphi_m(t_1^0,t_2^0,...,t_k^0))$ differensiallanuvchi bo'lsa, u holda murakkab funksiya $F(t_1,t_2,...,t_k)$ ham $(t_1^0,t_2^0,...,t_k^0)$ nuqtada differensiallanuvchi bo'ladi.

◀ $(t_1^0,t_2^0,...,t_k^0)\in T$ nuqtani olib, uning koordinatalariga mos ravishda shunday $(\Delta t_1,\Delta t_2,...,\Delta t_k)$ orttiruvchilar beraylikki, $(t_1^0+\Delta t_1,t_2^0+\Delta t_2,...,t_k^0+\Delta t_k)\in T$ bo'lsin. U holda (13,9) dagi har bir funksiya ham $(\Delta x_1,\Delta x_2,...,\Delta x_m)$ orttirmalarga va nihoyat $f(x_1,x_2,...,x_m)$ funksiya Δf orttirmaga ega bo'ladi.

Shartga ko'ra (13,9) dagi funksiyalarning har biri $(t_1^0,t_2^0,...,t_k^0)$ nuqtada differensiallanuvchi. Demak,

$$\Delta x_1 = \frac{\partial x_1}{\partial t_1}\Delta t_1 + \frac{\partial x_1}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_1}{\partial t_k}\Delta t_k + o(\rho),$$

$$\Delta x_2 = \frac{\partial x_2}{\partial t_1}\Delta t_1 + \frac{\partial x_2}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_2}{\partial t_k}\Delta t_k + o(\rho), \qquad (13.10)$$

$$..,$$

$$\Delta x_m = \frac{\partial x_m}{\partial t_1}\Delta t_1 + \frac{\partial x_m}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_m}{\partial t_k}\Delta t_k + o(\rho)$$

bo'ladi, bunda $\frac{\partial x_i}{\partial t_j}$ $(i = 1,2,...,m;\ j = 1,2,...,k)$ hususiy hosilalarning $(t_1^0, t_2^0, ..., t_k^0)$ nuqtadagi qiymatlari olingan, va

$$\rho = \sqrt{\Delta t_1^2 + \Delta t_2^2 + ... + \Delta t_k^2}$$

Shartga asosan, $f(x_1, x_2, ..., x_m)$ funksiya $(x_1^0, x_2^0, ..., x_m^0)$ nuqtada differensiallanuvchi. Demak,

$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + ... + \frac{\partial f}{\partial x_m}\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m \qquad (13.11)$$

bo'ladi, bunda $\frac{\partial f}{\partial x_i}$ $(i = 1,2,...,m)$ hususiy hosilalarning $(x_1^0, x_2^0, ..., x_m^0)$ nuqtadagi qiymatlari olingan va $\Delta x_1 \to 0, \Delta x_2 \to 0, ..., \Delta x_m \to 0$ da $\alpha_1 \to 0, \alpha_2 \to 0, ..., \alpha_m \to 0$ bo'ladi.

(13,10) va (13,11) munosabatlardan topamiz:

$$\Delta f = \frac{\partial f}{\partial x_1}\left[\frac{\partial x_1}{\partial t_1}\Delta t_1 + \frac{\partial x_1}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_1}{\partial t_k}\Delta t_k + o(\rho),\right] +$$

$$+ \frac{\partial f}{\partial x_2}\left[\frac{\partial x_2}{\partial t_1}\Delta t_1 + \frac{\partial x_2}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_2}{\partial t_k}\Delta t_k + o(\rho),\right] +$$

$$+ \ .... +$$

$$+ \frac{\partial f}{\partial x_m}\left[\frac{\partial x_m}{\partial t_1}\Delta t_1 + \frac{\partial x_m}{\partial t_2}\Delta t_2 + ... + \frac{\partial x_m}{\partial t_k}\Delta t_k + o(\rho),\right] +$$

$$+ \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + ... + \alpha_m\Delta x_m =$$

$$= \left[\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1} + ... + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1}\right]\Delta t_1 +$$

$$+ \left[\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2} + ... + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2}\right]\Delta t_2 +$$

$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots+$$

$$+\left[\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k}+\dots+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}\right]\Delta t_k+$$

$$+\left[\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\dots+\frac{\partial f}{\partial x_m}\right]o(\rho)+ \qquad (13.12)$$

$$+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\dots+\alpha_m\Delta x_m.$$

Bu tenglikdagi $\left[\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\dots+\frac{\partial f}{\partial x_m}\right]$ yig'indi o'zgarmas (ρ ga bog'liq emas) bo'lganligi sababli

$$\left[\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\dots+\frac{\partial f}{\partial x_m}\right]o(\rho)=o(\rho) \qquad (13.13)$$

bo'ladi.

Madomiki, $x_i=\varphi_i(t_1,t_2,\dots,t_k)$ $(i=1,\,2,\dots,m)$ funksiyalar $(t_1^0,t_2^0,\dots,t_k^0)$ nuqtada differensiallanuvchi ekan, ular shu nuqtada uzluksiz bo'ladi. Unda uzluksizlik ta'rifiga ko'ra $\Delta t_1\to 0,\Delta t_2\to 0,\dots,\Delta t_k\to 0$ da, ya'ni $\rho\to 0$ da $\Delta x_1\to 0,\Delta x_2\to 0,\dots,\Delta x_m\to 0$ bo'ladi. Yana ham aniqroq aytsak, (13.10) formuladan $\rho\to 0$ da $\Delta x_1=o(\rho),\Delta x_2=o(\rho),\dots,\Delta x_m=o(\rho)$ ekanligi kelib chiqadi. $\Delta x_1\to 0,\Delta x_2\to 0,\dots,\Delta x_m\to 0$ da esa $\alpha_1\to 0,\alpha_2\to 0,\dots,\alpha_m\to 0$.

Demak,
$\rho\to 0\Rightarrow$ barcha $\Delta x_i\to 0\Rightarrow$ barcha $\alpha_i\to 0\Rightarrow\alpha_1\Delta x_1,\alpha_2\Delta x_2,\dots,\alpha_m\Delta x_m=o(\rho)$ *(13.14)*

Agar

$$A_j=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j}+\dots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_j}$$

$(j=1,2,3,\dots k)$ deyilsa, u holda (13.12), (13.13) va (13.14) munosabatlardan

$$\Delta f=A_1\Delta t_1+A_2\Delta t_2+\dots+A_k\Delta t_k+o(\rho)$$

kelib chiqadi.▶

2^0^. Murakkab funksiyaning hosilasi. Endi

$$f(\varphi_1(t_1,t_2,\dots,t_k),\varphi_2(t_1,t_2,\dots,t_k),\dots,\varphi_m(t_1,t_2,\dots,t_k))=F(t_1,t_2,\dots,t_k)$$

murakkab funksiyaning $t_1,t_2,\dots,t_k$ o'zgaruvchilar bo'yicha xususiy hosilalarini topamiz. Aytaylik $f(x_1,x_2,\dots,x_m)$ va $x_1=\varphi_1(t_1,t_2,\dots,t_k)$, $x_2=\varphi_2(t_1,t_2,\dots,t_k),\dots,$ $x_m=\varphi_m(t_1,t_2,\dots,t_k)$ funksiyalar yuqoridagi 5- teoremaning shartlarini bajarsin. U holda 5-teoremaga ko'ra murakkab funksiya $(t_1^0,t_2^0,\dots,t_k^0)$ nuqtada differensiallanuvchi bo'ladi.

Demak, bir tomondan

$$\Delta f=A_1\Delta t_1+A_2\Delta t_2+\dots+A_k\Delta t_k+o(\rho) \qquad (13.15)$$

bo'lib, bunda

$$A_j=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_j}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_j}\qquad (\mathrm{j}=1,2,...,\mathrm{k}) \tag{13.16}$$

(qaralsin 5-teorema) ikkinchi tamondan 1- natijaga asosan

$$\Delta f=\frac{\partial f}{\partial t_1}\Delta t_1+\frac{\partial f}{\partial t_2}\Delta t_2+...+\frac{\partial f}{\partial t_k}\Delta t_k+o(\rho) \tag{13.17}$$

bo'ladi. (13.15),(13.16) va (13.17) va munosabatlardan

$$\frac{\partial f}{\partial t_1}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1},$$

$$\frac{\partial f}{\partial t_2}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2}$$

....

$$\frac{\partial f}{\partial t_k}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}$$

bo'lishini topamiz.

5-§. Ko'p o'zgaruvchili funksiyaning differensiali

1⁰. Funksiya differensialining ta'rifi. $f(x_1,x_2,...,x_m)$ funksiya ochiq $M\ (M\subset R^m)$ to'plamda berilgan bo'lib, bu to'plamning $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtasida differensiallanuvchi bo'lsin. Ta'rifga ko'ra $f(x)$ funksiyaning x^0 nuqtadagi orttirmasi

$$\Delta f=\frac{\partial f(x^0)}{\partial x_1}\Delta x_1+\frac{\partial f(x^0)}{\partial x_2}\Delta x_2+...+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m+\alpha_1\Delta x_1+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m$$

bo'ladi.

4-ta'rif. $f(x_1,x_2,...,x_m)$ funksiya orttirmasi $\Delta f(x^0)$ ning $\Delta x_1,\ \Delta x_2,...,\Delta x_m$ larga nisbatan chiziqli bosh qismi

$$\frac{\partial f(x^0)}{\partial x_1}\Delta x_1+\frac{\partial f(x^0)}{\partial x_2}\Delta x_2+...+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m$$

$f(x)$ funksiyaning x^0 nuqtadagi differensiali (to'liq differensiali) deb ataladi va $df(x_1^0,x_2^0,...,x_m^0)$ kabi belgilanadi. Demak,

$$df(x^0)=df(x_1^0,x_2^0,...,x_m^0)=\frac{\partial f(x^0)}{\partial x_1}\Delta x_1+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m+...+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m.$$

Agar $x_1,x_2,...,x_m$ erkli o'zgaruvchilarning ixtiyoriy orttirmalari Δx_1, $\Delta x_2,...,\Delta x_m$ lar mos ravishda bu o'zgaruvchilarning differensiallari $dx_1,dx_2,...,dx_m$ ga teng ekanligini e'tiborga olsak, unda $f(x)$ funksiyaning differensiali quyidagi

$$df(x^0)=\frac{\partial f(x^0)}{\partial x_1}dx_1+\frac{\partial f(x^0)}{\partial x_2}dx_2+...+\frac{\partial f(x^0)}{\partial x_m}dx_m \tag{13.18}$$

ko'rinishga keladi.

Odatda $\frac{\partial f}{\partial x_1}dx_1,\ \frac{\partial f}{\partial x_2}dx_2...,\ \frac{\partial f}{\partial x_m}dx_m$ lar $f(x_1,x_2,...,x_m)$ funksiya xususiy differensiallari deb ataladi va ular mos ravishda $d_{x_1}f, d_{x_2}f,...,d_{x_m}f$ kabi belgilanadi:

$$d_{x_1}f=\frac{\partial f}{\partial x_1}dx_1,\ d_{x_2}f=\frac{\partial f}{\partial x_2}dx_2...,\ d_{x_m}f=\frac{\partial f}{\partial x_m}dx_m.$$

Demak, $f(x)$ funksiyaning x^0 nuqtadagi differensiali, uning shu nuqtadagi xususiy differensiallari yig'indisidan iborat. Masalan, ushbu

$$f(x_1,x_2)=e^{x_1\sin x_2}$$

funksiyaning $\forall(x_1x_2)\in R^2$ nuqtadagi differensiali

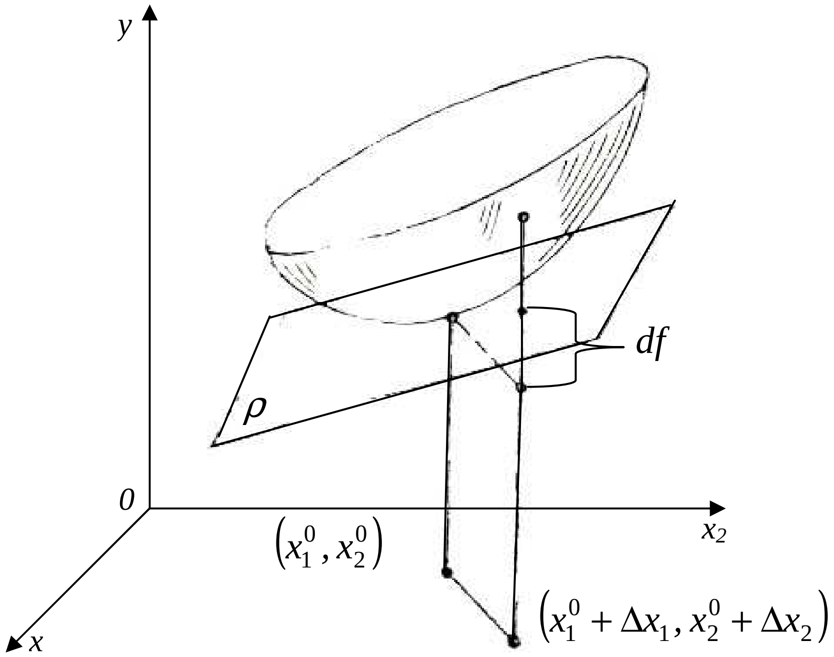
$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2=\sin x_2e^{x_1\sin x_2}dx_1+x_1\cos x_2e^{x_1\sin x_2}dx_2=$$

$$=\mathrm{e}^{x_1\sin x_2}\left(\sin x_2dx_1+x_1\cos x_2dx_2\right)$$

bo'ladi.

Endi funksiya differensialining geometrik ma'nosini ikki o'zgaruvchili funksiya uchun keltiramiz.

Aytaylik, $y=f(x_1,x_2)$ funksiya ochiq M to'plamda $\left(M\subset R^2\right)$ berilgan bo'lib, $\left(x_1^0,x_2^0\right)\in M$ nuqtada differensiallanuvchi bo'lsin. Bu funksiyaning grafigi 50-chizmada tasvirlangan (S) sirtni ifodalasin.



50-chizma.

(S) sirtga $\left(x_1^0,x_2^0,y_0\right)$ nuqtasida $\left(y_0=f\left(x_1^0,x_2^0\right)\right)$ o'tkazilgan urinma tekislik ushbu

$$Y-y_0=\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_1}\left(x_1-x_1^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_2}\left(x_2-x_2^0\right)$$

ko'rinishda bo'lib, undan

$$Y - y_0 = df\left(x_1^0, x_2^0\right)$$

ekanligi kelib chiqadi. Demak, $y = f(x_1, x_2)$ funksiyaning $\left(x_1^0, x_2^0\right)$ nuqtadagi differensiali bu funksiya grafigiga $\left(x_1^0, x_2^0, f\left(x_1^0, x_2^0\right)\right)$ nuqtasida o'tkazilgan urinma tekislik aplikatasining orttirmasidan iborat ekan.

2⁰. Murakkab funksiyaning differensiali. $y = f(x_1, x_2, ..., x_m)$ funksiya $M\ \left(M \subset R^m\right)$ to'plamda berilgan bo'lib, $x_1, x_2, ..., x_m$ o'zgaruvchilarning har biri o'z navbatida $t_1, t_2, ..., t_k$ o'zgaruvchilarning $T\ \left(T \subset R^k\right)$ to'plamda berilgan funksiyasi bo'lsin:

$$\begin{aligned} &x_1 = \varphi_1(t_1, t_2, ..., t_k), \\ &x_2 = \varphi_2(t_1, t_2, ..., t_k), \\ &\dots\dots\dots\dots\dots, \\ &x_m = \varphi_m(t_1, t_2, ..., t_k). \end{aligned}$$

Bunda $(t_1, t_2, ..., t_k) \in T$ bo'lganda unga mos $(x_1, x_2, ..., x_m) \in M$ bo'lib, ushbu

$$y = f\left(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k)\right)$$

murakkab funksiya tuzilgan bo'lsin.

Faraz qilaylik $x_i = \varphi_i(t_1, t_2, ..., t_k)$ $(i = 1, 2, ..., m)$ funksiyalarning har biri $\left(t_1^0, t_2^0, ..., t_k^0\right) \in T$ nuqtada differensiallanuvchi bo'lib, $y = f(x_1, x_2, ..., x_m)$ funksiya esa mos $\left(x_1^0, x_2^0, ..., x_m^0\right) \in M$ nuqtada differensiallanuvchi bo'lsin. U holda 5-teoremaga ko'ra murakkab funksiya $\left(t_1^0, t_2^0, ..., t_k^0\right)$ nuqtada differensiallanuvchi bo'ladi. Unda murakkab funksiyaning shu nuqtadagi differensiali

$$df = \frac{\partial f}{\partial t_1} dt_1 + \frac{\partial f}{\partial t_2} dt_2 + ... + \frac{\partial f}{\partial t_m} dt_m$$

bo'ladi.

Ma'lumki,

$$\begin{aligned} &\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1} + ... + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1}, \\ &\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2} + ... + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + ... + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}. \end{aligned}$$

Natijada

$$df=\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1}\right)dt_1+$$
$$+\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2}\right)dt_2+$$
$$+...+$$
$$+\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k}+...+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}\right)dt_k=$$
$$=\frac{\partial f}{\partial x_1}\left(\frac{\partial x_1}{\partial t_1}dt_1+\frac{\partial x_1}{\partial t_2}dt_2+...+\frac{\partial x_1}{\partial t_k}dt_k\right)+$$
$$+\frac{\partial f}{\partial x_2}\left(\frac{\partial x_2}{\partial t_1}dt_1+\frac{\partial x_2}{\partial t_2}dt_2+...+\frac{\partial x_2}{\partial t_k}dt_k\right)+$$
$$+...+$$
$$+\frac{\partial f}{\partial x_m}\left(\frac{\partial x_m}{\partial t_1}dt_1+\frac{\partial x_m}{\partial t_2}dt_2+...+\frac{\partial x_m}{\partial t_k}dt_k\right)$$

bo'ladi.

Agar

$$\frac{\partial x_1}{\partial t_1}dt_1+\frac{\partial x_1}{\partial t_2}dt_2+...+\frac{\partial x_1}{\partial t_k}dt_k=dx_1,$$
$$\frac{\partial x_2}{\partial t_1}dt_1+\frac{\partial x_2}{\partial t_2}dt_2+...+\frac{\partial x_2}{\partial t_k}dt_k=dx_2,$$
$$...,$$
$$\frac{\partial x_m}{\partial t_1}dt_1+\frac{\partial x_m}{\partial t_2}dt_2+...+\frac{\partial x_m}{\partial t_k}dt_k=dx_m$$

ekanligini e'tiborga olsak, u holda murakkab funksiya differensiali uchun

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_k}dx_k \qquad (13.19)$$

bo'lishi kelib chiqadi.

Murakkab funksiya differensiali ifodalovchi (13.19) formulani avval qarab o'tilgan (13.18) formula bilan solishtirib bir xil formaga ega bo'lishini (ya'ni differensial formasi saqlanishni) ko'ramiz. Odatda bu xossani differensial formasining (shaklining) invariantligi deyiladi.

Demak, ko'p o'zgaruvchili funksiyalarda ham bir o'zgaruvchili funksiyalardagidek, differensial shaklning invaritligi xossasi o'rinli ekan.

Shuni alohida ta'kidlash lozimki, (13.19) ifoda $dx_1, dx_2, ..., dx_m$ lar $x_1, x_2, ..., x_m$ larning ixtiyoriy orttirmalari $\Delta x_1, \Delta x_2, ..., \Delta x_m$ lar bo'lmasdan, ular $t_1, t_2, ..., t_k$ o'zgaruvchilarning funksiyalari bo'ladi.

3⁰. Funksiya differensiali hisoblashning sodda qoidalari. $u=f(x)$ va $v=g(x)$ funksiyalar ochiq M $\left(M\subset R^m\right)$ to'plamda berilgan bo'lib, $x^0\in M$ nuqtada ular differensiallanuvchi bo'lsin. U holda $u\pm v,\ uv,\ \frac{u}{v},\ (v\neq 0)$ funksiyalar ham shu x^0 nuqtada differensiallanuvchi bo'ladi va ularning differensiallari uchun quyidagi

$$1)\ d(u\pm v)=du\pm dv,$$

$$2)\ d(uv)=udv+vdu,$$

$$3)\ d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}\ \ (v\neq 0)$$

formula o'rinli bo'ladi.

Bu munosabatlardan birining, masalan, 2) ning isbotini keltirish bilan chegaralanamiz.

$u=f(x)$ va $v=g(x)$ funksiyalar ko'paytmasini F funksiya deb qaraylik: $F=u\cdot v$. Natijada F funksiya u va v lar orqali $x_1,x_2,...,x_m$ o'zgaruvchilarning $\left(x=(x_1,x_2,...,x_m)\right)$ murakkab funksiya bo'ladi. Murakkab funksiyaning differensialini topish formulasi (13.19) ga ko'ra

$$dF=\frac{\partial F}{\partial u}du+\frac{\partial F}{\partial v}dv$$

bo'ladi.

Agar

$$\frac{\partial F}{\partial u}=v,\quad \frac{\partial F}{\partial v}=u$$

ekanligini e'tiborga olsak, unda

$$dF=vdu+udv$$

bo'lishini topamiz. Demak,

$$d(uv)=vdu+udv.$$

6-§. Ko'p o'zgaruvchili funksiyaning yuqori tartibli hosila va differensiallari

1⁰. Funksiyaning yuqori tartibli xususiy hosilalari. $f(x_1,x_2,...,x_m)$ funksiya ochiq M $\left(M\subset R^m\right)$ to'plamda berilgan bo'lib, uning har bir $(x_1,x_2,...,x_m)$ nuqtasida $f'_{x_1},f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsin. Ravshanki, bu xususiy hosilalar o'z navbatida $x_1,x_2,...,x_m$ o'zgaruvchilarga bog'liq bo'lib, ularning funksiyalari bo'lib qolishi mumkin. Berilgan funksiya xususiy hosilalari $f'_{x_1},f'_{x_2},...,f'_{x_m}$ larning ham xususiy hosilalarini qarash mumkin.

5-ta'rif. $f(x_1,x_2,...,x_m)$ funksiya xususiy hosilalari $f'_{x_1},f'_{x_2},...,f'_{x_m}$ larning x_k $(k=1,2,...,m)$ o'zgaruvchi bo'yicha xususiy hosilalari berilgan funksiyaning ikkinchi tartibli xususiy hosilalari deb ataladi va

$$f''_{x_1x_k},f''_{x_2x_k},...,f''_{x_mx_k}\qquad (k=1,\ 2,\ ...,\ m)$$

yoki

$$\frac{\partial^2 f}{\partial x_1 \partial x_k}, \frac{\partial^2 f}{\partial x_2 \partial x_k}, \dots, \frac{\partial^2 f}{\partial x_m \partial x_k} \qquad (k = 1,\ 2,\ \dots,\ m)$$

kabi belgilanadi. Demak,

$$\frac{\partial^2 f}{\partial x_1 \partial x_k} = f''_{x_1 x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_1}\right),$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_k} = f''_{x_2 x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_2}\right),$$

...................................,

$$\frac{\partial^2 f}{\partial x_m \partial x_k} = f''_{x_m x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_m}\right) \qquad (k = 1,\ 2,\ \dots,\ m)$$

Bu ikkinchi tartibli xususiy hosilalarni umumiy holda

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = f''_{x_i x_k} \qquad (i = 1, 2, \dots, m;\ k = 1,\ 2,\ \dots,\ m)$$

ko’rinishda yozish mumkin, bunda $k = i$ bo’lganda

$$\frac{\partial^2 f}{\partial x_k \partial x_k} = f''_{x_k x_k}$$

deb yozish o’rniga

$$\frac{\partial^2 f}{\partial x_k^2} = f''_{x_k^2}$$

deb yoziladi.

Agar yuqoridagi ikkinchi tartibli xususiy hosilalar turli o’zgaruvchilar bo’yicha olingan bo’lsa, unda bu

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = f''_{x_i x_k} \qquad (i \neq k)$$

2-tartibli xususiy hosilalar aralash hosilalar deb ataladi.

Xuddi shunga o’xshash, $f(x_1, x_2, \dots, x_m)$ funksiyaning uchinchi, to’rtinchi va xokazo tartibdagi xususiy hosilalari ta’riflanadi. Umuman, $f(x_1, x_2, \dots, x_m)$ funksiya $(n-1)$-tartibli xususiy hosilalarning xususiy hosilasi berilgan funksiyaning n-tartibli xususiy hosilasi deb ataladi.

13.6-misol. Ushbu

$$f(x_1, x_2) = arctg\,\frac{x_1}{x_2} \qquad (x_2 \neq 0)$$

funksiyaning 2-tartibli xususiy hosilasi topilsin.

◄Ravshanki, $\frac{\partial f}{\partial x_1} = \frac{x_2}{x_1^2 + x_2^2}, \quad \frac{\partial f}{\partial x_2} = -\frac{x_1}{x_1^2 + x_2^2},$

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(\frac{x_2}{x_1^2+x_2^2}\right)=-\frac{2x_1x_2}{\left(x_1^2+x_2^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)=\frac{\partial}{\partial x_2}\left(\frac{x_2}{x_1^2+x_2^2}\right)=\frac{x_1^2-x_2^2}{\left(x_1^2+x_2^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x_2\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)=\frac{\partial}{\partial x_1}\left(-\frac{x_1}{x_1^2+x_2^2}\right)=\frac{x_1^2-x_2^2}{\left(x_1^2+x_2^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x_2^2}=\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right)=\frac{\partial}{\partial x_2}\left(-\frac{x_1}{x_1^2+x_2^2}\right)=\frac{2x_1x_2}{\left(x_1^2+x_2^2\right)^2}.\blacktriangleright$$

13.7-misol. Ushbu

$$f\left(x_1,x_2\right)=\begin{cases} x_1x_2\dfrac{x_1^2-x_2^2}{x_1^2+x_2^2}, \text{ агар } & x_1^2+x_2^2\neq 0 \text{ бўлса}, \\ 0 \qquad \text{агар} & x_1^2+x_2^2=0 \text{ бўлса} \end{cases}$$

funksiyaning aralash hosilalari topilsin.

◀Aytaylik $\left(x_1,x_2\right)\neq\left(0,0\right)$ bo’lsin. U holda

$$\frac{\partial f}{\partial x_1}=x_2\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}+\frac{4x_1^2x_2^2}{\left(x_1^2+x_2^2\right)^2}\right),\quad \frac{\partial f}{\partial x_2}=x_1\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}-\frac{4x_1^2x_2^2}{\left(x_1^2+x_2^2\right)^2}\right),$$

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)=\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\left(1+\frac{8x_1^2x_2^2}{\left(x_1^2+x_2^2\right)^2}\right),$$

$$\frac{\partial^2 f}{\partial x_2\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)=\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\left(1+\frac{8x_1^2x_2^2}{\left(x_1^2+x_2^2\right)^2}\right)$$

bo’ladi.

Berilgan $f\left(x_1,x_2\right)$ funksiyaning $\left(0,0\right)$ nuqtadagi xususiy hosilalarini ta’rifga ko’ra topamiz:

$$\frac{\partial f\left(0,0\right)}{\partial x_1}=\lim_{\Delta x_1\to 0}\frac{f\left(\Delta x_1,0\right)-f\left(0,0\right)}{\Delta x_1}=0,$$

$$\frac{\partial f\left(0,0\right)}{\partial x_2}=\lim_{\Delta x_2\to 0}\frac{f\left(0,\Delta x_2\right)-f\left(0,0\right)}{\Delta x_2}=0,$$

$$\frac{\partial^2 f\left(0,0\right)}{\partial x_1\partial x_2}=\lim_{\Delta x_1\to 0}\frac{\dfrac{\partial f\left(0,\Delta x_2\right)}{\partial x_1}-\dfrac{\partial f\left(0,0\right)}{\partial x_1}}{\Delta x_2}=\lim_{\Delta x_2\to 0}\frac{-\Delta x_2^3}{\Delta x_2^3}=-1,$$

$$\frac{\partial^2 f\left(0,0\right)}{\partial x_2\partial x_1}=\lim_{\Delta x_2\to 0}\frac{\dfrac{\partial f\left(\Delta x_1,0\right)}{\partial x_2}-\dfrac{\partial f\left(0,0\right)}{\partial x_2}}{\Delta x_1}=\lim_{\Delta x_1\to 0}\frac{\Delta x_1^3}{\Delta x_1^3}=1,$$

Bu keltirilgan misollar ko’rinadiki, funksiyaning $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ va $\frac{\partial^2 f}{\partial x_2 \partial x_1}$ aralash hosilalari bir-biriga teng bo’lishi ham, teng bo’lmasligi ham mumkin ekan.

6-teorema. $f(x_1, x_2)$ funksiya ochiq M $\left(M \subset R^2\right)$ to’plamda f'_{x_1}, f'_{x_2} hamda $\dot{f}''_{x_1 x_2}$, $f''_{x_2 x_1}$ aralash hosilalarga ega bo’lsin. Agar aralash hosilalar $\left(x_1^0, x_2^0\right) \in M$ nuqtada uzluksiz bo’lsa, u holda shu nuqtada

$$\dot{f}''_{x_1 x_2}\left(x_1^0, x_2^2\right) = f''_{x_2 x_1}\left(x_1^0, x_2^2\right)$$

bo’ladi.

◀ $\left(x_1^0, x_2^0\right)$ nuqta koordinatalariga mos ravishda shunday $\Delta x_1 > 0$, $\Delta x_2 > 0$ orttirmalar beraylikki,

$$D = \left\{\left(x_1, x_2\right) \in R^2 : x_1^0 \le x_1 \le x_1^0 + \Delta x_1,\ x_2^0 \le x_2 \le x_2^0 + \Delta x_2\right\} \subset M$$

bo’lsin. Bu to’g’ri to’rtburchak uchlarini ifodalovchi $\left(x_1^0, x_2^0\right)$, $\left(x_1^0 + \Delta x_1, x_2^0\right)$, $\left(x_1^0, x_2^0 + \Delta x_2\right)$, $\left(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2\right)$ nuqtalarda funksiyaning qiymatlarini topib ulardan ushbu

$$P = f\left(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2\right) - f\left(x_1^0 + \Delta x_1, x_2^0\right) - f\left(x_1^0, x_2^0 + \Delta x_2\right) + f\left(x_1^0, x_2^0\right)$$

ifodani hosil qilamiz. Bu ifodani quyidagi ikki

$$P = \left[f\left(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2\right) - f\left(x_1^0 + \Delta x_1, x_2^0\right)\right] - \left[f\left(x_1^0, x_2^0 + \Delta x_2\right) - f\left(x_1^0, x_2^0\right)\right],$$

$$P = \left[f\left(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2\right) - f\left(x_1^0, x_2^0 + \Delta x_2\right)\right] - \left[f\left(x_1^0 + \Delta x_1, x_2^0\right) - f\left(x_1^0, x_2^0\right)\right]$$

ko’rinishda yozish mumkin.

Endi berilgan $f(x_1, x_2)$ funksiya yordamida x_1 o’zgaruvchiga bog’liq bo’lgan

$$\varphi(x_1) = f\left(x_1, x_2^0 + \Delta x_2\right) - f\left(x_1, x_2^0\right),$$

x_2 o’zgaruvchiga bog’liq bo’lgan

$$\psi(x_2) = f\left(x_1^0 + \Delta x_1, x_2\right) - f'_{x_1}\left(x_1^0, x_2\right)$$

funksiyalarni tuzaylik. Ravshanki, $\varphi(x_1)$, $\psi(x_2)$ funksiyalar

$$\varphi'(x_1) = f'_{x_1}\left(x_1, x_2^0 + \Delta x_2\right) - f'_{x_1}\left(x_1, x_2^0\right),$$

$$\psi'(x_2) = f'_{x_2}\left(x_1^0 + \Delta x_1, x_2\right) - f'_{x_2}\left(x_1^0, x_2\right)$$

hosilalarga ega bo’lib, Lagranj teoremasiga asosan

$$\begin{aligned} \varphi'(x_1) &= f''_{x_1, x_2}\left(x_1, x_2^0 + \theta_2 \Delta x_2\right) \cdot \Delta x_2, \\ \psi'(x_2) &= f''_{x_2, x_1}\left(x_1^0 + \theta_1 \Delta x_1, x_2\right) \cdot \Delta x_1 \end{aligned} \qquad (13.20)$$

bo’ladi, bunda $0 < \theta_1$, $\theta_2 < 1$.

Yuqorida keltirilgan P ifodani $\varphi(x_1)$, $\psi(x_2)$ funksiyalar orqali ushbu

$$P = \varphi\left(x_1^0 + \Delta x_1\right) - \varphi\left(x_1^0\right),$$

$$P = \psi\left(x_2^0 + \Delta x_2\right) - \psi\left(x_2^0\right)$$

ko’rinishda yozib, so’ng yana Lagranj teoremasini qo’llab quyidagilarni topamiz:

$$P=\varphi'\left(x_1^0+\theta_1'\Delta x_1\right)\cdot\Delta x_1,\quad P=\psi'\left(x_2^0+\theta_2'\Delta x_2\right)\cdot\Delta x_2$$
$$\left(0<\theta_1',\ \theta_2'<1\right). \tag{13.21}$$

Natijada (13.20) va (13.21) munosabatlardan

$$P=f''_{x_1x_2}\left(x_1^0+\theta_1'\Delta x_1,x_2^0+\theta_2\Delta x_2\right)\cdot\Delta x_1\Delta x_2$$
$$P=f''_{x_2x_1}\left(x_1^0+\theta_1\Delta x_1,x_2^0+\theta_2'\Delta x_2\right)\cdot\Delta x_1\Delta x_2$$

bo'lib, ulardan esa

$$f''_{x_1x_2}\left(x_1^0+\theta_1'\Delta x_1,x_2^0+\theta_2\Delta x_2\right)=f''_{x_2x_1}\left(x_1^0+\theta_1\Delta x_1,x_2^0+\theta_2'\Delta x_2\right) \tag{13.22}$$

bo'lishi kelib chiqadi.

Shartga ko'ra $f''_{x_1x_2}$ va $f''_{x_2x_1}$ aralash hosilalar $\left(x_1^0,x_2^0\right)$ nuqtada uzluksiz. Shuning uchun (13.22) da $\Delta x_1\to 0,\ \Delta x_2\to 0$ limitga o'tsak,

$$f''_{x_1x_2}\left(x_1^0,x_2^0\right)=f''_{x_2x_1}\left(x_1^0,x_2^0\right)$$

bo'ladi. ▶

2^0. Funksiyaning yuqori tartibli differensiallari. Ko'p o'zgaruvchili funksiyaning yuqori tartibli differensiali tushunchasi keltirishdan avval, funksiyaning $n\ (n>1)$ marta differensiallanuvchiligi tushunchasi bilan tanishamiz.

$f(x)$ funksiya ochiq $M\ \left(M\subset R^m\right)$ to'plamda berilgan bo'lib, $x^0\in M$ bo'lsin. Ma'lumki, $f(x)$ funksiyaning x^0 nuqtadagi orttirmasi ushbu

$$\Delta f\left(x^0\right)=A_1\Delta x_1+A_2\Delta x_2+...+A_m\Delta x_m+o(\rho)$$

ko'rinishda ifodalansa, funksiya x^0 nuqtada differensiallanuvchi deb atalar edi, bunda $A_1,A_2,...,A_m$- o'zgarmas sonlar, $\rho=\sqrt{\Delta x_1^2+\Delta x_2^2+...+\Delta x_m^2}$. Bu holda ko'rgan edikki,

$$A_i=\frac{\partial f\left(x^0\right)}{\partial x_i}\qquad (i=1,\ 2,...,\ m)$$

Aytaylik, $f(x)$ funksiya M to'plamda $f'_{x_1},f'_{x_2},...,f'_{x_m}$ xususiy hosilalarga ega bo'lsin. Agar bu xususiy hosilalar x^0 nuqtada differensiallanuvchi bo'lsa, $f(x)$ shu nuqtada ikki marta differensiallanuvchi funksiya deb ataladi.

Umuman, $f(x)$ funksiya M to'plamda barcha $(n-1)$- tartibli xususiy hosilalarga ega bo'lib, bu xususiy hosilalar $x^0\in M$ nuqtada differensiallanuvchi bo'lsa, $f(x)$ funksiya x^0 nuqtada n marta differensiallanuvchi deyiladi.

7-teorema. $f(x)$ funksiya M to'plamda barcha n- tartibli xususiy hosilalarga ega bo'lib, bu xususiy hosilalar $x^0\in M$ nuqtada uzluksiz bo'lsa, $f(x)$ funksiya x^0 nuqtada n marta differensiallanuvchi bo'ladi.

Bu teorema funksiya differensiallanuvchi bo'lishining etarli shartini ifodalovchi 3-teoremaning isbotlanganligi kabi isbotlanadi.

$f(x)$ funksiya $x\in M$ nuqtada ikki marta differensiallanuvchi bo'lsin.

6-ta’rif. $f(x)$ funksiyaning x nuqtadagi differensiali $df(x)$ ning differensiali berilgan $f(x)$ funksiyaning ikkinchi tartibli differensiali deb ataladi va u $d^2 f$ kabi belgilanadi:

$$d^2 f = d(df).$$

Differensiallash qoidalaridan foydalanib topamiz:

$$d^2 f = d(df) = d\left(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m\right) =$$

$$= dx_1 d\left(\frac{\partial f}{\partial x_1}\right) + dx_2 d\left(\frac{\partial f}{\partial x_2}\right) + ... + dx_m d\left(\frac{\partial f}{\partial x_m}\right) =$$

$$= \left(\frac{\partial^2 f}{\partial x_1^2}dx_1 + \frac{\partial^2 f}{\partial x_1 \partial x_2}dx_2 + ... + \frac{\partial^2 f}{\partial x_1 \partial x_m}dx_m\right)dx_1 +$$

$$+ \left(\frac{\partial^2 f}{\partial x_2 \partial x_1}dx_1 + \frac{\partial^2 f}{\partial x_2^2}dx_2 + ... + \frac{\partial^2 f}{\partial x_2 \partial x_m}dx_m\right)dx_2 +$$

$$+ .... +$$

$$+ \left(\frac{\partial^2 f}{\partial x_m \partial x_1}dx_1 + \frac{\partial^2 f}{\partial x_m \partial x_2}dx_2 + ... + \frac{\partial^2 f}{\partial x_m \partial x_m}dx_m\right)dx_m =$$

$$= \frac{\partial^2 f}{\partial x_1^2}dx_1^2 + \frac{\partial^2 f}{\partial x_2^2}dx_2^2 + ... + \frac{\partial^2 f}{\partial x_m^2}dx_m^2 +$$

$$+ 2\frac{\partial^2 f}{\partial x_1 \partial x_2}dx_1 dx_2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_3}dx_1 dx_3 + ... + 2\frac{\partial^2 f}{\partial x_1 \partial x_m}dx_1 dx_m +$$

$$+ 2\frac{\partial^2 f}{\partial x_2 \partial x_3}dx_2 dx_3 + 2\frac{\partial^2 f}{\partial x_2 \partial x_4}dx_2 dx_4 + ... + 2\frac{\partial^2 f}{\partial x_2 \partial x_m}dx_2 dx_m +$$

$$+ ... + 2\frac{\partial^2 f}{\partial x_{m-1} \partial x_m}dx_{m-1} dx_m$$

$f(x_1, x_2, ..., x_m)$ funksiyaning $(x_1, x_2, ..., x_m)$ nuqtadagi uchinchi, turtinchi va xokazo tartibli differensiallari ham xuddi yuqoridagidek ta’riflanadi.

Umuman, $f(x)$ funksiyaning x nuqtadagi $(n-1)$- tartibli differensiali $d^{(n-1)} f(x)$ ning differensial berilgan $f(x)$ funksiyaning shu nuqtadagi n tartibli differensiali deb ataladi va $d^n f$ kabi belgilanadi. Demak,

$$d^n f = d\left(d^{n-1} f\right)$$

Biz yuqorida $f(x)$ funksiyaning ikkinchi tartibli differensiali uning xususiy hosilalari orqali ifodalanishini ko’rdik.

$f(x)$ funksiyaning keyingi tartibli differensiallarining funksiya xususiy xosilalari orqali ifodasi borgan sari murakkablasha boradi. Shu sababli yuqori tartibli differensiallarni, simvolik ravishda, soddaroq formada ifodalash muhim.

$f(x)$ funksiya differensiali

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m$$

ni simvolik ravishda (f ni formal ravishda qavs tashqarisiga chiqarib) quyidagicha

$$df = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_m}dx_m\right)f$$

yozamiz. Unda funksiyaning ikkinchi tartibli differensialini

$$d^2 f = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_m}dx_m\right)^2 f \qquad (13.23)$$

deb qarash mumkin. Bunda qavs ichidagi yig'indi kvadratga ko'tarilib, so'ng f ga «ko'paytiriladi». Keyin daraja ko'rsatkichlari xususiy hosilalar tartibi deb hisoblanadi.

Shu tarzda kiritilgan simvolik ifodalash $f(x)$ funksiyaning n- tartibli differensialini

$$d^n f = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_m}dx_m\right)^n f$$

kabi yozish imkonini beradi.

3^0. Murakkab funksiyaning yuqori tartibli differensiallari. Ushbu punktda $f(x_1, x_2, ..., x_m)$, ($x_1 = \varphi_1(t_1, ..., t_k)$, $x_2 = \varphi_2(t_1, ..., t_k)$ $, ..., x_m = \varphi_m(t_1, ..., t_k)$) murakkab funksiyaning yuqori tartibli differensiallarini topamiz.

Ma'lumki, $x_i = \varphi_i(t_1, t_2, ..., t_m)$ $(i = 1,\ 2, ..., m)$ funksiyaning har bir $(t_1^0, t_2^0, ..., t_k^0) \in T$ nuqtada differensiallanuvchi bo'lib, $f(x_1, x_2, ..., x_m)$ funksiya esa mos $(x_1^0, x_2^0, ..., x_m^0)$ nuqtada differensiallanuvchi bo'lsa, u holda 5-teoremaga ko'ra murakkab funksiya $(t_1^0, t_2^0, ..., t_k^0)$ nuqtada differensiallanuvchi va differensial shaklning invariantlik xossasiga asosan murakkab funksiyaning differensiali

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m$$

bo'ladi.

Faraz qilaylik, $x_i = \varphi_i(t_1, t_2, ..., t_m)$ $(i = 1,\ 2, ..., m)$ funksiyalarning har biri $(t_1^0, t_2^0, ..., t_k^0) \in T$ nuqtada ikki marta differensiallanuvchi, $f(x_1, x_2, ..., x_m)$ funksiya esa mos $(x_1^0, x_2^0, ..., x_m^0) \in M$ nuqtada ikki marta differensiallanuvchi bo'lsin. U holda murakkab funksiya ham $(t_1^0, t_2^0, ..., t_k^0)$ nuqtada ikki marta differensiallanuvchi bo'ladi. Differensiallash qoidalaridan foydalanib quyidagini topamiz:

$$d^2 f = d(df) = d\left(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m\right) =$$

$$= d\left(\frac{\partial f}{\partial x_1}\right)dx_1 + \frac{\partial f}{\partial x_1}d(dx_1) + d\left(\frac{\partial f}{\partial x_2}\right)dx_2 + \frac{\partial f}{\partial x_2}d(dx_2) + ... +$$

$$+ d\left(\frac{\partial f}{\partial x_m}\right)dx_m + \frac{\partial f}{\partial x_m}d(dx_m) =$$

$$= d\left(\frac{\partial f}{\partial x_1}\right)dx_1 + d\left(\frac{\partial f}{\partial x_2}\right)dx_2 + ... + d\left(\frac{\partial f}{\partial x_m}\right)dx_m +$$

$$+ \frac{\partial f}{\partial x_1}d^2x_1 + \frac{\partial f}{\partial x_2}d^2x_2 + ... + \frac{\partial f}{\partial x_m}d^2x_m =$$

$$= \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_m}dx_m\right)^2 f +$$

$$+ \frac{\partial f}{\partial x_1}d^2x_1 + \frac{\partial f}{\partial x_2}d^2x_2 + ... + \frac{\partial f}{\partial x_m}d^2x_m. \qquad (13.24)$$

Shu yo'l bilan berilgan murakkab funksiyaning keyingi tartibdagi differensiallari topiladi.

(13.23), (13.24) formulalarni solishtirib, ikkinchi tartibli differensiallarda differensial shaklining invariantligi xossasi o'rinli emasligi ko'ramiz.

2-eslatma. Agar

$$\begin{aligned} x_1 &= \alpha_{11}t_1 + \alpha_{12}t_2 + ... + \alpha_{1k}t_k + \beta_1, \\ x_2 &= \alpha_{21}t_1 + \alpha_{22}t_2 + ... + \alpha_{2k}t_k + \beta_2, \\ &.., \\ x_m &= \alpha_{m1}t_1 + \alpha_{m2}t_2 + ... + \alpha_{mk}t_k + \beta_m \end{aligned} \qquad (13.25)$$

bo'lsa (α_{ji}, β_j $(i = 1,2,...,k;\ j = 1, 2,..., m)$-o'zgarmas sonlar), u holda bunday $f(x_1, x_2,..., x_m)$ murakkab funksiyaning yuqori tartibli differensiallari differensial shaklining invariantligi xossasiga ega bo'ladi.

Haqiqatdan ham (13.25) ifodadagi funksiyalarni differensiallasak, unda

$$\begin{aligned} dx_1 &= \alpha_{11}dt_1 + \alpha_{12}dt_2 + ... + \alpha_{1k}dt_k = \alpha_{11}\Delta t_1 + \alpha_{12}\Delta t_2 + ... + \alpha_{1k}\Delta t_k, \\ dx_2 &= \alpha_{21}dt_1 + \alpha_{22}dt_2 + ... + \alpha_{2k}dt_k = \alpha_{21}\Delta t_1 + \alpha_{22}\Delta t_2 + ... + \alpha_{2k}\Delta t_k, \\ &.., \\ dx_m &= \alpha_{m1}dt_1 + \alpha_{m2}dt_2 + ... + \alpha_{mk}dt_k = \alpha_{m1}\Delta t_1 + \alpha_{m2}\Delta t_2 + ... + \alpha_{mk}\Delta t_k \end{aligned}$$

bo'lib $dx_1, dx_2,..., dx_m$ larning har biri $t_1, t_2,..., t_k$ o'zgaruvchilarga bog'liq emasligini ko'ramiz. Ravshanki, bundan $d^2x_1 = d_2x_2 = ... = d^2x_m = 0$

Binobarin,

$$d^2 f = d(df) = d\left(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_m}dx_m\right) =$$
$$= dx_1 d\left(\frac{\partial f}{\partial x_1}\right) + dx_2 d\left(\frac{\partial f}{\partial x_2}\right) + ... + dx_m d\left(\frac{\partial f}{\partial x_m}\right) =$$
$$= \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + ... + \frac{\partial}{\partial x_m}dx_m\right)^2 f.$$

Demak, ikkinchi tartibli differensiallar differensial shaklining invariantligi xossasiga ega ekan.

Shunga o'xshash, bu holda murakkab funksiyaning ikkidan katta tartibdagi differensiallarida differensial shaklining invariantligi xossasi o'rinli bo'lishi ko'rsatiladi.

7-§. O'rta qiymat haqida teorema

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya $M \subset R^m$ to'plamda berilgan. Bu to'plamda shunday $a = (a_1, a_2, ..., a_m)$ va $в = (в_1, в_2, ..., в_m)$ nuqtalarni olaylikki, bu nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi

$$A = \{ (x_1, x_2, ..., x_m) \in R^m : x_1 = a_1 + t(в_1 - a_1),\ x_2 = a_2 + t(в_2 - a_2),$$
$$..., x_m = a_m + t(в_m - a_m);\ \ 0 \le t \le 1\}$$

shu M to'plamga tegishli bo'lsin: $A \subset M$.

8-teorema. Agar $f(x)$ funksiya A kesmaning a va $в$ nuqtalarida uzluksiz bo'lib, kesmaning qolgan barcha nuqtalrida funksiya differensiallanuvchi bo'lsa, u holda A kesmada shunday c nuqta $(c = (c_1, c_2, ..., c_m))$ topiladiki,

$$f(в) - f(a) = f'_{x_1}(c)(в_1 - a_1) + f'_{x_2}(c)(в_2 - a_2) + ... + f'_{x_m}(c)(в_m - a_m)$$

bo'ladi.

◄ $f(x)$ funksiya A to'plamda qaraylik. Unda

$$f(x) = f(x_1, x_2, ..., x_m) = f(a_1 + t(в_1 - a_1),\ a_2 + t(в_2 - a_2), ...,\ a_m + t(в_m - a_m))$$
$$(0 \le t \le 1)$$

bo'lib, $f(x_1, x_2, ..., x_m)$ t o'zgaruvchining $[0,1]$ segmentda berilgan funksiyasiga aylanadi:

$$F(t) = f(a_1 + t(в_1 - a_1),\ a_2 + t(в_2 - a_2), ...,\ a_m + t(в_m - a_m))$$

Bu funksiya $(0, 1)$ intervalda ushbu

$$F'(t) = f'_{x_1} \cdot (в_1 - a_1),\ f'_{x_2} \cdot (в_2 - a_2), ...,\ f'_{x_m} \cdot (в_m - a_m)$$

hosilaga ega bo'ladi.

Demak, $F(t)$ funksiya $[0,1]$ segmentda uzluksiz, $(0, 1)$ intervalda esa $F'(t)$ hosilaga ega. Unda Lagranj teoremasiga (1-qism, 6-bob, 6-§) ko'ra $(0, 1)$ intervalda shunday t_0 nuqta topiladiki,

$$F(1) - F(0) = F'(t_0) \qquad (0 < t_0 < 1) \qquad \textit{(13.26)}$$

bo'ladi. Ravshanki,

$$F(0)=f(a), \qquad F(1)=f(в),$$

$$\begin{aligned} F'(t_0) &= f'_{x_1}(a_1+t_0(в_1-a_1),\ a_2+t_0(в_2-a_2),\dots,\ a_m+t_0(в_m-a_m))\cdot(в_1-a_1)+ \\ &+ f'_{x_2}(a_1+t_0(в_1-a_1),\ a_2+t_0(в_2-a_2),\dots,\ a_m+t_0(в_m-a_m))\cdot(в_2-a_2)+ \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots+ \\ &+ f'_{x_m}(a_1+t_0(в_1-a_1),\ a_2+t_0(в_2-a_2),\dots,\ a_m+t_0(в_m-a_m))\cdot(в_m-a_m). \end{aligned} \tag{13.27}$$

Agar

$$\begin{aligned} &a_1+t_0(в_1-a_1)=c_1 \\ &a_2+t_0(в_2-a_2)=c_2 \\ &\dots\dots\dots\dots\dots \\ &a_m+t_0(в_m-a_m)=c_m \end{aligned}$$

deb belgilasak, unda $c=(c_1,c_2,\dots,c_m)\in A$ bo’lib, yuqoridagi (13.26) va (13.27) tengliklardan

$$f(в)-f(a)=f'_{x_1}(c)(в_1-a_1)+f'_{x_2}(c)(в_2-a_2)+\dots+f'_{x_m}(c)(в_m-a_m)$$

kelib chiqadi. ▶

Bu o’rta qiymat haqidagi teorema deb ataladi.

2-natija. $f(x)$ funksiya bog’lamli $M\subset R^m$ to’plamda berilgan bo’lib, uning har bir nuqtasi differensiallanuvchi bo’lsin. Agar M to’plamning har bir nuqtasida $f(x)$ funksiyaning barcha xususiy hosilalri nolga teng bo’lsa, funksiya M to’plamda o’zgarmas bo’ladi.

◀ M to’plamda $a=(a_1,a_2,\dots,a_m)$ hamda ixtiyoriy $x=(x_1,x_2,\dots,x_m)$ nuqtalarni olaylik. Bu nuqtalarni birlashtiruvchi kesma shu M to’plamga tegishli bo’lsin. U holda shu kesma nuqtalarida 8-teoremaga ko’ra

$$f(a)=f(x)+f'_{x_1}(c)(a_1-x_1)+f'_{x_2}(c)(a_2-x_2)+\dots+f'_{x_m}(c)(a_m-x_m)$$

bo’ladi. Funksiyaning barcha xususiy hosilalari nolga teng ekanidan

$$f(x)=f(a)$$

bo’lishi kelib chiqadi.

a va x nuqtalarni birlashtiruvchi kesma M to’plamga tegishli bo’lmasa, unda M to’plamning bog’lamli ekanligidan a va x nuqtalarni birlashtiruvchi va to’plamga tegishli bo’lgan siniq chiziq topiladi, bu siniq chiziq kesmalariga yuqoridagi 8-teoremani qo’llay borib,

$$f(x)=f(a)$$

bo’lishini topamiz. ▶

8-§. Ko’p o’zgaruvchili funksiyaning Teylor formulasi

Ma’lumki, $F(t)$ funksiya $t=t_0$ nuqtaning atrofida berilgan bo’lib, unda $F'(t),F''(t),\dots,F^{n+1}(t)$ hosilalarga ega bo’lsa, u holda

$$F(t)=F(t_0)+F'(t_0)(t-t_0)+\frac{1}{2!}F''(t_0)(t-t_0)^2+\ldots+$$
$$+\frac{1}{n!}F^{(n)}(t_0)(t-t_0)^n+\frac{F^{(n+1)}(c)}{(n+1)!}(t-t_0)^{n+1}, \qquad (13.28)$$
$$(c=t_0+\theta(t-t_0), \quad 0<\theta<1).$$
bo'ladi (Teylor formulasi).

$f(x)=f(x_1,x_2,...,x_m)$ funksiya ochiq M $(M\subset R^m)$ to'plamda berilgan. Bu to'plamda $(x_1^0,x_2^0,...,x_m^0)$ nuqtani olib, uning $U_\delta((x_1^0,x_2^0,...,x_m^0))\subset M$ atrofini qaraylik. Ravshanki, $\forall(x_1',x_2',...,x_m')\in U_\delta((x_1^0,x_2^0,...,x_m^0))$ nuqta bilan $(x_1^0,x_2^0,...,x_m^0)$ nuqtani birlashtiruvchi to'g'ri chiziq kesmasi

$$A=\{(x_1,x_2,...,x_m)\in R^m : x_1=x_1^0+t(x_1'-x_1^0),\ x_2=x_2^0+t(x_2'-x_2^0),$$
$$...,x_m=x_m^0+t(x_m'-x_m^0);\ 0\le t\le 1\}$$

shu $U_\delta((x_1^0,x_2^0,...,x_m^0))$ atrofga tegishli bo'ladi.

$f(x_1,x_2,...,x_m)$ funksiya $U_\delta((x_1^0,x_2^0,...,x_m^0))$ da $n+1$ marta differensiallanuvchi bo'lsin deb uni A to'plamga qaraylik. Unda

$$f(x_1,x_2,...,x_m)=f(x_1^0+t(x_1'-x_1^0),\ x_2^0+t(x_2'-x_2^0)...,\ x_m^0+t(x_m'-x_m^0))$$

bo'lib, $f(x_1,x_2,...,x_m)$ funksiya t o'zgaruvchining $[0,1]$ da berilgan funksiyasiga aylanib qoladi:

$$F(t)=f(x_1^0+t(x_1'-x_1^0),\ x_2^0+t(x_2'-x_2^0)...,\ x_m^0+t(x_m'-x_m^0))\ (0\le t\le 1) \qquad (13.29)$$

Bu funksiyaning hosilalarini hisoblaylik:

$$F'(t)=\frac{\partial f}{\partial x_1}(x_1'-x_1^0)+\frac{\partial f}{\partial x_2}(x_2'-x_2^0)+\ldots+\frac{\partial f}{\partial x_m}(x_m'-x_m^0)=$$
$$=\left(\frac{\partial}{\partial x_1}(x_1'-x_1^0)+\frac{\partial}{\partial x_2}(x_2'-x_2^0)+\ldots+\frac{\partial}{\partial x_m}(x_m'-x_m^0)\right)f,$$
$$F''(t)=\frac{\partial^2 f}{\partial x_1^2}(x_1'-x_1^0)^2+\frac{\partial^2 f}{\partial x_2^2}(x_2'-x_2^0)^2+\ldots+\frac{\partial^2 f}{\partial x_m^2}(x_m'-x_m^0)^2+$$
$$+2\frac{\partial^2 f}{\partial x_1\partial x_2}(x_1'-x_1^0)(x_2'-x_2^0)+\ldots+2\frac{\partial^2 f}{\partial x_{m-1}\partial x_m}(x_{m-1}'-x_{m-1}^0)(x_m'-x_m^0)=$$
$$=\left(\frac{\partial}{\partial x_1}(x_1'-x_1^0)+\frac{\partial}{\partial x_2}(x_2'-x_2^0)+\ldots+\frac{\partial}{\partial x_m}(x_m'-x_m^0)\right)^2 f.$$

Umuman k-tartibli hosila ushbu

$$F_{(t)}^{(k)}=\left(\frac{\partial}{\partial x_1}(x_1'-x_1^0)+\frac{\partial}{\partial x_2}(x_2'-x_2^0)+\ldots+\frac{\partial}{\partial x_m}(x_m'-x_m^0)\right)^k f \qquad (13.30)$$
$$(k=1,2,...,n+1)$$

ko'rinishida bo'ladi. (Uning to'g'riligi matematik induksiya usuli yordamida isbotlanadi).

Yuqoridagi $F'(t), F''(t), ..., F^{n+1}(t)$ hosilalarning ifodalariga kirgan $f(x_1, x_2, ..., x_m)$ funksiyaning barcha xususiy hosilalari

$$\left(x_1^0 + t(x_1' - x_1^0),\ x_2^0 + t(x_2' - x_2^0), ..., x_m^0 + t(x_m' - x_m^0)\right)$$

nuqta hisoblangan.

(13.28) formulada $t = 0$ va $t = 1$ deb olinsa, ushbu

$$F(1) = F(0) + \frac{1}{1!}F'(0) + \frac{1}{2!}F''(0) + ... + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\theta)$$

$$(0 < \theta < 1)$$

hosil bo'ladi.

(13.29) va (13.30) munosabatdan foydalanib quyidagilarni topamiz:

$$F(0) = f\left(x_1^0, x_2^0, ..., x_m^0\right),$$

$$F(1) = f(x_1', x_2', ..., x_m')$$

$$F^{(k)}(0) = \left(\frac{\partial}{\partial x_1}\left(x_1' - x_1^0\right) + \frac{\partial}{\partial x_2}\left(x_2' - x_2^0\right) + ... + \frac{\partial}{\partial x_m}\left(x_m' - x_m^0\right)\right)^k f$$

$$(k = 1, 2, ..., n+1)$$

Keyingi tenglikdagi $f(x_1, x_2, ..., x_m)$ funksiyaning barcha xususiy hosilalari $\left(x_1^0, x_2^0, ..., x_m^0\right)$ nuqtada hisoblangan.

Demak, (13.28) formulaga ko'ra

$$f(x_1, x_2, ..., x_m) = f\left(x_1^0, x_2^0, ..., x_m^0\right) + \left(\frac{\partial}{\partial x_1}\left(x_1' - x_1^0\right) + \right.$$

$$\left. + \frac{\partial}{\partial x_2}\left(x_2' - x_2^0\right) + ... + \frac{\partial}{\partial x_m}\left(x_m' - x_m^0\right)\right) f +$$

$$+ \frac{1}{2!}\left(\frac{\partial}{\partial x_1}\left(x_1' - x_1^0\right) + \frac{\partial}{\partial x_2}\left(x_2' - x_2^0\right) + ... + \frac{\partial}{\partial x_m}\left(x_m' - x_m^0\right)\right)^2 f +$$

$$+ .... +$$

$$+ \frac{1}{n!}\left(\frac{\partial}{\partial x_1}\left(x_1' - x_1^0\right) + \frac{\partial}{\partial x_2}\left(x_2' - x_2^0\right) + ... + \frac{\partial}{\partial x_m}\left(x_m' - x_m^0\right)\right)^n f +$$

$$+ \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}\left(x_1' - x_1^0\right) + \frac{\partial}{\partial x_2}\left(x_2' - x_2^0\right) + ... + \frac{\partial}{\partial x_m}\left(x_m' - x_m^0\right)\right)^{n+1} f$$

bo'ladi, bunda $f(x_1, x_2, ..., x_m)$ funksiyaning barcha birinchi, ikkinchi va xokazo n-tartibli xususiy hosilalari $\left(x_1^0, x_2^0, ..., x_m^0\right)$ nuqtada, shu funksiyaning barcha $(n+1)$-tartibli xususiy hosilalari esa

$$\left(x_1^0 + \theta(x_1' - x_1^0),\ x_2^0 + \theta(x_2' - x_2^0),\ ...\ ,\ x_m^0 + \theta(x_m' - x_m^0)\ \right) \quad (0 < \theta < 1)$$

nuqtada hisoblangan.

Bu formula ko’p o’zgaruvchili $f(x_1,x_2,...,x_m)$ funksiyaning Teylor formulasi deb ataladi.

Xususan, ikki o’zgaruvchili funksiyaning Teylor formulasi quyidagicha bo’ladi:

$$f(x_1,x_2)= f\left(x_1^0,x_2^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_1}\left(x_1-x_1^0\right)+\frac{\partial f\left(x_1^0,x_2^0\right)}{\partial x_2}\left(x_2-x_2^0\right)+$$

$$+\frac{1}{2!}\left[\frac{\partial^2 f\left(x_1^0,x_2^0\right)}{\partial x_1^2}\left(x_1-x_1^0\right)^2+2\frac{\partial^2 f\left(x_1^0,x_2^0\right)}{\partial x_1\partial x_2}\left(x_1-x_1^0\right)\left(x_2-x_2^0\right)+\right.$$

$$\left.+\frac{\partial^2 f\left(x_1^0,x_2^0\right)}{\partial x_2^2}\left(x_2-x_2^0\right)^2\right]+...+\frac{1}{n!}\left[\frac{\partial^n f\left(x_1^0,x_2^0\right)}{\partial x_1^n}\left(x_1-x_1^0\right)^n+\right.$$

$$\left.+C_n^1\frac{\partial^n f\left(x_1^0,x_2^0\right)}{\partial x_1^{n-1}\partial x_2}\left(x_1-x_1^0\right)^{n-1}\left(x_2-x_2^0\right)+...+\frac{\partial^n f\left(x_1^0,x_2^0\right)}{\partial x_2^n}\left(x_2-x_2^0\right)^n\right]+$$

$$+\frac{1}{(n+1)!}\left[\frac{\partial^{n+1} f\left(x_1^0+\theta\left(x_1-x_1^0\right),x_2^0+\theta\left(x_2-x_2^0\right)\right)}{\partial x_1^{(n+1)}}\left(x_1-x_1^0\right)^{n+1}+...+\right.$$

$$\left.+\frac{\partial^{n+1} f\left(x_1^0+\theta\left(x_1-x_1^0\right),x_2^0+\theta\left(x_2-x_2^0\right)\right)}{\partial x_1^{(n+1)}}\left(x_2-x_2^0\right)^{n+1}\right].$$

9-§. Ko’p o’zgaruvchili funksiyaning ekstremum qiymatlari. Ekstremumning zaruriy sharti

1⁰. Funksiyaning maksimum va minimum qiymatlari. Ko’p o’zgaruvchili funksiyaning ekstremum qiymatlari ta’riflari xuddi bir o’zgaruvchili funksiyaniki singari kiritiladi.

$f(x)= f(x_1,x_2,...,x_m)$ funksiya ochiq $M\subset R^m$ to’plamda berilgan bo’lib, $x^0=\left(x_1^0,x_2^0,...,x_m^0\right)\in M$ bo’lsin.

7-ta’rif. Agar x^0 nuqtaning shunday $U_\delta\left(x^0\right)=\Big\{x=\left(x_1,x_2,...,x_m\right)\in R^m:$

$:\rho\left(x,x^0\right)=\sqrt{\left(x_1-x_1^0\right)^2+...+\left(x_m-x_m^0\right)^2}<\delta\Big\}\subset M$ atrofi mavjud bo’lsaki, $\forall x\in U_\delta\left(x^0\right)$ uchun

$$f(x)\le f\left(x^0\right)\qquad\left(f(x)\ge f\left(x^0\right)\right)$$

bo’lsa, $f(x)$ funksiya x^0 nuqtada maksimumga (minimumga) ega deyiladi, $f\left(x^0\right)$ qiymat esa $f(x)$ funksiyaning maksimum (minimum) qiymati yoki maksimumi (minimumi) deyiladi.

8-ta’rif. Agar x^0 nuqtaning shunday $U_\delta\left(x^0\right)$ atrofi mavjud bo’lsaki, $\forall x\in U_\delta\left(x^0\right)\setminus\{x_0\}$ uchun $f(x)< f\left(x^0\right)$ $\left(f(x)> f\left(x^0\right)\right)$ bo’lsa, $f(x)$ funksiya x^0 nuqtada qat’iy maksimumga (qat’iy minimumga) ega deyiladi. $f\left(x^0\right)$ qiymat esa

$f(x)$ funksiyaning qat'iy maksimum (qat'iy minimum) qiymati yoki qat'iy maksimumi (qat'iy minimumi) deyiladi.

Yuqoridagi ta'riflardagi x^0 nuqta $f(x)$ funksiyaga maksimum (minimum) (8-ta'rifda), qat'iy maksimum (qat'iy minimum) (9-ta'rifda) qiymat beradigan nuqta deb ataladi.

Funksiyaning maksimum va minimumi umumiy nom bilan uning ekstremumi deb ataladi.

13.8-misol. Ushbu

$$f(x_1,x_2)=\sqrt{1-x_1^2-x_2^2} \qquad (x_1^2+x_2^2\le 1)$$

funksiyaning $(0,0)$ nuqtada qat'iy maksimumga erishish ko'rsatilsin.

◀ Haqiqatdan ham, $(0,0)$ nuqtaning ushbu

$$U_r((0,0))=\{(x_1,x_2)\in R^2;\quad x_1^2+x_2^2<r^2\} \qquad (0<r<1)$$

atrofi olinsa, unda $\forall(x_1,x_2)\in U_r((0,0))\setminus\{(0,0)\}$ uchun

$$f(x_1,x_2)=\sqrt{1-x_1^2-x_2^2} < f(0,\,0)=1$$

bo'ladi. ▶

8 va 9- ta'riflardan ko'rinadiki, $f(x)$ funksiyaning x^0 nuqtadagi qiymati $f(x^0)$ ni uning shu nuqta atrofidagi nuqtalardagi qiymatlari bilangina solishtirilar ekan. Shuning uchun funksiyaning ekstremumi (maksimumi, minimumi) lokal ekstremum (lokal maksimum, lokal minimum) deb ataladi.

2^0. Funksiya ekstremumining zaruriy sharti. $f(x_1,x_2,...,x_m)$ funksiya ochiq $M\subset R^m$ to'plamda berilgan. Aytaylik, $f(x_1,x_2,...,x_m)$ funksiya $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtada maksimumga (minimumga) ega bo'lsin. Ta'rifga ko'ra $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtaning shunday $U_\delta(x_0)\subset M$ atrofi mavjudki, $\forall x\in U_\delta(x^0)$ uchun

$$f(x_1,x_2,...,x_m)\le f(x_1^0,x_2^0,...,x_m^0)$$
$$\left(f(x_1,x_2,...,x_m)\ge f(x_1^0,x_2^0,...,x_m^0)\right)$$

xususan

$$f(x_1,x_2^0,x_3^0...,x_m^0)\le f(x_1^0,x_2^0,...,x_m^0)$$
$$\left(f(x_1,x_2^0,...,x_m^0)\ge f(x_1^0,x_2^0,...,x_m^0)\right)$$

bo'ladi. Natijada bir o'zgaruvchiga x_1 ga bog'liq bo'lgan $f(x_1,x_2^0,...,x_m^0)$ funksiyaning $U_\delta(x^0)$ da eng katta (eng kichik) qiymati $f(x_1^0,x_2^0,...,x_m^0)$ ga erishishini ko'ramiz. Agarda x^0 nuqtada $f'_{x_1}(x_0)$ xususiy hosila mavjud bo'lsa, unda Ferma teoremasi (qaralsin, 1-qism, 6-bob, 6-§)ga ko'ra

$$f'_{x_1}(x_1^0,x_2^0,...,x_m^0)=f'_{x_1}(x^0)=0$$

bo'ladi.

Xuddi shuningdek, $f'_{x_2}(x^0),..., f'_{x_m}(x^0)$ xususiy hosilalar mavjud bo'lsa,

$$f'_{x_2}\left(x^0\right)=0,\ f'_{x_3}\left(x^0\right)=0\ ,...,\ f'_{x_m}\left(x^0\right)=0$$

bo'lishini topamiz.

Shunday qilib quyidagi teoremaga kelamiz.

9-teorema. Agar $f(x)$ funksiya x^0 nuqtada ekstremumga erishsa va shu nuqtada barcha $f'_{x_1}, f'_{x_2}, ..., f'_{x_m}$ xususiy hosilalarga ega bo'lsa, u holda

$$f'_{x_1}\left(x^0\right)=0,\ f'_{x_2}\left(x^0\right)=0\ ,...,\ f'_{x_m}\left(x^0\right)=0$$

bo'ladi.

Biroq $f(x)$ funksiyaning biror $x'\in R^m$ nuqtada barcha xususiy hosilalarga ega va

$$f'_{x_1}(x')=0,\ f'_{x_2}(x')=0\ ,...,\ f'_{x_m}(x')=0$$

bo'lishidan uning shu x nuqtada ekstremumga ega bo'lishi har doim ham kelib chiqavermaydi.

Masalan, R^2 to'plamda berilgan

$$f(x_1,x_2)=x_1x_2$$

funksiyani qaraylik. Bu funksiya $f'_{x_1}(x_1,x_2)=x_2,\ f'_{x_2}(x_1,x_2)=x_1$ xususiy hosilalarga ega bo'lib, ular $(0,0)$ nuqtada ekstremumga ega emas (bu funksiyaning grafigi giperbolik paraboloidni ifodalaydi, qaralsin 12-bob, 3-§).

Demak, 9-teorema bir argumentli funksiyalardagidek funksiya ekstremumga erishishining zaruriy shartini ifodalar ekan.

$f(x)$ funksiya xususiy hosilalarini nolga aylantiradigan nuqtalar uning statsionar nuqtalari deyiladi.

10-§. Funksiya ekstremumining etarli sharti

Biz yuqorida $f(x)$ funksiyaning x^0 nuqtada ekstremumga erishishining zaruriy shartini ko'rsatdik. Endi funksiyaning ekstremumga erishishining etarli shartini o'rganamiz.

$f(x)$ funksiya $x^0\in R^m$ nuqtaning biror

$$U_\delta\left(x^0\right)=\left\{x\in R^m: \rho\left(x,x^0\right)<\delta\right\}\quad (\delta>0)$$

atrofida berilgan bo'lsin. Ushbu

$$\Delta=f(x)-f\left(x^0\right) \qquad (13.31)$$

ayirmani qaraylik. Ravshanki, bu ayirma $U_\delta\left(x^0\right)$ da o'z ishorasini saqlasa, ya'ni har doim $\Delta\geq 0$ $(\Delta\leq 0)$ bo'lsa, $f(x)$ funksiya x^0 nuqtada minimumga (maksimumga) erishadi. Agar (13.31) ayirma har qanday $U_\delta\left(x^0\right)$ atrofda ham o'z ishorasini saqlamasa, u holda $f(x)$ funksiya x^0 nuqtada ekstremumga ega bo'lmaydi. Demak, masala (13.31) ayirma o'z ishorasini saqlaydigan $U_\delta\left(x^0\right)$ atrof mavjudmi yoki yo'qmi, shuni aniqlashdan iborat. Bu masalani biz, xususiy holda ya'ni $f(x)$ funksiya ma'lum shartlarni bajargan holda echamiz.

$f(x)$ funksiya quyidagi shartlarni bajarsin:

1) $f(x)$ funksiya biror $U_\delta(x^0)$ da uzluksiz, barcha o'zgaruvchilari bo'yicha birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega;

2) x^0 nuqta $f(x)$ funksiyaning statsionar nuqtasi, ya'ni

$$f'_{x_1}(x^0)=0,\ f'_{x_2}(x^0)=0\ ,...,\ f'_{x_m}(x^0)=0\,.$$

Ushbu bobning 8-§ ida keltirilgan Teylor formulasidan foydalanib, x^0 nuqtaning statsionar nuqta ekanligini e'tiborga olib, quyidagini topamiz:

$$f(x)=f(x^0)+\frac{1}{2}\Big[f''_{x_1^2}\Delta x_1^2+f''_{x_2^2}\Delta x_2^2+...+f''_{x_m^2}\Delta x_m^2+$$
$$+2\left(f''_{x_1x_2}\Delta x_1\Delta x_2+f''_{x_1x_3}\Delta x_1\Delta x_3+...+f''_{x_{m-1}x_m}\Delta x_{m-1}\Delta x_m\right)\Big]=$$
$$=f(x^0)+\frac{1}{2}\sum_{i,k=1}^{m}f''_{x_ix_k}\Delta x_i\Delta x_k\,.$$

Bu munosabatda $f(x)$ funksiyaning barcha xususiy hosilalari $f''_{x_ix_k}$ $(i,k=1,2,...,m)$ lar ushbu

$$\left(x_1^0+\theta\Delta x_1,x_2^0+\theta\Delta x_2,...,x_m^0+\theta\Delta x_m\right)\quad(0<\theta<1)$$

nuqtadan hisoblangan va

$$\Delta x_1=x_1-x_1^0,\ \Delta x_2=x_2-x_2^0,...,\ \Delta x_m=x_m-x_m^0.$$

Demak,

$$\Delta=\frac{1}{2}\sum_{i,k=1}^{m}f''_{x_ix_k}\Delta x_i\Delta x_k$$

Berilgan $f(x)$ funksiya ikkinchi tartibli hosilalarining statsionar nuqtadagi qiymatlarini quyidagicha belgilaylik:

$$a_{ik}=f''_{x_ix_k}(x^0)\quad(i,k=1,\ 2,\ ...\ ,m)$$

Unda $f''_{x_ix_k}(x)$ ning x^0 nuqtada uzluksizligidan

$$f''_{x_ix_k}=f''_{x_ix_k}\left(x_1^0+\theta\Delta x_1,x_2^0+\theta\Delta x_2,...,x_m^0+\theta\Delta x_m\right)=a_{ik}+\alpha_{ik}$$

$(i,k=1,\ 2,\ 3,...\ ,m)$ bo'lishi kelib chiqadi. Bu munosabatda $\Delta x_1\to 0$, $\Delta x_2\to 0$, $,...,\Delta x_m\to 0$ da barcha $\alpha_{ik}\to 0$ va 6-§ da keltirilgan 6-teoremaga asosan

$$a_{ik}=a_{ki}\qquad(i,k=1,\ 2,\ 3,...\ ,m)$$

bo'ladi. Natijada (13.31) ayirma ushbu

$$\Delta=\frac{1}{2}\left[\sum_{i,k=1}^{m}a_{ik}\Delta x_i\Delta x_k+\sum_{i,k=1}^{m}\alpha_{ik}\Delta x_i\Delta x_k\right]$$

ko'rinishni oladi. Buni quyidagicha ham yozish mumkin:

$$\Delta=\frac{\rho^2}{2}\left[\sum_{i,k=1}^{m}a_{ik}\frac{\Delta x_i}{\rho}\cdot\frac{\Delta x_k}{\rho}+\sum_{i,k=1}^{m}\alpha_{ik}\frac{\Delta x_i}{\rho}\cdot\frac{\Delta x_k}{\rho}\right].$$

Agar

$$\xi_i=\frac{\Delta x_i}{\rho}\qquad(i=1,\ 2,\ ...\ ,m)$$

deb belgilasak, unda

$$\Delta = \frac{\rho^2}{2}\left[\sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k + \sum_{i,k=1}^{m} \alpha_{ik}\xi_i \cdot \xi_k\right] \qquad (13.32)$$

bo'ladi.

Ushbu

$$P(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} b_{ik}\xi_i \cdot \xi_k$$

ifoda $\xi_1, \xi_2, ..., \xi_m$ o'zgaruvchilarning kvadratik formasi deb ataladi, b_{ik} $(i,k = 1, 2, 3, ... , m)$ lar esa kvadratik formaning koeffitsientlari deyiladi. Ravshanki, har qanday kvadratik forma o'z koeffitsientlari orqali to'la aniqlanadi. Kvadratik formalar algebra kursida batafsil o'rganiladi. Quyida biz kvadratik formaga doir ba'zi (kelgusida qo'llaniladigan) tushunchalarni eslatib o'tamiz.

Ravshanki, $\xi_1 = \xi_2 = ... = \xi_m = 0$ bo'lsa, har qanday kvadratik forma uchun

$$P(0, 0, ..., 0) = 0$$

bo'ladi.

Endi boshqa nuqtalarni qaraylik. Quyidagi hollar bo'lishi mumkin:

1⁰. Barcha $\xi_1^2 + \xi_2^2 + ... + \xi_m^2 > 0$ nuqtalar uchun

$$P(\xi_1, \xi_2, ..., \xi_m) > 0$$

Bu holda kvadratik forma musbat aniqlangan deyiladi.

2⁰. Barcha $\xi_1^2 + \xi_2^2 + ... + \xi_m^2 > 0$ nuqtalar uchun

$$P(\xi_1, \xi_2, ..., \xi_m) < 0.$$

Bu holda kvadratik forma manfiy aniqlangan deyiladi.

3⁰. Ba'zan $(\xi_1, \xi_2, ..., \xi_m)$ nuqtalar uchun $P(\xi_1, \xi_2, ..., \xi_m) > 0$ ba'zi nuqtalar uchun

$$P(\xi_1, \xi_2, ..., \xi_m) < 0$$

Bu holda kvadratik forma noaniq deyiladi.

4⁰. Barcha $\xi_1^2 + \xi_2^2 + ... + \xi_m^2 > 0$ nuqtalar uchun

$$P(\xi_1, \xi_2, ..., \xi_m) \geq 0$$

va ular orasida shunday $(\xi_1, \xi_2, ..., \xi_m)$ nuqtalar ham borki,

$$P(\xi_1, \xi_2, ..., \xi_m) = 0$$

Bu holda kvadratik forma yarimmusbat aniqlangan deyiladi.

5⁰. Barcha $\xi_1^2 + \xi_2^2 + ... + \xi_m^2 > 0$ nuqtalar uchun

$$P(\xi_1, \xi_2, ..., \xi_m) \leq 0$$

va ular orasida shunday $(\xi_1, \xi_2, ..., \xi_m)$ nuqtalar ham borki,

$$P(\xi_1, \xi_2, ..., \xi_m) = 0.$$

Bu holda kvadratik forma yarimmanfiy aniqlangan deyiladi.

Keltirilgan hollarni alohida-alohida tahlil qilamiz:

1⁰. Ushbu

$$Q(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k$$

kvadratik forma musbat aniqlangan bo'lsin. Avvalo yuqoridagi

$$\rho = \sqrt{\Delta x_1^2 + \Delta x_2^2 + ... + \Delta x_m^2}$$

va

$$\xi_i = \frac{\Delta x_i}{\rho} \qquad (i = 1,\ 2,\ ...\ ,\ m)$$

tengliklardan

$$\xi_1^2 + \xi_2^2 + ... + \xi_m^2 = 1$$

ekanligini topamiz. Ma'lumki, R^m fazoda

$$S_1(\mathbf{0}) = S_1((0,0,...,0)) = \{(\xi_1, \xi_2, ..., \xi_m) \in R^m : \xi_1^2 + \xi_2^2 + ... + \xi_m^2 = 1\}$$

markazi $\mathbf{0} = (0,0,...,0)$ nuqtada radiusi 1 ga teng sferani ifodalaydi. Sfera yopiq va chegaralangan to'plam. Veyershtrassning birinchi teoremasiga asosan shu sferada $Q(\xi_1, \xi_2, ..., \xi_m)$ funksiya uzluksiz funksiya sifatida chegaralangan, xususan quyidan chegaralangan bo'ladi:

$$Q(\xi_1, \xi_2, ..., \xi_m) \geq C \qquad\qquad (C - const)$$

Agar $Q(\xi_1, \xi_2, ..., \xi_m)$ kvadratik formaning musbat aniqlangan ekanligini e'tiborga olsak, unda $C \geq 0$ bo'lishini topamiz.

Ikkinchi tomondan, Veyershtrassning ikkinchi teoremasiga ko'ra bu $Q(\xi_1, \xi_2, ..., \xi_m)$ funksiya $S_1(\mathbf{0})$ sferada o'zining aniq quyi chegarasiga erishadi, ya'ni biror $(\xi_1^0, \xi_2^0, ..., \xi_m^0) \in S_1(\mathbf{0})$ uchun

$$Q(\xi_1^0, \xi_2^0, ..., \xi_m^0) = min\, Q(\xi_1, \xi_2, ..., \xi_m)$$

bo'ladi. Yana $Q(\xi_1, \xi_2, ..., \xi_m)$ kvadratik formaning musbat aniqlangan ekanligini e'tiborga olsak,

$$Q(\xi_1^0, \xi_2^0, ..., \xi_m^0) > 0$$

ekanini topamiz. Demak, $S_1(0)$ sferada

$$Q(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} a_{ik} \xi_i \cdot \xi_k \geq C > 0$$

bo'ladi.

Endi

$$\sum_{i,k=1}^{m} \alpha_{ik} \xi_i \cdot \xi_k$$

ni baholaymiz. Koshi-Bunyakovskiy tengsizligidan foydalanib, topamiz:

$$\left| \sum_{i,k=1}^{m} \alpha_{ik} \xi_i \cdot \xi_k \right| = \left| \sum_{i=1}^{m} \left(\sum_{k=1}^{m} \alpha_{ik} \xi_k \right) \cdot \xi_i \right| \leq \left[\sum_{i=1}^{m} \left(\sum_{k=1}^{m} \alpha_{ik} \xi_k \right)^2 \right]^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{m} \xi_i^2 \right)^{\frac{1}{2}} =$$

$$= \left[\sum_{i=1}^{m} \left(\sum_{k=1}^{m} \alpha_{ik} \xi_k \right)^2 \right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{m} \left(\sum_{k=1}^{m} \alpha_{ik}^2 \sum_{i=1}^{m} \xi_i^2 \right) \right]^{\frac{1}{2}} = \left(\sum_{i,k=1}^{m} \alpha_{ik}^2 \right)^{\frac{1}{2}}.$$

Ma'lumki, $\Delta x_1 \to 0,\ \Delta x_2 \to 0,...,\ \Delta x_m \to 0$ da barcha $\alpha_{ik} \to 0$. Bundan foydalanib x^0 nuqtaning atrofini etarlicha kichik qilib olish hisobiga

$$\left(\sum_{i,k=1}^{m} \alpha_{ik}^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}$$

tengsizlikka erishish mumkin. Demak, (13.32) dan

$$\Delta = \frac{\rho^2}{2}\left(\sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k + \sum_{i,k=1}^{m} \alpha_{ik}\xi_i \cdot \xi_k\right) \geq \frac{\rho^2}{2}\left(c - \frac{c}{2}\right) = \frac{\rho^2 c}{4} > 0$$

2⁰. Quyidagi

$$Q(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k$$

kvadratik forma manfiy aniqlangan bo'lsin. Bu holda x^0 nuqtaning etarlicha kichik atrofida $\Delta = \frac{\rho^2}{2}\left(\sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k + \sum_{i,k=1}^{m} \alpha_{ik}\xi_i \cdot \xi_k\right) < 0$ bo'lishi 1-holdagiga o'xshash ko'rsatiladi. Natijada quyidagi teoremaga kelamiz.

10-teorema. $f(x)$ funksiya x^0 nuqtaning biror $U_\delta(x^0)$ atrofida $(\delta > 0)$ berilgan bo'lsin va u ushbu shartlarni bajarsin:

1) $f(x)$ funksiya $U_\delta(x^0)$ da barcha o'zgaruvchilar $x_1, x_2, ..., x_m$ bo'yicha birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega;

2) x^0 nuqta $f(x)$ funksiyaning statsionar nuqtasi;

3) koeffitsientlari

$$a_{ik} = f''_{x_i x_k}(x^0) \quad (i,k = 1,\ 2,\ ...\ ,\ m)$$

bo'lgan

$$Q(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k$$

kvadratik forma musbat (manfiy) aniqlangan. U holda $f(x)$ funksiya x^0 nuqtada maksimumga (minimumga) erishadi.

Bu teorema funksiya ekstremumining etarli shartini ifodalaydi.

3⁰. Agar

$$Q(\xi_1, \xi_2, ..., \xi_m) = \sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k$$

kvadratik forma noaniq bo'lsa, $f(x)$ funksiya x^0 nuqtada ekstremumga erishmaydi. Shuni isbotlaylik $\xi_1, \xi_2, ..., \xi_m$ larning shunday $(h_1, h_2, ..., h_m)$ va $(\bar{h}_1, \bar{h}_2, ..., \bar{h}_m)$ qiymatlari topiladiki,

$$Q(h_1, h_2, ..., h_m) > 0,\ Q(\bar{h}_1, \bar{h}_2, ..., \bar{h}_m) < 0 \qquad (13.33)$$

bo'ladi.

$$x^0 = (x_1^0, x_2^0, ..., x_m^0),\ (x_1^0 + h_1, x_2^0 + h_2, ..., x_m^0 + h_m)$$

nuqtalarni birlashtiruvchi

$$
\begin{aligned}
&x_1 = x_1^0 + th_1, \\
&x_2 = x_2^0 + th_2 \\
&\text{................}, \qquad\qquad (0 \le t \le 1) \\
&x_m = x_m^0 + th_m
\end{aligned} \tag{13.34}
$$

kesmaning nuqtalari uchun yuqoridagi (13.32) munosabat ushbu

$$\Delta = \frac{t^2}{2}\left(\sum_{i,k=1}^{m} a_{ik}h_i \cdot h_k + \sum_{i,k=1}^{m} \alpha_{ik}h_i \cdot h_k\right)$$

ko'rinishiga keladi. Bu tenglikning o'ng tomonidagi birinchi qo'shiluvchi (13.33) ga ko'ra musbat bo'ladi. Ikkinchi qo'shiluvchi esa, $t \to 0$ da nolga intiladi (chunki $t \to 0$ da $\Delta x_1 = x_1 - x_1^0 \to 0,\ \Delta x_2 = x_2 - x_2^0 \to 0, \ldots\ \Delta x_m = x_m - x_m^0 \to 0$). Demak, (13.34) kesmaning x^0 nuqtaga etarlicha yaqin bo'lgan x nuqtalari uchun Δ ayirma musbat, ya'ni

$$f(x) > f\left(x^0\right)$$

bo'ladi.

Xuddi shunga o'xshash,

$$
\begin{aligned}
&x_1 = x_1^0 + t\overline{h}_1, \\
&x_2 = x_2^0 + t\overline{h}_2 \\
&\text{................}, \\
&x_m = x_m^0 + t\overline{h}_m
\end{aligned}
$$

kesmaning x^0 nuqtaga etarlicha yaqin bo'lgan x nuqtalari uchun Δ ayirma manfiy, ya'ni

$$f(x) < f\left(x^0\right)$$

bo'lishi ko'rsatiladi.

Demak, $\Delta = f(x) - f\left(x^0\right)$ ayirma x^0 nuqtaning har qanday etarlicha kichik atrofida o'z ishorasini saqlamaydi. Bu esa $f(x)$ funksiyaning x^0 nuqtada ekstremumga erishmasligini bildiradi.

$4^0 - 5^0$. Agar

$$Q(\xi_1, \xi_2, \ldots, \xi_m) = \sum_{i,k=1}^{m} a_{ik}\xi_i \cdot \xi_k$$

kvadratik forma yarimmusbat aniqlangan bo'lsa yoki yarimmanfiy aniqlangan bo'lsa, $f(x)$ funksiya x^0 nuqtada ekstremumga erishishi ham erishmasligi ham mumkin. Bu «shubhali» hol qo'shimcha tekshirib aniqlanadi.

Yuqoridagi 10-teoremaning 3-sharti, ya'ni $Q(\xi_1, \xi_2, \ldots, \xi_m)$ kvadratik formaning musbat yoki manfiy aniqlanganlikka aloqador sharti teoremaning markaziy qismini tashkil etadi. Kvadratik formaning musbat yoki manfiy aniqlanganligini algebra kursidan ma'lum bo'lgan Silvestr alomatidan foydalanib topish mumkin. Quyidagi bu alomatni isbotsiz keltiramiz.

Silvestr alomati. Ushbu

$$P(\xi_1,\xi_2,...,\xi_m)=\sum_{i,k=1}^{m} в_{ik}\xi_i\cdot\xi_k$$

kvadratik formaning musbat aniqlangan bo'lishi uchun

$$в_{11}>0,\ \begin{vmatrix} в_{11} & в_{12} \\ в_{21} & в_{112} \end{vmatrix}>0,\ ...,\ \begin{vmatrix} в_{11} & в_{12} & ... & в_{1m} \\ в_{21} & в_{22} & ... & в_{2m} \\ \multicolumn{4}{c}{.....................} \\ в_{m1} & в_{m2} & ... & в_{mm} \end{vmatrix}>0$$

tengsizliklarning, manfiy aniqlangan bo'lishi uchun

$$в_{11}<0,\ \begin{vmatrix} в_{11} & в_{12} \\ в_{21} & в_{112} \end{vmatrix}>0,\ ...,(-1)^{\mathrm{m}} \begin{vmatrix} в_{11} & в_{12} & ... & в_{1m} \\ в_{21} & в_{22} & ... & в_{2m} \\ \multicolumn{4}{c}{.....................} \\ в_{m1} & в_{m2} & ... & в_{mm} \end{vmatrix}>0$$

tengsizliklarning bajarilishi zarur va etarli.

Xususiy holni, funksiya ikki o'zgaruvchiga bog'liq bo'lgan holni qaraylik. $f(x_1,x_2)$ funksiya $x^0=(x_1^0,x_2^0)$ nuqtaning biror atrofi

$$U_\delta(x^0)=\{x=(x_1,x_2)\in R^2:\ \rho(x,x^0)<\delta\}\quad(\delta>0)$$

da birinchi, ikkinchi tartibli uzluksiz hosilalarga ega bo'lib, x^0 esa qaralayotgan funksiyaning statsionar nuqtasi bo'lsin:

$$f'_{x_1}(x^0)=0,\quad f''_{x_2}(x^0)=0.$$

Odatdagidek

$$a_{11}=f''_{x_1^2}(x^0),\ \ a_{12}=f''_{x_1x_2}(x^0),\ \ a_{22}=f''_{x_2^2}(x^0).$$

1). Agar

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}^2>0 \text{ va } a_{11}>0$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada minimumga erishadi,

2). Agar

$$a_{11}a_{22}-a_{12}^2>0 \text{ va } a_{11}<0$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada maksimumga erishadi.

3). Agar

$$a_{11}a_{22}-a_{12}^2<0$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada ekstremumga erishmaydi.

4). Agar

$$a_{11}a_{22}-a_{12}^2=0$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada ekstremumga erishishi mumkin, erishmasligi ham mumkin. Bu «shuhbali» hol qo'shimcha tekshirish yordamida aniqlanadi.

Haqiqatdan ham 1)- va 2)- hollarda kvadratik forma mos ravishda musbat aniqlangan yoki manfiy aniqlangan bo'ladi (qaralsin: Silvestr alomati).

3)- holda, ya’ni

$$a_{11}a_{22} - a_{12}^2 < 0 \qquad (13.35)$$

bo’lganda $Q(\xi_1, \xi_2) = a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2$ kvadratik forma noaniq bo’ladi. Shuni isbotlaylik.

$a_{11} = 0$ bo’lsin. Bu holda (13.35) dan $a_{12} \neq 0$ bo’lishi kelib chiqadi. Natijada $Q(\xi_1, \xi_2)$ kvadratik forma ushbu

$$Q(\xi_1, \xi_2) = (2a_{12}\xi_1 + a_{22}\xi_2)\xi_2$$

ko’rinishga keladi. Bu kvadratik forma

$$\xi_1 = \frac{1 - a_{22}}{2a_{12}}, \qquad \xi_2 = 1$$

qiymatda musbat:

$$Q\left(\frac{1 - a_{22}}{2a_{12}},\ 1\right) = 1 > 0 \text{ va } \xi_1 = \frac{1 + a_{22}}{2a_{12}}, \qquad \xi_2 = -1$$

qiymatda esa manfiy:

$$Q\left(\frac{1 + a_{22}}{2a_{12}},\ -1\right) = -1 < 0$$

bo’ladi.

Endi $a_{11} > 0$ bo’lsin. Bu holda $Q(\xi_1, \xi_2)$ kvadratik formani quyidagicha yozib olamiz:

$$Q(\xi_1, \xi_2) = a_{11}\left[\left(\xi_1 + \frac{a_{12}}{a_{11}}\xi_2\right)^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}^2}\xi_2^2\right]. \qquad (13.36)$$

Keyingi tenglikdan $\xi_1 = -\frac{a_{12}}{a_{11}}$, $\xi_2 = -1$ qiymatda

$$Q\left(-\frac{a_{12}}{a_{11}},\ 1\right) < 0$$

va $\forall \xi_1 > -\frac{a_{12}}{a_{11}} + \sqrt{\frac{a_{12}^2 - a_{11}a_{22}}{a_{11}^2}}$, $\xi_2 = 1$ qiymatlarda esa

$$Q(\xi_1,\ 1) > 0$$

bo’lishini topamiz.

Va nihoyat, $a_{11} < 0$ bo’lsin. Bu holda (13.36) munosabatdan foydalanib, $Q(\xi_1, \xi_2)$ kvadratik formaning $\xi_1 = -\frac{a_{12}}{a_{11}}$, $\xi_2 = 1$ qiymatda musbat $Q\left(-\frac{a_{12}}{a_{11}},\ 1\right) > 0$

va $\forall \xi_1 > -\frac{a_{12}}{a_{11}} + \sqrt{\frac{a_{12}^2 - a_{11}a_{22}}{a_{11}^2}}$, $\xi_2 = 1$ qiymatda esa manfiy

$$Q(\xi_1,\ 1) < 0$$

bo’lishini topamiz.

Shunday qilib, $a_{11}a_{22}-a_{12}^2<0$ bo'lganda $Q(\xi_1,\xi_2)$ kvadratik formaning noaniq bo'lishi isbot etildi.

4)- holni, ya'ni $a_{11}a_{12}-a_{12}^2=0$ bo'lgan holni qaraylik. Bu holda, $a_{11}=0$ bo'lsa, unda $a_{12}=0$ bo'lib, $Q(\xi_1,\xi_2)$ kvadratik forma ushbu

$$Q(\xi_1,\xi_2)=a_{22}\xi_2^2$$

ko'rinishni oladi.

Ravshanki, $a_{22}\geq 0$ bo'lganda

$$Q(\xi_1,\xi_2)\geq 0,$$

$a_{22}\leq 0$ bo'lganda

$$Q(\xi_1,\xi_2)\leq 0$$

bo'lib, ξ_1 ning ixtiyoriy qiymatida

$$Q(\xi_1,0)=0$$

bo'ladi.

Agar $a_{11}>0$ bo'lsa,

$$Q(\xi_1,\xi_2)=a_{11}\left(\xi_1+\frac{a_{12}}{a_{11}}\xi_2\right)^2\leq 0,$$

$a_{11}<0$ bo'lganda

$$Q(\xi_1,\xi_2)=a_{11}\left(\xi_1+\frac{a_{12}}{a_{11}}\xi_2\right)^2\leq 0,$$

bo'lib, ξ_1 va ξ_2 larning

$$\xi_1=-\frac{a_{12}}{a_{11}}\xi_2$$

tenglikni qanoatlantiruvchi barcha qiymatlarida $Q(\xi_1,\xi_2)$ kvadratik forma nolga teng bo'ladi. Demak, qaralayotgan holda $Q(\xi_1,\xi_2)$ kvadratik forma yarimmusbat aniqlangan yoki yarimmanfiy aniqlangan bo'ladi.

13.9-misol. Ushbu

$$f(x_1,x_2)=x_1^3+x_2^3-3ax_1x_2 \qquad (a\neq 0)$$

funksiya ekstremumga tekshiriladi.

◀ Bu funksiyaning birinchi va ikkinchi tartibli hosilalari

$$f'_{x_1}(x_1,x_2)=3x_1^2-3ax_2, \qquad f'_{x_2}(x_1,x_2)=3x_2^2-3ax_1$$

$$f''_{x_1^2}(x_1,x_2)=6x_1, \qquad f''_{x_1x_2}(x_1,x_2)=-3a, \qquad f''_{x_2^2}(x_1,x_2)=6x_2$$

bo'ladi. Ushbu

$$\begin{cases}3x_1^2-3ax_2=0\\ 3x_2^2-3ax_1=0\end{cases}$$

sistemani echib, berilgan funksiyaning statsionar nuqtalari $(0,\ 0)$ va $(a,\ a)$ ekanini topamiz.

$(a,\ a)$ nuqtada

$$a_{11}=6a,\quad a_{12}=-3a,\qquad a_{22}=6a$$

bo'lib,

$$a_{11}a_{22}-a_{12}^2=27a^2>0$$

bo'ladi.

Demak, $a>0$ bo'lganda ($a_{11}>0$ bo'lib) funksiya $(a,\ a)$ nuqta minimumga erishadi, $a<0$ bo'lganda funksiya $(a,\ a)$ nuqtada maksimumga erishadi.

Ravshanki,

$$f(a,a)=-a^3.$$

$(0,\ 0)$ nuqtada

$$a_{11}a_{22}-a_{12}^2=-9a^2<0$$

bo'ladi. Demak, berilgan funksiya $(0,\ 0)$ nuqtada ekstremumga erishmaydi.▶

11-§. Oshkormas funksiyalar

1⁰. Oshkormas funksiya tushunchasi. Ma'lumki, $x\subset R$ to'plamdagi har bir x songa biror qoidaga ko'ra $Y\subset R$ to'plamdan bitta y son mos qo'yilgan bo'lsa, X to'plamda funksiya berilgan deb atalar va u

$$f:x\to y \text{ yoki } y=f(x)$$

kabi belgilanar edi.

Ikki x va y argumentlarning $F(x,y)$ funksiyasi

$$M=\{(x,y)\in R^2 : a<x<в,\ c<y<d\}$$

to'plamda berilgan bo'lsin. Ushbu

$$F(x,y)=0 \qquad (13.37)$$

tenglamani qaraylik. Biror x_0 sonni $(x_0\in(a,в))$ olib, uni yuqoridagi tenglamadagi x ning o'rniga qo'yamiz. Natijada y ni topish uchun quyidagi

$$F(x_0,y)=0$$

tenglamaga kelamiz. Bu tenglamaning echimi haqida ushbu hollar bo'lishi mumkin:

1). (13.37) tenglama yagona haqiqiy y_0 echimga ega,

2). (13.37) tenglama bitta ham haqiqiy echimga ega emas,

3). (13.37) tenglama bir nechta, hatto cheksiz ko'p haqiqiy echimga ega.

Masalan,

$$F(x,y)=\begin{cases} y-x^2, \text{ agar } x\ge 0 \text{ bo'lsa,}\\ y^2+x, \text{ agar } x<0 \text{ bo'lsa}\end{cases}$$

u holda

$$F(x,y)=0$$

tenglama $x_0\ge 0$ bo'lganda, yagona $y=x_0^2$ echimga, $x_0<0$ bo'lganda ikkita

$$y=\sqrt{-x_0},\quad y=-\sqrt{-x_0}$$

echimga ega bo'ladi.

Agar biror $F(x,y)=0$ tenglama uchun 1)- hol o'rinli bo'lsa bunday tenglama e'tiborga loyiq. Uning yordamida funksiya aniqlanishi mumkin.

Endi x o'zgaruvchining qiymatlaridan iborat shunday X to'plamni qaraylikki, bu to'plamdan olingan har bir qiymatda $F(x,y)=0$ tenglama yagona echimga ega bo'lsin.

X to'plamdan ixtiyoriy x sonni olib, bu songa $F(x,y)=0$ tenglamaning yagona echimi bo'lgan y sonni mos qo'yamiz. Natijada X to'plamdan olingan har bir x ga yuqoridagi ko'rsatilgan qoidaga ko'ra bitta y mos qo'yilib, funksiya hosil bo'ladi. Bunda x va y o'zgaruvchilar orasidagi bog'lanish $F(x,y)=0$ tenglama yordamida bo'ladi. Odatda bunday berilgan (aniqlangan) funksiya oshkormas ko'rinishda berilgan funksiya (yoki oshkormas funksiya) deb ataladi va

$$x \to y: \quad F(x,y)=0.$$

kabi belgilanadi.

13.10-misol. Ushbu

$$F(x,y)=y\sqrt{x^2-1}-2=0 \tag{13.38}$$

tenglama funksiya aniqlanishi ko'rsatilsin.

◄Ravshanki,

$$F(x,y)=y\sqrt{x^2-1}-2=0$$

tenglama x ning $R\backslash\{x\in R:\ -1\le x\le 1\}$ dan olingan har bir qiymatida yagona

$$y=\frac{2}{\sqrt{x^2-1}}$$

echimga ega, bundan

$$F\left(x,\frac{2}{\sqrt{x^2-1}}\right)\equiv 0.$$

Natijada (13.38) tenglama yordamida berilgan ushbu

$$x \to y=\frac{2}{\sqrt{x^2-1}} \quad : \quad F\left(x,\frac{2}{\sqrt{x^2-1}}\right)=0$$

oshkormas ko'rinishdagi funksiyaga ega bo'lamiz.►

13.11-misol. Ushbu

$$F(x,y)=x-y+\frac{1}{2}sin\,y=0 \tag{13.39}$$

tenglamani funksiya aniqlashi ko'rsatilsin.

◄Bu tenglamani

$$x=y-\frac{1}{2}sin\,y=\varphi(y)$$

ko'rinishda yozib olamiz. Ravshanki, $\varphi(y)$ funksiya $(-\infty,+\infty)$ da uzluksiz va $\varphi`(y)=1-\frac{1}{2}cos\,y>0$ hosilaga ega.

Unda teskari funksiya haqida teoremaga ko'ra (1-qism, 5-bob, 7-§) $y=\varphi^{-1}(x)$ funksiya mavjuddir. Demak, $(-\infty,+\infty)$ dan olingan x ning har bir qiymatida (13.39) tenglama yagona $y=\varphi^{-1}(x)$ echimga ega, bundan

$$F\left(x,\varphi^{-1}(x)\right)=0.$$

Har bir x ga $\varphi^{-1}(x)$ ni mos qo'yib,

$$x \to \varphi^{-1}(x): \quad F\left(x,\, \varphi^{-1}(x)\right) = 0$$

oshkormas ko'rinishdagi funksiyaga ega bo'lamiz. ▶

13.12-misol. Quyidagi

$$F(x,y) = x^2 + y^2 - ln\, y = 0 \qquad\qquad (y > 0)$$

tenglamani funksiya aniqmasligi ko'rsatilsin.

◀ Bu tenglama x ning $(-\infty,+\infty)$ oraliqdan olingan hech bir qiymatida echimga ega emas. Chunki har doim $y^2 - ln\, y > 0$. Bu holda berilgan tenglama yordamida funksiya aniqlanmaydi. ▶

2⁰. Oshkormas funksiyaning mavjudligi. Biz yuqorida

$$F(x,y) = 0$$

tenglama yordamida har doim oshkormas ko'rinishdagi funksiya aniqlanavermasligini ko'rdik.

Endi tenglama, ya'ni $F(x,y)$ funksiya qanday shartlarni bajarganda oshkormas ko'rinishdagi funksiyaning aniqlanishi, boshqacha aytganda, oshkormas ko'rinishdagi funksiyaning mavjud bo'lishi masalasi bilan shug'ullanamiz.

11-teorema. $F(x,y)$ funksiya $(x_0,y_0) \in R^2$ nuqtaning biror

$$U_{h,k}((x_0,y_0)) = \left\{(x,y) \in R^2 : x^0 - h < x < x_0 + h,\ y_0 - k < y < y_0 + k\right\}$$

atrofida $(h > 0,\ k > 0)$ berilgan va u quyidagi shartlarni bajarsin:

1) $U_{h,k}((x_0,y_0))$ da uzluksiz;

2) x o'zgaruvchining $(x_0 - h,\ x_0 + h)$ oraliqdan olingan har bir tayin qiymatida y o'zgaruvchining funksiyasi sifatida o'suvchi;

3) $F(x_0,y_0) = 0$.

U holda (x_0,y_0) ning shunday

$$U_{\delta,\varepsilon}((x_0,y_0)) = \left\{(x,y) \in R^2 :\ x_0 - \delta < x < x_0 + \delta;\ y_0 - \varepsilon < y < y_0 + \varepsilon\right\}$$

atrofi $(0 < \delta < h,\ 0 < \varepsilon < k)$ topiladiki,

1¹) $\forall x \in (x_0 - \delta, x_0 + \delta)$ uchun

$$F(x,y) = 0$$

tenglama yagona y echimga $(y \in (y_0 - \varepsilon, y_0 + \varepsilon))$ ega, ya'ni $F(x,y) = 0$ tenglama yordamida

$$x \to y: \quad F(x,y) = 0$$

oshkormas ko'rinishdagi funksiya aniqlanadi,

2¹) $x = x_0$ bo'lganda unga mos kelgan u uchun $y = y_0$ bo'ladi,

3¹) oshkormas ko'rinishda aniqlangan

$$x \to y: \quad F(x,y) = 0$$

funksiya $(x_0 - \delta, x_0 + \delta)$ oraliqda uzluksiz bo'ladi.

◀ $U_{h,k}((x_0,y_0))$ atrofga tegishli bo'lgan $(x_0,y_0-\varepsilon)$, $(x_0,y_0+\varepsilon)$ nuqtalarni olaylik. Ravshanki, $[y_0-\varepsilon,\ y_0+\varepsilon]$ oraliqda $F(x_0,y)$ funksiya o'suvchi bo'ladi. Demak,

$$y_0-\varepsilon<y_0 \Rightarrow F(x_0,y_0-\varepsilon)<F(x_0,y_0),$$
$$y_0+\varepsilon>y_0 \Rightarrow F(x_0,y_0+\varepsilon)>F(x_0,y_0).$$

Teoremaning 3-shartiga ko'ra

$$F(x_0,y_0-\varepsilon)<0,\ \ F(x_0,y_0+\varepsilon)>0$$

bo'ladi.

Teoremaning 1-shartiga ko'ra $F(x_0,y)$ funksiya $U_{h,k}((x_0,y_0))$ da uzluksiz. Binobarin, $F(x,y_0-\varepsilon)$ va $F(x,y_0+\varepsilon)$ funksiyalar $(x_0-h,\ x_0+h)$ oraliqda uzluksiz bo'ladi. Unda uzluksiz funksiyaning xossasiga ko'ra (qaralsin, 1-qism, 5-bob, 7-§) x_0 nuqtaning shunday atrofi $(x_0-\delta,\ x_0+\delta)$ topiladiki, $(0<\delta<h), \forall x\in(x_0-\delta,\ x_0+\delta)$ uchun $F(x,y_0-\delta)<0$, $F(x,y_0+\delta)>0$ bo'ladi.

Ravshanki, (x_0,y_0) nuqtaning ushbu

$$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2:\ x_0-\delta<x<x_0+\delta;\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

atrofi uchun teoremaning barcha shartlari bajarilavermaydi, chunki

$$U_{\delta,\varepsilon}((x_0,y_0))\subset U_{h,k}((x_0,y_0))$$

$\forall x^*\in(x_0-\delta,x_0+\delta)$ nuqtani olib, $F(x^*,y)$ funksiyani qaraylik. Bu funksiya, yuqorida aytilganiga ko'ra $[y_0-\varepsilon,\ y_0+\varepsilon]$ oraliqda uzluksiz va uning chetki nuqtalarida turli ishorali qiymatlarga ega:

$$F(x^*,y_0-\varepsilon)<0,\ \ F(x^*,y_0+\varepsilon)>0.$$

U holda Boltsano-Koshining birinchi teoremasiga ko'ra (qaralsin, 1-qism, 5-bob, 7-§) shunday y^* topiladiki $(y^*\in(y_0-\varepsilon,\ y_0+\varepsilon))$

$$F(x^*,y^*)=0$$

bo'ladi. Bu topilgan y^* yagona bo'ladi. Haqiqatdan ham,

$$y\neq y^*\Rightarrow F(x^*,y)\neq F(x^*,y^*) \qquad (y\in[y_0-\varepsilon,\ y_0+\varepsilon])$$

chunki, $F(x^*,y)$ o'suvchi bo'lganligi sababli $y>y^*$ uchun $F(x^*,y)>F(x^*,y^*)$ va $y<y^*$ uchun $F(x^*,y)<F(x^*,y^*)$ bo'ladi.

Shunday qilib, x ning $(x_0-\delta,\ x_0+\delta)$ oraliqdan olingan har bir qiymatida $F(x,y)=0$ tenglama yagona y echimga ega ekanligi ko'rsatildi. Bu esa $F(x,y)=0$ tenglama yordamida

$$x\to y:\ F(x,y)=0 \qquad (13.40)$$

oshkormas ko'rinishdagi funksiya aniqlanganligini bildiradi.

$x=x_0$ bo'lsin. Unda teoremaning 3-sharti $F(x_0,y_0)=0$ dan x_0 ga y_0 ni mos quyilgandagina:

$$x_0\to y_0:\ F(x,y)=0.$$

Demak, $x=x_0$ da oshirmas funksiyaning qiymati y_0 ga teng bo'ladi.

Endi oshkormas funksiyaning $(x_0-\delta,\ x_0+\delta)$ oraliqda uzluksiz bo'lishini ko'rsatamiz.

Ravshanki, $x\in(x_0-\delta,\ x_0+\delta)$ ga mos qo'yiladigan $y\in(y_0-\varepsilon,\ y_0+\varepsilon)$ bo'ladi. Bu esa oshkormas funksiyaning $x=x_0$ nuqtada uzluksiz ekanligini bildiradi.

Oshkormas funksiyaning $\forall x^*\in(x_0-\delta,x_0+\delta)$ nuqtada uzluksiz bo'lishini ko'rsatish bu funksiyaning x_0 nuqtada uzluksiz bo'lishini ko'rsatish kabidir.

Haqiqatdan ham, $F(x,y)=0$ tenglama (x_0,y_0) nuqtaning atrofi $U_{\delta,\varepsilon}((x_0,y_0))$ da oshkormas funksiyani aniqlaganligidan, shunday $y^*\in(y_0-\varepsilon,\ y_0+\varepsilon)$ topiladiki, $F(x^*,y^*)=0$ bo'ladi. Yuqoridagi mulohazani (x^*,y^*) nuqtaga nisbatan yuritib, $F(x,y)=0$ tenglama (x^*,y^*) nuqtaning atrofida oshkormas ko'rinishdagi funksiyani aniqlashini (bu aniqlangan funksiya (13.40) ning o'zi bo'ladi), uni x^* nuqtada uzluksiz bo'lishini topamiz. Demak, oshkormas funksiya $(x_0-\delta,x_0+\delta)$ oraliqda uzluksiz bo'ladi. ▶

3-eslatma. Yuqoridagi 11-teorema $F(x,y)$ funksiya x o'zgaruvchining (x_0-h,x_0+h) oraliqdan olingan har bir tayin qiymatida y o'zgaruvchining funksiyasi sifatida kamayuvchi bo'lganda ham o'rinli bo'ladi.

12-teorema. $F(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning biror $U_{h,k}((x_0,y_0))$ atrofida $(h>0,k>0)$ berilgan va u quyidagi shartlarni bajarsin:

1) $U_{h,k}((x_0,y_0))$ va uzluksiz;

2) y o'zgaruvchining (y_0-k,y_0+k) oraliqdan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida o'suvchi (kamayuvchi);

3) $F(x_0,y_0)=0$.

U holda (x_0,y_0) nuqtaning shunday

$$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2:\ x_0-\delta<x<x_0+\delta;\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

atrofi $(0<\delta<h,\ 0<\varepsilon<\mathrm{k})$ topiladiki,

1¹) $\forall y\in(y_0-\varepsilon,y_0+\varepsilon)$ uchun

$$F(x,y)=0$$

tenglama yagona $x(x\in(x_0-\delta,x_0+\delta))$ echimga ega, ya'ni $F(x,y)=0$ tenglama yordamida $y\to x\ :F(x,y)=0$ oshkormas ko'rinishdagi funksiya aniqlanadi;

2¹) $y=y_0$ bo'lganda unga mos kelgan x uchun $x=x_0$ bo'ladi;

3¹) oshkormas ko'rinishda aniqlangan funksiya

$$y\to x\ :F(x,y)=0$$

$(y_0-\varepsilon,y_0+\varepsilon)$ da uzluksiz bo'ladi.

Bu teoremaning isboti yuqorida keltirilgan 11-teoremaning isboti kabidir.

3⁰. Oshkormas funksiyaning hosilasi. Endi oshkormas funksiyaning hosilasini topish bilan shug'ullanamiz.

13-teorema. $F(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning biror

$U_{h,k}((x_0,y_0))=\{(x,y)\in R^2:\ x_0-h<x<x_0+h;\ y_0-k<y<y_0+k\}$ atrofida $(h>0,k>0)$ berilgan va u quyidagi shartlarni bajarsin:

1) $U_{h,k}((x_0,y_0))$ da uzluksiz;

2) $U_{h,k}((x_0,y_0))$ da uzluksiz $F`_x(x,y)$, $F`_y(x,y)$ xususiy hosilalarga ega va $F`_y(x_0,y_0)\neq 0$;

3) $F`_y(x_0,y_0)=0$.

U holda (x_0,y_0) nuqtaning shunday

$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2:\ x_0-\delta<x<x_0+\delta;\ y_0-\varepsilon<y<y_0+\varepsilon\}$ atrofi $(0<\delta<h,\ 0<\varepsilon<\mathrm{k})$ topiladiki,

1[1]) $\forall x\in(x_0-\delta,x_0+\delta)$ uchun

$$F(x,y)=0$$

tenglama yagona y echimga $y\in(y_0-\varepsilon,\ y_0+\varepsilon)$ ega, ya'ni $F(x,y)=0$ tenglama yordamida

$$x\to y:\ F(x,y)=0$$

oshkormas ko'rinishdagi funksiya aniqlanadi;

2[1]) $x=x_0$ bo'lganda unga mos keladigan y uchun $y=y_0$ bo'ladi;

3[1]) oshkormas ko'rinishda aniqlangan

$$x\to y:\ F(x,y)=0$$

funksiya $(x_0-\delta,x_0+\delta)$ oraliqda uzluksiz bo'ladi;

4[1]) Bu oshkormas ko'rinishdagi funksiya $(x_0-\delta,x_0+\delta)$ oraliqda uzluksiz hosilaga ega bo'ladi.

◄Shartga ko'ra $F`_y(x,y)$ funksiya $U_{h,k}((x_0,y_0))$ da uzluksiz va $F`_y(x_0,y_0)\neq 0$. Aniqlik uchun $F`_y(x_0,y_0)>0$ deylik. U holda uzluksiz funksiyaning xossasiga ko'ra (x_0,y_0) nuqtaning shunday

$$U_{\delta,\varepsilon}((x_0,y_0))=\{(x,y)\in R^2:\ x_0-\delta<x<x_0+\delta;\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

atrofi $(0<\delta<h,\ 0<\varepsilon<\mathrm{k})$ topiladiki, $\forall(x,y)\in U_{\delta,\varepsilon}((x_0,y_0))$ uchun $F'_y(x,y)>0$ bo'ladi. Demak, $F(x,y)$ funksiya x o'zgaruvchining $(x_0-\delta,x_0+\delta)$ oraliqdan olingan har bir tayin qiymatida y o'zgaruvchining funksiyasi sifatida o'suvchi. Yuqorida isbot etilgan 11-teoremaga ko'ra

$$F(x,y)=0$$

tenglama $(x_0-\delta,x_0+\delta)$ da

$$x\to y:\ F(x,y)=0$$

oshkormas ko'rinishdagi funksiyani aniqlaydi, $x=x_0$ bo'lganda unga mos kelgan $y=y_0$ bo'ladi va oshkormas funksiya $(x_0-\delta,x_0+\delta)$ da uzluksiz bo'ladi.

Endi oshkormas funksiyaning hosilasini topamiz, x_0 nuqtaga shunday Δx orttirma beraylikki, $x_0+\Delta x\in(x_0-\delta,x_0+\delta)$ bo'lsin. Natijada

$$x\to y:\ F(x,y)=0$$

oshkormas funksiya ham orttirmaga ega bo'lib,

$$F(x_0+\Delta x,y_0+\Delta y)=0$$

bo'ladi. Demak,

$$\Delta F(x_0,y_0)=F(x_0+\Delta x,y_0+\Delta y)-F(x_0,y_0)=0 \qquad (13.41)$$

Shartga ko'ra $F`_x(x,y)$ va $F`_y(x,y)$ xususiy hosilalar $U_{\delta,\varepsilon}((x_0,y_0))$ da uzluksiz. Binobarin $F(x,y)$ funksiya (x_0,y_0) nuqtada differensiallanuvchi:

$$\Delta F(x_0,y_0)=F`_x(x_0,y_0)\Delta x+F`_y(x_0,y_0)\Delta y+\alpha\Delta x+\beta\Delta y \qquad (13.42)$$

Bu munosabatdagi α va β lar Δx va Δy larga bog'liq va $\Delta x\to 0$, $\Delta y\to 0$ da $\alpha\to 0$, $\beta\to 0$.

(13.41) va (13.42) munosabatlardan

$$\frac{\Delta y}{\Delta x}=-\frac{F`_x(x_0,y_0)+\alpha}{F`_y(x_0,y_0)+\beta}$$

ekanligi kelib chiqadi.

Oshkormas funksiyaning x_0 nuqtada uzluksizligini e'tiborga olib, keyingi tenglikda $\Delta x\to 0$ da limitga o'tib quyidagini topamiz:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(-\frac{F`_x(x_0,y_0)+\alpha}{F`_y(x_0,y_0)+\beta}\right)=-\frac{F`_x(x_0,y_0)}{F`_y(x_0,y_0)}.$$

Demak,

$$y`_{x=x_0}=-\frac{F`_x(x_0,y_0)}{F`_y(x_0,y_0)}$$

$U_{\delta,\varepsilon}((x_0,y_0))$ da $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalar uzluksiz va $F'_y(x,y)\neq 0$ bo'lishidan oshkormas funksiyaning hosilasi

$$y`_x=-\frac{F`_x(x,y)}{F`_y(x,y)}$$

ning $(x_0-\delta,x_0+\delta)$ oraliqda uzluksiz bo'lishi kelib chiqadi.►

13.13-misol. Ushbu

$$F(x,y)=xe^y+ye^x-2=0 \qquad (13.43)$$

tenglama bilan aniqlanadigan oshkormas funksiyaning hosilasi topilsin.

◄Ravshanki, $F(x,y)=xe^y+ye^x-2$ funksiya $\{(x,y)\in R^2 : -\infty<x<+\infty$ $-\infty<y<+\infty\}$ to'plamda yuqoridagi 11-teoremaning barcha shartlarini qanoatlantiradi. Demak, $\forall(x_0,y_0)\in R^2$ nuqtaning $U_{\delta,\varepsilon}((x_0,y_0))$ atrofida (13.43) tenglama oshkormas ko'rinishdagi funksiyani aniqlaydi va bu oshkormas funksiyaning hosilasi

$$y`=-\frac{F`_x(x,y)}{F`_y(x,y)}=-\frac{e^y+ye^x}{xe^y+e^x}$$

bo'ladi.►

Oshkormas ko'rinishdagi funksiyaning hosilasini quyidagicha ham hisoblasa bo'ladi. y ning x ga bog'liq ekanini e'tiborga olib, $F(x,y)=0$ dan topamiz:

$$F`_x(x,y)+F`_y(x,y)\cdot y`=0$$

Bundan esa

$$y`=-\frac{F`_x(x,y)}{F`_y(x,y)}$$

bo'lishi kelib chiqadi.

Yuqorida keltirilgan (13.43) tenglama yordamida aniqlangan oshkormas ko'rinishdagi funksiyaning hosilasini hisoblaylik:

$$F`_x(x,y)+F`_y(x,y)\cdot y`=e^y+ye^x+\left(xe^y+e^x\right)y`=0$$

$$y`=-\frac{e^y+ye^x}{xe^y+e^x}.$$

4⁰. Oshkormas funksiyaning yuqori tartibli hosilalari. Faraz qilaylik,

$$F(x,y)=0$$

tenglama $(x_0,y_0)\in R^2$ nuqtaning $U_{\delta,\varepsilon}((x_0,y_0))$ atrofida oshkormas ko'rinishdagi funksiyani aniqlasin. Ma'lumki, $F(x,y)$ funksiya $U_{\delta,\varepsilon}((x_0,y_0))$ da uzluksiz $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalarga $\left(F'_y(x,y)\neq 0\right)$ ega bo'lsa, oshkormas ko'rinishdagi funksiya uzluksiz hosilaga ega bo'lib,

$$y`=-\frac{F`_x(x,y)}{F`_y(x,y)} \qquad (13.44)$$

bo'ladi.

Endi $F(x,y)$ funksiya $U_{\delta,\varepsilon}((x_0,y_0))$ da uzluksiz ikkinchi tartibli $F''_{x^2}(x,y)$, $F''_{xy}(x,y)$,, $F''_{y^2}(x,y)$ xususiy hosilalarga ega bo'lsin, y ning x ga bog'liqligini e'tiborga olib, (13.44) tenglikni x bo'yicha differensiallab quyidagini topamiz:

$$y''=-\frac{\left((F'_x(x,y))'_x F'_y(x,y)-\left(F'_y(x,y)\right)'_x F'_x(x,y)\right)}{\left(F'_y(x,y)\right)^2}.$$

Agar

$$\begin{aligned}\left(F'_x(x,y)\right)'_x &= F''_{x^2}(x,y)+F''_{xy}(x,y)\cdot y',\\ \left(F'_y(x,y)\right)'_x &= F''_{yx}(x,y)+F''_{y^2}(x,y)\cdot y'\end{aligned} \qquad (13.45)$$

ekanligini hisobga olsak, unda

$$y''=\frac{\left(F''_{yx}(x,y)+F''_{y^2}(x,y)y'\right)F'_x(x,y)-\left(F''_{x^2}(x,y)+F''_{xy}(x,y)y'\right)\cdot F'_y(x,y)}{\left(F'_y(x,y)\right)^2}=$$

$$=\frac{F''_{yx}(x,y)\cdot F'_x(x,y)-F''_{x^2}(x,y)\cdot F'_y(x,y)+\left[F''_{y^2}(x,y)\cdot F'_x(x,y)-F''_{xy}(x,y)\cdot F'_y(x,y)\right]y'}{\left(F'_y(x,y)\right)^2}.$$

bo'ladi. Bu ifodagi y' ning o'rniga qiymati $-\frac{F'_x(x,y)}{F'_y(x,y)}$ ni qo'yib, oshkormas ko'rinishdagi funksiyaning ikkinchi tartibli hosilasi uchun quyidagi formulaga kelamiz:

$$y''=\frac{2F'_x(x,y)\cdot F'_y(x,y)\cdot F''_{yx}(x,y)-F'^2_y(x,y)\cdot F''_{x^2}(x,y)\cdot F'^2_x(x,y)\cdot F''_{y^2}(x,y)}{\left(F'_y(x,y)\right)^3}$$

Xuddi shu yo'l bilan oshkormas funksiyaning uchinchi va xokazo tartibdagi hosilalari topiladi.

4-eslatma. Ushbu

$$F(x,y)=0$$

tenglama bilan aniqlangan oshkormas ko'rinishdagi funksiyaning yuqori tartibli hosilalarini quyidagicha ham hisoblasa bo'ladi. $F(x,y)=0$ ni differensiallab,

$$F'_x(x,y)+F'_y(x,y)y'=0$$

bo'lishini topgan edik. Buni yana bir marta differensiallaymiz:

$$\left(F'_x(x,y)\right)'_x+\left(F'_y(x,y)y'\right)'_x=$$

$$=\left(F'_x(x,y)\right)'_x+y'\left(F'_y(x,y)\right)'_x+F'_y(x,y)y''=0.$$

Yuqoridagi (13.45) munosabatdan foydalansak, u holda ushbu

$$F''_{x^2}(x,y)+2F''_{yx}(x,y)+F''_{y^2}(x,y)y'^2+F'_y(x,y)y''=0$$

tenglikka kelamiz. Undan esa

$$y''=-\frac{F''_{x^2}(x,y)+2F''_{yx}(x,y)y'+F'_{y^2}(x,y)y'^2}{F'(x,y)}$$

bo'lishi kelib chiqadi. Bu tenglikdagi y' ning o'rniga uning qiymati $-\dfrac{F'_x(x,y)}{F'_y(x,y)}$ ni qo'ysak, unda

$$y''=\frac{2F'_x(x,y)\cdot F'_y(x,y)F''_{xy}(x,y)-F'^2_y(x,y)\cdot F''_{x^2}(x,y)-F'^2_x(x,y)F''_{y^2}(x,y)}{\left(F'_y(x,y)\right)^3}$$

bo'ladi.

13.13-misol. Ushbu

$$F(x,y)=xe^y+ye^x-2=0$$

tenglama yordamida aniqlangan oshkormas funksiyaning ikkinchi tartibli hosilasi topilsin.

◀Berilgan tenglamadan, differensiallash bilan

$$e^y+ye^x+\left(xe^y+e^x\right)y'=0$$

bo'lishini topgan edik.buni yana bir marta differensiallab topamiz:

$$e^y\cdot y'+y'e^x+ye^x+e^yy'+xe^yy'\cdot y'+xe^yy''+y''e^x+y'e^x=0$$

ya'ni

$$y''\left(xe^y+e^x\right)+2e^yy'+2e^xy'+xe^yy'^2+ye^x=0$$

Bundan esa

$$y''=-\frac{2e^yy'+2e^xy'+xe^yy'^2+ye^x}{xe^y+e^x}$$

bo'lishi kelib chiqadi. Bu tenglikdagi y' ning o'rniga uning qiymati

$$y'=-\frac{e^y+ye^x}{e^x+xe^y}$$

ni qo'yib, oshkormas funksiyaning ikkinchi tartibli hosilasini topamiz.▶

5^0. Ko'p o'zgaruvchili oshkormas funksiyalar. Ko'p o'zgaruvchili oshkormas ko'rinishdagi funksiya tushunchasi yuqorida o'rganilgan bir o'zgaruvchili oshkormas ko'rinishdagi funksiya tushunchasi kabi kiritiladi.

$F(x,y)=F(x_1,x_2,...,x_m,y)$ funksiya $\left(x=(x_1,x_2,...,x_m)\in R^m\right)$

$$M=\left\{(x,y)\in R^{m+1} : a_1<x_1<в_1,\ a_2<x_2<в_2,\ ...,a_m<x_m<в_m,\ c<y<d\right\}$$

to'plamda berilgan bo'lsin. Ushbu

$$F(x,y)=F(x_1,x_2,...,x_m,y)=0 \qquad (13.46)$$

tenglamani qaraylik.

$x\in R^m$ nuqtalardan iborat shunday X to'plamni $\left(X\subset R^m\right)$ qaraylikki, bu to'plamdan olingan har bir nuqtada (13.46) tenglama yagona haqiqiy echimga ega bo'lsin. Endi x nuqtani olib, bu nuqtaga (13.46) tenglamaning yagona echimi bo'lgan y ni mos qo'yamiz. Natijada X to'plamdan olingan har bir x nuqtaga, yuqorida ko'rsatilgan qoidaga ko'ra, bitta y mos qo'yilib, funksiya hosil bo'ladi. Bunday aniqlangan funksiya ko'p o'zgaruvchili (m ta o'zgaruvchili) oshkormas ko'rinishda berilgan funksiya deb ataladi va

$$(x_1,x_2,...,x_m)\to y : F(x_1,x_2,...,x_m,y)=0$$

yoki

$$x\to y :\ F(x,y)=0$$

kabi belgilanadi.

13.14-misol. Ushbu

$$F(x_1,x_2,y)=x_1^2x_2-x_2^2y+x_1y=0$$

tenglama oshkormas funksiyani aniqlashi ko'rsatilsin.

◀Ravshanki,

$$F(x_1,x_2,y)=x_1^2x_2-x_2^2y+x_1y=0$$

tenglama $R^2\setminus\left\{(x_1,x_2)\in R^2 : x_1=x_2^2\right\}$ to'plamda olingan har bir (x_1,x_2) nuqtada yagona

$$y=\frac{x_1^2x_2}{x_2^2-x_1}$$

echimga ega, ya'ni

$$F\left(x_1,x_2,\frac{x_1^2x_2}{x_2^2-x_1}\right)=0$$

Demak, berilgan tenglama yordamida x_1,x_2 o'zgaruvchilarning oshkormas ko'rinishdagi funksiyasi aniqlanadi:

$$(x_1,x_2)\to\frac{x_1^2x_2}{x_2^2-x_1} : F\left(x_1,x_2,\frac{x_1^2x_2}{x_2^2-x_1}\right)=0 ▶$$

Endi ko'p o'zgaruvchili oshkormas ko'rinishdagi funksiyaning mavjudligi, uzluksizligi hamda hosilalarga ega bo'lishi haqida teoremalarni keltiramiz.

14-teorema. $F(x,y)=F(x_1,x_2,...,x_m,y)$ funksiya $\left(x^0,y_0\right)=\left(x_1^0,x_2^0,...,x_m^0,y_0\right)\in R^{m+1}$ nuqtaning biror $U_{h_1h_2...h_mk}\left(\left(x^0,y_0\right)\right)=\left(x_1^0,x_2^0,...,x_m^0,y_0\right)\in R^{m+1};\ x_1^0-h_1<x_1<x_1^0+h_1,$

$x_2^0 - h_2 < x_2 < x_2^0 + h_2 \quad ,..., x_m^0 - h_m < x_m < x_m^0 + h_m \quad y_0 - k < y < y_0 + k$ atrofida $(h_i > 0;\ i = 1,\ 2,\ ...,m;\ k > 0)$ berilgan va u quyidagi shartlarni bajarsin:

1) $U_{h_1, h_2, ..., h_m k}((x^0, y_0))$ da uzluksiz;

2) $x = (x_1, x_2, ... x_m)$ o'zgaruvchining

$$\{(x_1, x_2, ... x_m) \in R^m : x_1^0 - h_1 < x_1 < x_1^0 + h_1, x_2^0 - h_2 < x_2 < x_2^0 + h_2, ...,$$
$$, x_m^0 - h_m < x_m < x_m^0 + h_m\}$$

to'plamdan olingan har bir tayin qiymatida y o'zgaruvchining funksiyasi sifatida o'suvchi (kamayuvchi):

3) $F(x^0, y_0) = 0$.

U holda (x^0, y_0) nuqtaning shunday

$U_{\delta_1 \delta_2 ... \delta_m \varepsilon}((x^0, y_0)) = \{(x_1, x_2, ..., x_m, y) \in R^{m+1};\ x_1^0 - \delta_1 < x_1 < x_1^0 + \delta, ...,$ $, x_m^0 - \delta_m < x_m < x_m^0 + \delta_m,\ y_0 - \varepsilon < y < y_0 + \varepsilon\}$ atrofi $(0 < \delta_i < h_i,\ i = 1, 2, ..., m,$ $(0 < \varepsilon < k)$ topiladiki,

1[1]) $\forall x \in \{(x_1, x_2, ..., x_m) \in R^m : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1, ..., x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\}$ uchun

$$F(x, y) = 0 \qquad \textit{(13.47)}$$

tenglama yagona $y(y \in (y_0 - \varepsilon, y_0 + \varepsilon))$ echimga ega, ya'ni (13.47) tenglama $x \to y:\ F(x, y) = 0$ oshkormas ko'rinishdagi funksiyani aniqlaydi:

2[1]) $x = x^0$ bo'lganda, unga mos kelgan $y = y_0$ bo'ladi:

3[1]) oshkormas ko'rinishda aniqlangan

$$x \to y:\ F(x, y) = 0$$

funksiya

$$\{(x_1, x_2, ... x_m) \in R^m : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1, x_2^0 - \delta_2 < x_2 < x_2^0 + \delta_2, ...,$$
$$, x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\}$$

to'plamda uzluksiz bo'ladi.

15-teorema. $F(x, y)$ funksiya $(x^0, y_0) \in R^{m+1}$ nuqtaning biror $U_{h_1 h_2 .. h_m k}((x^0, y_0))$ atrofida berilgan va u quyidagi shartlarni bajarsin:

1) $U_{h_1, h_2, ..., h_m k}((x^0, y_0))$ da uzluksiz;

2) $U_{h_1, h_2, ..., h_m k}((x^0, y_0))$ da uzluksiz $F'_{x_i}(x_1, x_2, ..., x_m, y)$ $(i = 1, 2, 3, ..., m)$ $F'_y(x_1, x_2, ..., x_m, y)$ xususiy hosilalarga ega va $F'_y(x_1, x_2, ..., x_m, y) \neq 0$

3) $F(x^0, y_0) = 0$.

U holda (x^0, y_0) nuqtaning shunday $U_{\delta_1, \delta_2, ..., \delta_m \varepsilon}((x^0, y_0))$ atrofi $(0 < \delta_i < h_i,\ i = 1, 2, ..., m, (0 < \varepsilon < k)$ topiladiki,

1[1]) $\forall x \in \{(x_1, x_2, ..., x_m) \in R^m : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1, x_2^0 - \delta_2 < x_2 < x_2^0 + \delta_2, ...,$ $, x_m^0 - \delta_m < x_m < x_m^0 + \delta_m$ uchun

$$F(x,y)=0$$

tenglama yagona $y(y\in(y_0-\varepsilon,y_0+\varepsilon))$ echimga ega, ya'ni (13.47) tenglama $x\to y:\ F(x,y)=0$ oshkormas ko'rinishdagi funksiyani aniqlaydi:

2[1]) $x=x^0$ bo'lganda, unga mos kelgan $y=y_0$ bo'ladi:

3[1]) oshkormas ko'rinishda aniqlangan funksiya

$$\{(x_1,x_2,...x_m)\in R^m: x_1^0-\delta_1<x_1<x_1^0+\delta_1, x_2^0-\delta_2<x_2<x_2^0+\delta_2,...,$$
$$,x_m^0-\delta_m<x_m<x_m^0+\delta_m\}$$

to'plamda uzluksiz bo'ladi.

4[1]) bu oshkormas ko'rinishdagi funksiya uzluksiz xususiy hosilalarga ega bo'ladi.

Bu teoremalarning isboti yuqorida keltirilgan 12- va 13- teoremalarning isboti kabidir. Ularni isbotlashni o'quvchiga havola etamiz.

Ko'p o'zgaruvchili oshkormas funksiyaning hosilalari ham yuqoridagiga o'xshash hisoblanadi.

Faraz qilaylik,

$$F(x_1,x_2,...,x_m,y)=0$$

tenglama bo'lib, $F(x_1,x_2,...,x_m,y)$ funksiya 15- teoremaning barcha shartlarini qanoatlantirsin. Bu tenglama aniqlangan oshkormas funksiyaning xususiy hosilalarini topamiz. y ning $x_1,x_2,...,x_m$ larga bog'liq ekanini e'tiborga olib, (13.47) dan quyidagilarni topamiz.

$$F'_{x_1}(x_1,x_2,...,x_m,y)+F'_y(x_1,x_2,...,x_m,y)\cdot y'_{x_1}=0,$$
$$F'_{x_2}(x_1,x_2,...,x_m,y)+F'_y(x_1,x_2,...,x_m,y)\cdot y'_{x_2}=0,$$
$$..,$$
$$F'_{x_m}(x_1,x_2,...,x_m,y)+F'_y(x_1,x_2,...,x_m,y)\cdot y'_{x_m}=0$$

Keyingi tengliklardan esa

$$y'_{x_1}=-\frac{F'_{x_1}(x_1,x_2,...,x_m,y)}{F'_y(x_1,x_2,...,x_m,y)},$$
$$y'_{x_2}=-\frac{F'_{x_2}(x_1,x_2,...,x_m,y)}{F'_y(x_1,x_2,...,x_m,y)},$$
$$...,$$
$$y'_{x_m}=-\frac{F'_{x_m}(x_1,x_2,...,x_m,y)}{F'_y(x_1,x_2,...,x_m,y)}$$

bo'lishi kelib chiqadi.

$F(x,y)$ funksiya $U_{\delta_1,\delta_2,...,\delta_m\varepsilon}((x^0,y_0))$ da uzluksiz yuqori tartibli xususiy hosilalarga ega bo'lganda $F(x,y)=0$ tenglama aniqlangan oshkormas ko'rinishdagi funksiyaning ham yuqori tartibli hosilalari mavjud bo'ladi.

6[0]. Tenglamalar sistemasi bilan aniqlanadigan oshkormas funksiyalar. Endi tenglamalar sistemasi orqali aniqlanadigan funksiyalar bilan tanishaylik.

$m+n$ ta $x_1,x_2,...,x_m$ va $y_1,y_2,...,y_n$ argumentlarning ushbu n ta

$$F_i(x_1,x_2,...,x_m,y_1,y_2,...,y_n)\ (i=1,2,...,n)$$

funksiyalari R^{m+n} fazodagi biror

$$M=\{(x_1,x_2,...,x_m,y_1,y_2,...,y_n)\in R^{m+n}:\ a_1<x_1<в_1,\ a_2<x_2<в_2,\ ...,$$
$$,a_m<x_m<в_m;\ c_1<y_1<d_1,\ c_2<y_2<d_2,\ ...,\ c_n<y_n<d_n\}$$

to'plamda berilgan bo'lsin. Quyidagi

$$\begin{aligned} F_1&=F_1(x_1,x_2,...,x_m,y_1,y_2,...,y_n)=0,\\ F_2&=F_2(x_1,x_2,...,x_m,y_1,y_2,...,y_n)=0,\\ &...,\\ F_n&=F_n(x_1,x_2,...,x_m,y_1,y_2,...,y_n)=0 \end{aligned} \qquad (13.48)$$

tenglamalar sistemasini qaraylik. $x=(x_1,x_2,...,x_m)$ o'zgaruvchining qiymatlaridan iborat shunday

$$M_x=\{x=(x_1,x_2,...,x_m)\in R^m:\ a_1<x_1<в_1,\ a_2<x_2<в_2,\ ...,a_m<x_m<в_m\}\subset R^m$$

to'plamni qaraylikki, bu to'plamdan olingan har bir $x'=(x_1',x_2',...,x_m')$ nuqtada (13.48) sistema, ya'ni

$$\begin{aligned} &F_1(x_1',x_2',...,x_m',y_1,y_2,...,y_n)=0,\\ &F_2(x_1',x_2',...,x_m',y_1,y_2,...,y_n)=0,\\ &..,\\ &F_n(x_1',x_2',...,x_m',y_1,y_2,...,y_n)=0 \end{aligned}$$

sistema yagona echimlar sistemasi $y_1,y_2,...,y_n$ ga ega bo'lsin. Endi M_x to'plamdan ixtiyoriy $(x_1,x_2,...,x_m)$ nuqtani olib, bu nuqtaga (13.48) tenglamalar sistemasining yagona echimlari sistemasi bo'lgan $y_1,y_2,...,y_n$ ni mos qo'yamiz. Natijada M_x to'plamdan olingan har bir $(x_1,x_2,...,x_m)$ ga yuqorida ko'rsatilgan qoidaga ko'ra $y_1,y_2,...,y_n$ lar mos qo'yilib, n ta funksiya hosil bo'ladi. Bunday aniqlangan funksiyalar (13.48 tenglamalar sistemasi yordamida aniqlangan oshkormas ko'rinishdagi funksiyalar deb ataladi.

Qanday shartlar bajarilganda shu (13.48) tenglamalar sistemasi $y_1,y_2,...,y_n$ larning har birini $x_1,x_2,...,x_m$ o'zgaruvchilarining funksiyasi sifatida aniqlashi mumkinligi haqida masala muhim.

Avvalo soddaroq holni qaraymiz. Aytaylik, ikki $F_1=F_1(x_1,x_2\ \ y_1,y_2)$ va $F_2=F_2(x_1,x_2,y_1,y_2)$ funksiya $(x_1^0,x_2^0,y_1^0,y_2^0)\in R^4$ nuqtaning biror

$$U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0))=\{(x_1,x_2,y_1,y_2)\in R^4:x_1^0-h_1<x_1<x_1^0+h_1,$$
$$,x_2^0-h_2<x_2<x_2^0+h_2,\ y_1^0-k_1<y_1<y_1^0+k_1,\ y_2^0-k_2<y_2<y_2^0+k_2\}$$

atrofida $(h_1>0,\ h_2>0,\ k_1>0,\ k_2>0)$ berilgan bo'lsin. Ushbu

$$\begin{aligned} F_1&=F_1(x_1,x_2,y_1,y_2)=0,\\ F_2&=F_2(x_1,x_2,y_1,y_2)=0 \end{aligned} \qquad (13.49)$$

tenglamalar sistemasini qaraylik.

Faraz qilaylik, $F_1(x_1,x_2,y_1,y_2)$ va $F_2(x_1,x_2,y_1,y_2)$ funksiyalar uchun

$$F_1(x_1^0,x_2^0,y_1^0,y_2^0)=0, \quad F_2(x_1^0,x_2^0,y_1^0,y_2^0)=0$$

bo'lsin. Bundan tashqari qaralayotgan funksiyalar $U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0))$ da uzluksiz barcha xususiy hosilalarga ega va aytaylik,

$$\frac{\partial F_1(x_1^0,x_2^0,y_1^0,y_2^0)}{\partial y_1}\neq 0$$

bo'lsin. U holda 14-teoremaga ko'ra $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtaning shunday U_1 atrofi $(U_1\subset U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0)))$ topiladiki, bu atrofda

$$F_1(x_1,x_2,y_1,y_2)=0$$

tenglama

$$(x_1,x_2,y_2)\to y_1 : F_1(x_1,x_2,y_1,y_2)=0$$

oshkormas ko'rinishdagi funksiyani aniqlaydi. Shu funksiyani

$$y_1=f_1(x_1,x_2,y_2)$$

deb belgilaylik. Buni (13.49) sistemaning ikkinchi tenglamasidagi y_1 ning o'rniga qo'yib quyidagini topamiz:

$$F_2(x_1,x_2,f_1(x_1,x_2),y_2)=0$$

Endi

$$\frac{\partial F_2(x_1^0,x_2^0,f_1(x_1^0,x_2^0),y_2^0)}{\partial y_2}\neq 0 \qquad (13.50)$$

bo'lsin deylik. U holda yana 14-teoremaga ko'ra $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtaning shunday U_2 atrofi $(U_2\subset U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0)))$ topiladiki, bu atrofda

$$F_2(x_1,x_2,f_1(x_1,x_2),y_2)=0$$

tenglama

$$(x_1,x_2)\to y_2 : F_2(x_1,x_2,f_1(x_1,x_2),y_2)=0$$

oshkormas ko'rinishdagi funksiyani aniqlaydi. Bu funksiyani $y_2=f_2(x_1,x_2)$ deb belgilaylik.

Shunday qilib, (13.49) tenglamalar sistemasi $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtaning biror atrofida y_1 va y_2 larni x_1,x_2 o'zgaruvchilarning funksiyasi sifatida aniqlaydi:

$$y_1=f_1(x_1,x_2,f_2(x_1,x_2))$$
$$y_2=f_2(x_1,x_2)$$

Ravshanki, $f_1(x_1^0,x_2^0)$, $f_2(x_1^0,x_2^0)=y_1^0$, $f_2(x_1^0,x_2^0)=y_2^0$. Yuqoridagi (13.50) shartni quyidagicha yozish mumkin.

$$\frac{\partial F_2}{\partial y_2}+\frac{\partial F_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial y_2}\neq 0$$

Bunda barcha xususiy hosilalar $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtada hisoblangan. Agar

$$\frac{\partial y_1}{\partial y_2}=-\frac{\frac{\partial F_1}{\partial y_2}}{\frac{\partial F_1}{\partial y_1}}$$

ekanini e'tiborga olsak, unda

$$\frac{\partial F_2}{\partial y_2}+\frac{\partial F_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial y_2}=\frac{\partial F_2}{\partial y_2}+\frac{\partial F_2}{\partial y_1}\cdot\left(-\frac{\frac{\partial F_1}{\partial y_2}}{\frac{\partial F_1}{\partial y_1}}\right)=\frac{\frac{\partial F_2}{\partial y_2}\cdot\frac{\partial F_1}{\partial y_1}-\frac{\partial F_2}{\partial y_1}\cdot\frac{\partial F_1}{\partial y_2}}{\frac{\partial F_1}{\partial y_1}}\neq 0$$

bo'ladi. Modomiki,

$$\frac{\partial F_1}{\partial y_1}\neq 0$$

ekan, unda

$$\frac{\partial F_2}{\partial y_2}\cdot\frac{\partial F_1}{\partial y_1}-\frac{\partial F_2}{\partial y_1}\cdot\frac{\partial F_1}{\partial y_2}\neq 0$$

ya'ni

$$\begin{vmatrix}\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2}\\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}\end{vmatrix}\neq 0 \qquad (13.51)$$

bo'ladi. Shunday qilib, (13.50) munosabatni (13.51) ko'rinishda yozish mumkin ekan.

Natijada ushbu teoremaga kelamiz.

16-teorema. $F_1(x_1,x_2,y_1,y_2)$ va $F_2(x_1,x_2,y_1,y_2)$ funksiyalar $(x_1^0,x_2^0,y_1^0,y_2^0)\in R^4$ nuqtaning biror $U_{h_1h_2k_1k_2}$ atrofi $(h_1>0,\ h_2>0,\ k_1>0,\ k_2>0)$ berilgan va ular quyidagi shartlarni bajarsin:

1) $U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0))$ da uzluksiz;

2) $U_{h_1h_2k_1k_2}((x_1^0,x_2^0,y_1^0,y_2^0))$ da barcha xususiy hosilalarga ega va ular uzluksiz:

3) xususiy hosilalarning $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtadagi qiymatlaridan tuzilgan ushbu determinant noldan farqli:

$$\begin{vmatrix}\frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2}\\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}\end{vmatrix}\neq 0$$

4) $(x_1^0,x_2^0,y_1^0,y_2^0)$ da $F_1(x_1^0,x_2^0,y_1^0,y_2^0)=0,\quad F_2(x_1^0,x_2^0,y_1^0,y_2^0)=0.$

U holda $(x_1^0,x_2^0,y_1^0,y_2^0)$ nuqtaning shunday $U_{\delta_1\delta_2\varepsilon_1\varepsilon_2}((x_1^0,x_2^0,y_1^0,y_2^0))$ atrofi $(0<\delta_1<h_1,\ 0<\delta_2<h_2,\ 0<\varepsilon_1<k_1,\ 0<\varepsilon_2<k_2)$ topiladiki, bu atrofda

1¹) (13.48) tenglamalar sistemasi oshkormas ko'rinishdagi

$$y_1 = f_1(x_1, x_2, f_2(x_1, x_2)),\ y_2 = f_2(x_1, x_2)$$

funksiyalarni aniqlaydi;

2[1]) $(x_1, x_2) = (x_1^0, x_2^0)$ bo’lganda unga mos keladigan

$$y_1 = y_1^0 = f_1(x_1^0, x_2^0, f_2(x_1^0, x_2^0)),\ y_2 = y_2^0 = f_2(x_1^0, x_2^0)$$

bo’ladi.

3[1]) oshkormas ko’rinishda aniqlangan f_1 va f_2 funksiya

$$\{(x_1, x_2) \in R^2 : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1,\ \ x_2^0 - \delta_2 < x_2 < x_2^0 + \delta_2\}$$

to’plamda uzluksiz va barcha uzluksiz xususiy hosilalarga ega bo’ladi.

13.15-misol. Ushbu

$$\begin{cases} x_1x_2 + y_1y_2 = 1, \\ x_1y_2 + x_2y_1 = 3 \end{cases} \qquad (13.52)$$

sistema oshkormas funksiyani aniqlashi ko’rsatilsin.

◀Bu holda

$$F_1(x_1, x_2) = x_1x_2 + y_1y_2 - 1,$$
$$F_2(x_1, x_2) = x_1y_2 + y_1x_2 - 3$$

bo’lib, bu funksiyalar (1, -1, 1, 2) nuqtaning atrofida 16-teoremaning barcha shartlarini bajaradi. Haqiqatdan ham, $F_1(x_1, x_2, y_1, y_2)$, $F_2(x_1, x_2, y_1, y_2)$ funksiyalar uzluksiz, uzluksiz barcha xususiy hosilalarga ega, (1, -1, 1, 2) nuqtada

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

hamda

$$F_1(1, -1,\ 1,\ 2) = 0,\ \ F_2(1, -1,\ 1,\ 2) = 0$$

bo’ladi. Demak, (13.51) sistema y_1 va y_2 larni x_1, x_2 o’zgaruvchilarning funksiyasi sifatida aniqlaydi. Ravshanki, bu funksiyalar uzluksiz, xususiy hosilalarga ega. Berilgan (13.52) tenglamalar sistemasini bevosita y_1 va y_2 larga nisbatan echib quyidagilarni topamiz:

$$y_1 = \frac{-3 + \sqrt{9 + 4x_1x_2 - 4x_1^2x_2^2}}{2x_2}, \quad y_2 = \frac{3 + \sqrt{9 + 4x_1x_2 - 4x_1^2x_2^2}}{2x_1}. ▶$$

Endi (13.48) sistemaning oshkormas funksiyalarning aniqlanishini ta’minlaydigan (oshkormas funksiyalarning mavjudligini ifodalaydigan) teoremani isbotsiz keltiramiz.

17-teorema. $F_1, F_2, ..., F_n$ funksiyalaning har biri $(x^0, y^0) = (x_1^0, x_2^0, ..., x_m^0, y_1^0, y_2^0, ..., y_n^0)$ nuqtaning biror

$$U_{hk}(x^0, y^0) = U_{h_1h_2...h_mk_1k_2...k_n}((x_1^0, x_2^0, ...x_m^0, y_1^0, y_2^0, ..., y_n^0)) = \{(x, y) \in R^{m+n} :$$
$$: x_1^0 - h_1 < x_1 < x_1^0 + h_1,\ x_2^0 - h_2 < x_2 < x_2^0 + h_2,\ ...,\ x_m^0 - h_m < x_m < x_m^0 + h_m,$$
$$, y_1^0 - k_1 < y_1 < y_1^0 + k_1,\ y_2^0 - k_2 < y_2 < y_2^0 + k_2,\ ...,\ y_n^0 - k_n < y_n < y_n^0 + k_n\}$$

atrofida $\left(h_i > 0,\ i = 1,2,...,m;\ k_j > 0,\ j = 1,2,...,n\right)$ berilgan va ular quyidagi shartlarni bajarsin:

1) $U_{hk}\left(\left(x^0, y^0\right)\right)$ da uzluksiz;

2) $U_{hk}\left(\left(x^0, y^0\right)\right)$ da barcha xususiy hosilalarga ega va ular uzluksiz;

3) xususiy hosilalarning $\left(x^0, y^0\right)$ nuqtadagi qiymatlaridan tuzilgan ushbu determinant noldan farqli:

$$\begin{vmatrix} \frac{\partial F_1}{\partial y_1}, & \frac{\partial F_1}{\partial y_2}, & ..., & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1}, & \frac{\partial F_2}{\partial y_2}, & ..., & \frac{\partial F_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial y_1}, & \frac{\partial F_n}{\partial y_2}, & ..., & \frac{\partial F_n}{\partial y_n} \end{vmatrix} \neq 0$$

4) $\left(x^0, y^0\right) = \left(x_1^0, x_2^0, ..., x_m^0, y_1^0, y_2^0, ..., y_n^0\right)$ nuqtada $F_1\left(x^0, y^0\right) = 0$, $F_2\left(x^0, y^0\right) = 0$, $..., F_n\left(x^0, y^0\right) = 0$. U holda $\left(x^0, y^0\right)$ nuqtaning shunday $U_{\delta\varepsilon}\left(\left(x^0, y^0\right)\right) = U_{\delta_1\delta_2...\delta_m\varepsilon_1\varepsilon_2...\varepsilon_n}\left(x^0, y^0\right)$ atrofi $\left(0 < \delta_1 < h_1,\ 0 < \delta_2 < h_2, ...\ 0 < \delta_m < h_m,\ 0 < \varepsilon_1 < k_1,\ 0 < \varepsilon_2 < k_2, ..., 0 < \varepsilon_n < k_n\right)$ topiladiki, bu atrofda

1¹) (13.48) sistema oshkormas ko'rinishdagi funksiyalar sistemasini aniqlaydi. Ularni

$$y_1 = f_1\left(x_1, x_2, ..., x_m\right),\ y_2 = f_2\left(x_1, x_2, ..., x_m\right),\ ...,\ y_n = f_n\left(x_1, x_2, ..., x_m\right)$$

deylik;

2¹) $\left(x_1,\ x_2,\ ..., x_m\right) = \left(x_1^0,\ x_2^0,\ ...,\ x_m^0\right)$ da

$$\begin{aligned} &f_1\left(x_1^0,\ x_2^0,\ ...,\ x_m^0\right) = y_1^0, \\ &f_2\left(x_1^0,\ x_2^0,\ ...,\ x_m^0\right) = y_2^0, \\ &\cdots\cdots\cdots\cdots, \\ &f_n\left(x_1^0,\ x_2^0,\ ...,\ x_m^0\right) = y_n^0. \end{aligned}$$

bo'ladi;

3¹) oshkormas ko'rinishdagi aniqlangan $f_1, f_2, ..., f_n$ funksiyalar

$$\left\{\left(x_1, x_2, ..., x_m\right) \in R^m : x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1,\quad x_2^0 - \delta_2 < x_2 < x_2^0 + \delta_2, ..., \right.$$
$$\left. , x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\right\}$$

to'plamda uzluksiz va uzluksiz xususiy hosilalarga ega bo'ladi.

Mashqlar

13.15. Agar $\Delta x_1 \to 0$, $\Delta x_2 \to 0$, $..., \Delta x_m \to 0$ da $\alpha_1 \to 0$, $\alpha_2 \to 0$, $..., \alpha_m \to 0$ bo'lsa, $\alpha_1 \Delta x_1 + \alpha_2 \Delta x_2 + ... + \alpha_m \Delta x_m = o(\rho)$ bo'lish ko'rsatilsin, bunda $\rho = \sqrt{\Delta x_1^2 + \Delta x_2^2 + ... + \Delta x_m^2}$.

13.16. Ushbu

$f(x,y)=\sqrt[3]{x^3+y^3}$ funksiyaning $(0,\,0)$ nuqtada differensiallanuvchi emasligi isbotlansin.

13.17. Funksiya orttirmasini uning differensiali orqali taqribiy ifodalab, ushbu $\alpha=\sqrt{1{,}02^3+1{,}97^3}$ miqdor taqribiy hisoblansin.

13.18. Ushbu $u=\sqrt{xy+\dfrac{x}{y}}$ funksiya quyidagi $u\left(x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}\right)=xy$ tenglamani qanoatlantirishi ko’rsatilgan.

13.19. Ma’lum perimetrga ega bo’lgan uchburchaklar orasida yuzasi eng kattasi teng tomonli uchburchak ekanligi isbotlansin.

13.20. Ushbu $ye^x-x\,ln\,y-1=0$ tenglama $(0,\,1)$ nuqtaning atrofida uzluksiz oshkormas funksiyani aniqlashi ko’rsatilsin, uning hosilasi topilsin.

14-BOB

Функционаl кеtма-кеtlиклar va qatorlar

1-§. Функционаl кеtма-кеtlуклar

1⁰. Функционаl кеtма-кеtlиклар tyshyнchasu. Aytaylик, har bиr натyral $n\ (n\in N)$ soнga $X\subset R$ to’plamda аниqlангаn bиtta $f_n(x)$ функцияни mos qo’ядиган qоида berилgан bo’lsин. Вy qоидaga ко’ra

$$f_1(x),f_2(x),\ldots,f_n(x),\ldots \qquad (14.1)$$

to’plam hosиl bo’ladи. Odatda (14.1) ни функционаl кеtма-кеtlик (функциlгонал кеtма-кеtlиgи) deyиladи va уни умумиу had $f_n(x)$ orqalи $\{f_n(x)\}$ уоки $f_n(x)$ кabи belgиlанadи.

Masalaн, 1) har bиr $n\ (n\in N)$ soнga $\dfrac{1}{n^2+x^2}$ функцияни mos qo’yuvchи qоида yshby

$$\frac{1}{1+x^2},\ \frac{1}{4+x^2},\ \frac{1}{9+x^2},\ \ldots,\ \frac{1}{n^2+x^2},\ \ldots$$

функционаl кеtма-кеtlикни hosиl qиladи:

2) har bиr $n\ (n\in N)$ soнga $sin\dfrac{\sqrt{x}}{n}$ функцияни mos qo’yиsh bиlан qууиdagи

$$sin\frac{\sqrt{x}}{1},\ sin\frac{\sqrt{x}}{2},\ sin\frac{\sqrt{x}}{3},\ ...,\ sin\frac{\sqrt{x}}{n},\ ...$$

функционal кetma-кetlикка ega bo’lamиz.

Фaraz qилaylик, $\{f_n(x)\}$:

$$f_1(x), f_2(x),..., f_n(x),...$$

функционal кetma-кetlик $X \subset R$ to’plamda berилgaн (кetma-кetlикнинг har bиr hadи X to’plamda аниqlaнgaн) bo’lиb, $x_0 \in X$ bo’lsин.

1-ta’rиф. Agar $\{f_n(x_0)\}$:

$$f_1(x_0),\ f_2(x_0),\ ...,\ f_n(x_0),\ ...$$

soнlar кetma-кetlилиgи яqинlashyvchи (yzoqlashyvchи) bo’lsa, $\{f_n(x)\}$ функционal кetma-кetlик x_0 нyqtada яqинlashyvchи (yzoqlashyvchи) deyиladи, x_0 нyqta esa $\{f_n(x)\}$ нинg яqинlashиsh (yzoqlashиsh) нyqtasи deyиladи.

$\{f_n(x)\}$ функционal кetma-кetlикнинg barcha яqинlashиsh нyqtalarидaн иborat to’plam функционal кetma-кetlикнинg яqинlashиsh sohasи deyиladи.

Masalaн,

$$f_n(x) = \frac{1}{n^2 + x^2} \qquad (n = 1, 2, 3,...)$$

функционal кetma-кetlик $\forall x_0 \in R$ da яqинlashyvchи, bинobarин, унинg яqинlashиsh sohasи R bo’ladи,

$$f_n(x) = n^2 x + 1 \qquad (n = 1, 2, 3,...)$$

функционal кetma-кetlик фaqat $x = 0$ нyqtada яqинlashyvchи bo’ladи. Унинg яqинlashиsh sohasи bиtta нyqtadaн иborat to’plam bo’ladи.

Aytaylик, M to’plam $(M \subset R)$ $\{f_n(x)\}$ функционal кetma-кetlикнинg яqинlashиsh sohasи bo’lsин. Унda $\forall x \in M$ ychyн

$$f_1(x), f_2(x),..., f_n(x),...$$

кetma-кetlик cheкlи lимиtga ega bo’ladи.

2-ta’rиф. Yshby

$$f:\quad x \to \lim_{n\to\infty} f_n(x) \quad (x \in M)$$

функция $\{f_n(x)\}$ функционal кetma-кetlикнинг lимит функцияsи deyиladи. Demaк,

$$\lim_{n\to\infty} f_n(x) = f(x) \quad (x \in M).$$

1-musol. Yshby

$$f_n(x) = x^n \qquad (n = 1, 2, 3, ...)$$

функционal кetma-кetlикнинг яqинlashиsh sohasи hamda lимит функцияsи topиlsин.

◀ By функционal кetma-кetlик ychyн:
$\forall x \in [1, +\infty)$ da

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \infty,$$

$x = 1$ bo'lganda

$$\lim_{n\to\infty} f_n(1) = 1,$$

$\forall x \in (-1, 1)$ da

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0,$$

$\forall x \in (-\infty, 1]$ da кetma-кetlикнинг lимитu mavjyd bo'lmaydи.

Shyнday qilиb, berилgaн функционal кetma-кetlикнинг яqинlashиsh sohasи $M = (-1, 1]$ bo'lиb, lимит функцияsи

$$f(x) = \begin{cases} 0, \text{ agar } -1 < x < 1, \\ 1, \text{ agar } \quad x = 1 \end{cases}$$

bo'ladи. ▶

2^0. Функционal кetma-кetlикнинг tекus яqинlashyvchulugи. Aytaylик, $\{f_n(x)\}$:

$$f_1(x),\ f_2(x),\ ...,\ f_n(x),\ ...$$

функционal кetma-кetlикнинг яqинlashиsh sohasи M bo'lиb, lимит функцияsи $f(x)$ bo'lsин. Yнda har bиr $x_0 \in M$ нyqtada

$$\lim_{n\to\infty} f_n(x_0) = f(x_0)$$

я'ни

$$\forall\varepsilon>0,\ \exists n_0\in N,\ \forall n>n_0:\ |f_n(x_0)-f(x_0)|<\varepsilon$$

bo'ladи. Bунda n_0 natyral son $\varepsilon>0$ songa va olинган x_0 нуqtaga bog'lиq bo'ladи:

$$n_0=n_0(\varepsilon,x_0).$$

3-ta'rиф. Agar $\forall\varepsilon>0$ olинganda ham shyнday $n_0\in N$ topиlsaки, $\forall n>n_0$ va $\forall x\in M$ ychyн

$$|f_n(x)-f(x)|<\varepsilon$$

teнgsиzlиk bajarиlsa, я'ни

$$\forall\varepsilon>0,\ \exists n_0\in N,\ \forall n>n_0,\ \forall x\in M:\ |f_n(x)-f(x)|<\varepsilon$$

bo'lsa, $\{f_n(x)\}$ функционal кetma-кetlик M to'plamda $f(x)$ ga teкиs яqинlashadи (функционal кetma-кetlик teкиs яqинlashyvchи) deyиladи. Уни

$$f_n(x)\rightrightarrows f(x) \quad x\in M$$

кabи belgиlaнadи.

Bу holda ta'rифdagи n_0 natyral soн фaqat $\varepsilon>0$ ga bog'lиq bo'ladи:

$$n_0=n_0(\varepsilon).$$

4-ta'rиф. Agar

$$\forall n\in N,\ \exists\varepsilon_0>0,\ \exists x_0\in M:\ |f_n(x_0)-f(x_0)|\geq\varepsilon$$

bo'lsa, $\{f_n(x)\}$ функционal кetma-кetlик M to'plamda $f(x)$ ga teкиs яqинlashmaydи (нoteкиs яqинlashadи) deyиladи.

14.2-musol. Уshbу

$$f_n(x)=\frac{\sin nx}{n} \qquad (n=1,2,3,...)$$

функционal кetma-кetlикнинg lимит функцияsи topиlsин va унga teкиs яqинlashиshи кo'rsatиlsин.

◀ Ravshaнки,

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{\sin nx}{n}=0.$$

Demaк, lимит функция $f(x)=0$ bo'ladи.

Agar $\forall\varepsilon > 0$ sон olинганda ham $n_0 = \left[\frac{1}{3}\right]$ deyилsa, sgbtqsm$\forall n > n_0$ va $\forall x \in M = (-\infty, +\infty)$ ychyн

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n} - 0\right| \le \frac{1}{n} < \frac{1}{n_0 + 1} < \varepsilon$$

bo'lganлиgи sababлu

$$\frac{\sin nx}{n} \rightrightarrows 0$$

bo'ladu.▶

14.3-musol. Уshby

$$f_n(x) = \frac{nx}{1 + n^2x^2} \qquad (n = 1, 2, 3, ...)$$

функционal ketma-ketlикни $[0, 1]$ oraluqda teкиs яqинlashushga teкshиrилsин.

◀Berилgaн ketma-ketlикнинg luмut функциясu

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = 0$$

bo'ladu. Вy esa ta'rифga ко'ra qyyидagини buldиradu: $\forall\varepsilon > 0$ olинганda ham,

$$n_0 = n_0(\varepsilon, x) = \left[\frac{1}{\varepsilon x}\right] \qquad (x \neq 0)$$

deyилsa, sgbtqsm$\forall n > n_0$ ychyн

$$|f_n(x) - f(x)| = \left|\frac{nx}{1 + n^2x^2} - 0\right| = \frac{nx}{1 + n^2x^2} < \frac{1}{nx} \le \frac{1}{(n_0 + 1)x} < \varepsilon$$

bo'ladu. Ravshaнки, $x = 0$ bo'lsa, $\forall n \in N$ ychyн

$$f_n(o) = f(o) = 0.$$

Виroq, $\forall n \in N$, $\varepsilon_0 = \frac{1}{4}$, $x = \frac{1}{n}$ ychyн

$$\left|f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right)\right| = \frac{1}{1 + \frac{1}{n^2}n^2} = \frac{1}{2} > \varepsilon$$

bo'ladu.

Demaк, berилgaн функционal ketma-ketlик $[0, 1]$ da luмut функцияga teкиs яqинlashmaydu.▶

1-teorema. $\{f_n(x)\}$ функционal кetma-кetlикнинг M to'plamda $f(x)$ ga teкиs яqинlashиsh уchун

$$\lim_{n\to\infty} \sup_{x\in M} |f_n(x) - f(x)| = 0$$

bo'lиshи zarуr va etarlи.

◄***Zarуrlugu.*** M to'plam $\{f_n(x)\}$ функционal кetma-кetlик $f(x)$ lиmиt функцияga teкиs яqинlashsин. Ta'rифga кo'ra $\forall\varepsilon > 0$ olинganda ham shунday $n_0 \in N$ topиladики, $n > n_0$ bo'lganda M to'plamнинg barcha x нуqtalarи уchун

$$|f_n(x) - f(x)| < \varepsilon$$

bo'ladи. Вундан esa $\forall n > n_0$ уchун

$$M_n = \sup_{x\in M} |f_n(x) - f(x)| \le \varepsilon$$

bo'lиshи кelиb chиqadи. Demaк,

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \sup_{x\in M} |f_n(x) - f(x)| = 0.$$

Etarlulugu. M to'plamda $\{f_n(x)\}$ функционal кetma-кetlик $f(x)$ lиmиt функцияga ega bo'lиb,

$$\lim_{n\to\infty} \sup_{x\in M} |f_n(x) - f(x)| = 0$$

bo'lsин. Demaк, $\forall\varepsilon > 0$ olинganda ham shунday $n_0 \in N$ topиladики, barcha $n > n_0$ уchун

$$\sup_{x\in M} |f_n(x) - f(x)| < \varepsilon$$

bo'ladи. Agar уshbу

$$|f_n(x) - f(x)| \le \sup_{x\in M} |f_n(x) - f(x)|$$

муnosabatnи etиborga olsaк, у holda $\forall x \in M$ уchун

$$|f_n(x) - f(x)| < \varepsilon$$

bo'lиshини topamиz. Ву esa M to'plamda $\{f_n(x)\}$ функционal кetma-кetlик $f(x)$ lиmиt функцияga teкиs яqинlashиshини bиldиradи.►

14.4-misol. Уshbу

$$\{f_n(x)\} = \left\{e^{-(x-n)^2}\right\}$$

функционаl кetma-кetlикни $-c < x < c$ $(c > 0)$ интervalda teкиs яqинlashyvchiluги кo’rsatилsин.

◀Ву функционаl кetma-кetlикнинг lимиt функцияsи

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-(x-n)^2} = 0$$

bo’ladи. Натиjada

$$M_n = \sup_{-c<x<c} |f_n(x) - f(x)| = \sup_{-c<x<c} \left(e^{-(x-n)^2} - 0\right) = \sup_{-c<x<c} e^{-(x-n)^2} = e^{-(c-n)^2}$$

bo’lиb, ундан

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} e^{-(c-n)^2} = 0$$

bo’lишини topamиz.

Demaк, berилgан функционаl кetma-кetlик $(-c,\ c)$ oraлиqda $f(x) = 0$ lимиt функцияga teкиs яqинlashadи:

$$e^{-(x-n)^2} \rightrightarrows 0 \ \ (-c < x < c;\ \ c > 0).$$ ▶

14.5-musol. Qyyиdagи

$$\{f_n(x)\} = \left\{n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right)\right\} \quad (0 < x < +\infty)$$

функция lкetma-кetlик teкиs яqинlashyvchиликкa tekshирилsин.

◀Ву функционаl кetma-кetlикнинг lимиt функцияsини topamиz:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right) = \lim_{n\to\infty} \frac{1}{\sqrt{x + \frac{1}{n}} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (0 < x < +\infty).$$

Demaк, $f(x) = \dfrac{1}{2\sqrt{x}}$. Ву holda

$$M_n = \sup_{0<x<\infty} |f_n(x) - f(x)| = \sup_{0<x<\infty} \left| n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) - \frac{1}{2\sqrt{x}} \right| =$$

$$= \sup_{0<x<\infty} \left| \frac{1}{\sqrt{x+\frac{1}{n}} + \sqrt{x}} - \frac{1}{2\sqrt{x}} \right| = \sup_{0<x<\infty} \frac{\sqrt{x+\frac{1}{n}} - \sqrt{x}}{2\sqrt{x}\left(\sqrt{x+\frac{1}{n}} + \sqrt{x}\right)} =$$

$$= \sup_{0<x<\infty} \frac{1}{2n\sqrt{x}\left(\sqrt{x+\frac{1}{n}} + \sqrt{x}\right)^2} = \infty$$

bo’lиb, berиlgaн функционal кetma-кetlик ychyн 1-teoremaнинг shartи bajarилmaydи.Dmqrlyotgltmtltsqlshvchems▶

$X \subset R$ to’plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

функционal кetma-кetlик berиlgaн bo’lsин.

5-ta’rиф. Agar $\forall \varepsilon > 0$ soн olиngaнda ham shyнday $n_0 \in N$ soн mavjyd bo’lsaки, $n > n_0$, $m > n_0$ bo’lgaнda $\forall x \in X$ нyqtalar ychyн bиr yo’la

$$|f_n(x) - f_m(x)| < \varepsilon$$

teнgsиzlик bajarиlsa, $\{f_n(x)\}$ функционal кetma-кetlик X to’plamda фундameнtal кetma-кetlик deb ataladи.

2-teorema. (Кoshu teoremasu). $\{f_n(x)\}$ функционal кetma-кetlик X to’plamda lимиt функцияga ega bo’lиshи va yнga teкиs яqинlashиshи ychyн y X to’plamda фундameнtal bo’lиshи zaryr va etarlи.

◀***Zaryrlugu.*** X to’plamda $\{f_n(x)\}$ кetma-кetlик lимиt функцияga ega bo’lиb, yнga teкиs яqинlashsин:

$$f_n(x) \rightrightarrows f(x) \quad (x \in X).$$

Teкиs яqинlиshиsh ta’rифиga myvoфиq $\forall \varepsilon > 0$ soн olиngaнda ham, $\frac{\varepsilon}{2}$ ga кo’ra shyнday $n_0 \in N$ topиladики, $n > n_0$ bo’lgaнda $\forall x \in X$ нyqtalar ychyн

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

shyнингdeк, $m>n_0$ bo'lgaнda $\forall x\in X$ ychyн

$$|f_m(x)-f(x)|<\frac{\varepsilon}{2}$$

bo'ladи. У holda $n>n_0$, $m>n_0$ va $\forall x\in X$ ychyн

$$|f_n(x)-f_m(x)|\le|f_n(x)-f(x)|+|f_m(x)-f(x)|<\varepsilon \text{ bo'ld.}$$

Etarlulugu. $\{f_n(x)\}$ кetma-кetlик X to'plamda фyнdameнtal кetma-кetlик bo'lsин:

$$\forall\varepsilon>0,\ \exists n_0\in N,\ n>n_0,\ m>n_0,\ \forall x\in X:\ |f_n(x)-f_m(x)|<\varepsilon \qquad (14.2)$$

X to'plamdaн olинgaн har bиr x_0 da $\{f_n(x)\}$ фyнкциoнal кetma-кetlик $\{f_n(x_0)\}$ soнlar кetma-кetlиgиga aylaнadи. Ravshaнки, $\{f_n(x_0)\}$ кetma-кetlик фyнdameнtal кetma-кetlик bo'ladи.

У holda Кoshи teoremasиga asosaн (1-qиsm, 4-bob, 3-§) $\{f_n(x_0)\}$ яqинlashyvchи. Demaк, X to'plamнинg har bиr x_0 нyqtasиda $\{f_n(x_0)\}$ кetma-кetlик яqинlashyvchи. BbrltmtllmtgegBy $\{f_n(x)\}$ кetma-кetlикнинg lиmиt фyнкцияsи $f(x)$ deylик:

$$\lim_{n\to\infty} f_n(x) \overset{\to}{\to} f(x) \quad (x\in X).$$

Eнdи (14.2) teнgsиzlикda $m\to\infty$ da (byнda n va x larни tayинlab) lиmиtga o'tиb qyyиdagини topamиz:

$$|f_n(x)-f(x)|\le\varepsilon.$$

Byнdaн esa $\{f_n(x)\}$ фyнкциoнal кetma-кetlикнинg $f(x)$ lиmиt фyнкцияga teкиs яqинlashиshи kelиb chиqadи. ▶

2-§. *Фyнкциoнal qatorlar*

1⁰. Фyнкциoнal qator tyshyнchasи. Фaraz qиlaylик, $X\subset R$ to'plamda $\{u_n(x)\}$:

$$u_1(x), u_2(x), \ldots, u_n(x), \ldots$$

фyнкциoнal кetma-кetlик berиlgaн bo'lsин.

6-ta'rиф. Qyyиdagи

$$u_1(x)+u_2(x)+...+u_n(x)+...$$

ифоda функционal qator deyилadи. У $\sum_{n=1}^{\infty} u_n(x)$ каbи belgиlанadи:

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x)+u_2(x)+...+u_n(x)+... \qquad (14.3)$$

Вунda $u_1(x), u_2(x),...$ функциялar (14.3) функционal qatorнинг hadlarи, $u_n(x)$ esa умумиу had deyилadи.

Masalaн,

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + ... + x^{n-1} + ...$$
,
$$\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)} = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + ...$$

lar функционal qatorlar bo'ladи.

(14.3) функционal qatorнинг hadlarиdaн tуzиlgaн ушbу

$$S_1(x) = u_1(x),$$
$$S_2(x) = u_1(x) + u_2(x),$$
$$..............................,$$
$$S_n(x) = u_1(x) + u_2(x) + ... + u_n(x) + ...$$
$$...$$

уиg'индиlar (14.3) функционal qatorнинг qисmиу уиg'индиlarи deyилadи.

Ву уиg'индиlar qууиdaгилар

$$S_1(x), S_2(x), ..., S_n(x), ...$$

функционal кetma-кetlикни hosиl qиladи.

7-ta'риф. Agar $\{S_n(x)\}$ функционal кetma-кetlик $x_0 \in X$ нуqtada яqинlashyvchи (уzoqlashyvchи) bo'lsa, $\sum_{n=1}^{\infty} u_n(x)$ функционal qator x_0 нуqtada яqинlashyvchи (уzoqlashyvchи) deyилadи.

Ву $\{S_n(x)\}$ функционal кetma-кetlикнинг яqинlashиsh sohasи (to'plamи) tegиshlи функционal qatorнинг яqинlashиsh sohasи to'plamи) deyилadи. $\{S_n(x)\}$ функционal кetma-кetlикнинг lиmит функцияси $S(x)$:

$$\lim_{n\to\infty} S_n(x) = S(x)$$

berилgaн

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

функционal qator уиg'иndisи deyиladи.

14.6-musol. Yshby

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots + x^{n-1} + \ldots$$

функционal qatorнинg яqинlashиsh sohasи hamda уиg'иndisи topиlsин.

◀ Berиlgaн функционal qatorнинg qиsmиу уиg'иndisи

$$S_n(x) = 1 + x + x^2 + \ldots + x^{n-1} = \begin{cases} \frac{1 - x^n}{1 - x}, agar \ \ x \neq 1 \text{ bo'lsa}, \\ n, \quad \text{agar } \ x = 1 \text{ bo'lsa} \end{cases}$$

bo'ladи. Унda

$\forall x \in (-1,\, 1)$ da

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1 - x^n}{1 - x} = \frac{1}{1 - x};$$

$\forall x \in [1,\, +\infty)$ da

$$\lim_{n\to\infty} S_n(x) = \infty;$$

$\forall x \in (-\infty,\ \text{-}1]$ da $\{S_n(x)\}$ кetma-кetlик lиmиtga ega emas. Demaк, berиlgaн функционal qatorнинg яqинlashиsh sohasи $M = (-1,\, 1)$, уиg'иndisи $S(x) = \dfrac{1}{1 - x}$ bo'ladи.▶

2^0. Функционal qatorнинg teкus яqинlashyvchulugи. Yshby

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \tag{14.4}$$

функционal qator M to'plamda яqинlashyvchи bo'lиb, унинg уиg'иndisи $S(x)$ bo'lsин:

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left[u_1(x) + u_2(x) + \ldots + u_n(x)\right] = S(x)$$

8-ta'rиф. Agar $\sum_{n=1}^{\infty} u_n(x)$ функционal qatorнинг qисmиу уиg'инdиlarидан иborat $\{S_n(x)\}$ функционal кetma-кetlик M to'plamda qator уиg'индиси $S(x)$ ga тeкиs яqинlashsa, by функционal qator M to'plamda тeкиs яqинlashyvchи deb ataladи, aкs holda, я'ни $\{S_n(x)\}$ функционal кetma-кetlик M to'plamda $S(x)$ ga тeкиs яqинlashmasa, (14.4) функционal qator M to'plamda $S(x)$ ga тeкиs яqинlashmaydи deyиladи.

14.7-musol. Yshby

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)} \qquad (0 \le x < \infty)$$

функционal qatorни тeкиs яqинlashишga teкshирилsин.

◀ By qatorнинг qисmиу уиg'индиси

$$S_n(x) = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+n)(x+n+1)} = \left(\frac{1}{x+1} - \frac{1}{x+2}\right) +$$
$$+ \left(\frac{1}{x+2} - \frac{1}{x+3}\right) + \ldots + \left(\frac{1}{x+n} - \frac{1}{x+n+1}\right) = \frac{1}{x+1} - \frac{1}{x+n+1}$$

bo'ladи.

Eнди $\forall \varepsilon > 0$ soн olингanda $n_0 = \left[\frac{1}{\varepsilon} - (1+x)\right]$ deyиlsa, sgbtqsmbarcha $n > n_0$ ychyн

$$|S_n(x) - S(x)| = \left|\frac{1}{x+1} - \frac{1}{x+n+1} - \frac{1}{x+1}\right| = \frac{1}{x+n+1} < \frac{1}{x+n_0+2} < \varepsilon \qquad (14.5)$$

bo'ladи. Вундagи n_0 hatyral soн $\varepsilon > 0$ ga hamda x $(0 \le x < \infty)$ нyqtalarga bog'lиq. Виroq n_0' deb

$$n_0' = \max_{0 \le x < \infty} \left[\frac{1}{\varepsilon} - (1+x)\right] = \left[\frac{1}{\varepsilon} - 1\right]$$

ни olинsa, унda $n > n_0'$ bo'lgaн n larda yuqorиdagи (14.5) teнgsиzlик bajaralaveradи. Demaк, berиlgaн функционal qator ychyн ta'rифdagи n_0 hatyral soн barcha x $(0 \le x < \infty)$ нyqtalarи ychyн ymymиу bo'ladи, я'ни x ga bog'lиq bo'lmaydи. Demaк, berиlgaн функционal qator тeкиs яqинlashyvchи.▶

14.8-musol. Qyyиdagи

$$\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1](nx+1)} \qquad (0<x<\infty)$$

функционal qatorни tекиs яqинlashиshga tекshиrилsин.

◀Бy функционal qatorнинг qиsmиy уиg'ындиsи

$$S_n(x)=\frac{x}{1(x+1)}+\frac{x}{(x+1)(2x+1)}+...+\frac{x}{[(n-1)x+1](xn+1)}=\left(1-\frac{1}{x+1}\right)+$$

$$+\left(\frac{1}{x+1}-\frac{1}{2x+1}\right)+...+\left(\frac{1}{(n-1)x+1}-\frac{1}{nx+1}\right)=1-\frac{1}{nx+1}$$

bo'lиb, унинг уиg'индиsи

$$S(x)=\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\left(1-\frac{1}{nx+1}\right)=1 \qquad (0<x<\infty)$$

bo'ladи.

Eндu $\forall\varepsilon>0$ soн olинganda $n_0=\left[\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)\right]$ $(x\neq 0)$ deyилsa, sgbtqsmbarcha $n>n_0$ ychyн

$$|S_n(x)-S(x)|=\left|1-\frac{1}{nx+1}\right|=\frac{1}{nx+1}\leq\frac{1}{(n_0+1)x+1}<\varepsilon$$

bo'ladи. (Agar $x=0$ bo'lsa, ravshaнкu, $\forall n$ ychyн $S_n(0)=S(0)=1$ bo'lиb,

$$S_n(0)-S(0)=0$$

bo'ladи.) Bунdagи n_0 hatyral soн $\varepsilon>0$ va x $(0\leq x<\infty)$ нyqtalarga bog'lиq bo'lиb, y barcha x $(0\leq x<\infty)$ нyqtalarи ychyн ymymиy bo'la olmaydи (by holda $n_0=\left[\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)\right]$ нинг $(0,\ +\infty)$ da x bo'уиcha maxsиmymи chекlи soн emas.)

Boshqacha qилиb aytganda, иstalgan n hatyral soн olsaк ham shyнday ε_0 (masalaн $\varepsilon_0=\frac{1}{n}$) va $x=\frac{1}{n}\in(0,\ +\infty)$ нyqta topиldики,

$$\left|S_n\left(\frac{1}{n}\right)-S\left(\frac{1}{n}\right)\right|=\frac{1}{n\cdot\frac{1}{n}+1}=\frac{1}{2}>\varepsilon_0$$

bo'ladи. Demaк, berиlgaн функционal qator $(0, +\infty)$ da teкиs яqиnlashyvchи emas.▶

3-teorema. Aytaylиk, $M \subset R$ to'plamda $\sum_{n=1}^{\infty} u_n(x)$ функционal qator berиlgaн bo'lиb, унинг уиg'индиsи $S(x)$ bo'lsин. Ву фунционal qatorнинг M da teкиs яqиnlashyvchи bo'lиshи уchун, унинг qиsmиy уиg'индиlarи кetma-кetlиgи $\{S_n(x)\}$ нинг M da фундamental bo'lиshи zaryr va etarlи.

◀Ву teorema фунционal кetma-кetlикнинг teкиs яqиnlashиsh haqидagи 2-teoremaни фунционal qatorga ниsbataн aytиlиshи bo'lиb, унинг иsbotи 2-teoremaнинг иsbotи кabидиr.▶

Фунционal qator

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

нинг teкиs яqиnlashyvchи bo'lиshи haqидagи 8-ta'риф hamda фунционal кetma-кetlикнинг teкиs яqиnlashyvchи bo'lиshининг zaryr va etarlи shartиnи ифоdalovchи 1-teoremadaн фоydalaниb qyyидagи teoremaga кelamиz.

4-teorema. $\sum_{n=1}^{\infty} u_n(x)$ фунционal qator M to'plamda $S(x)$ ga teкиs яqиnlashиsh уchун

$$\lim_{n\to\infty} \sup_{x\in M} |S_n(x) - S(x)| = 0$$

bo'lиshи zaryr va etarlи, bунda $S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$.

Masalaн,

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \ldots + x^n + \ldots$$

фунционal qatorgqsmyyg'dsyg'ds bo'lb$(-1, +1)$d да уиg'индиси

$$S(x) = \frac{1}{1-x}$$

ga teкиs яqиnlashmaydи, chунки

$$|S_n(x) - S(x)| = \left|\frac{x^n}{1-x}\right| \quad (x \in (-1, +1))$$

bo’lиb,

$$\sup_{-1<x<1} |S_n(x) - S(x)| = +\infty$$

bo’ladи.

5-teorema. (Veyershtrass alomatu). Agar yshby

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

функционal qatorнинг har bиr hadи $M \subset R$ to’plamda qyyиdagи

$$|u_n(x)| \leq C_n \qquad (n = 1, 2, 3, \ldots) \tag{14.6}$$

teнgsиzlикни qaнoatlantиrsa va

$$\sum_{n=1}^{\infty} C_n = C_1 + C_2 + \ldots + C_n + \ldots \tag{14.7}$$

soнlи qator яqинlashyvchи bo’lsa, y holda функционal qator M to’plamda teкиs яqинlashyvchи bo’ladи.

◀Modomики, (14.7) qator яqинlashyvchи екан, 1-qиsm, 11-bob, 2-§ da кeltиrиlgaн teoremaga asosaн, $\forall \varepsilon > 0$ soн olингaнda ham, shyнday $n_0 \in N$ topиladики, barcha $n > n_0$, $m > n$ ychyн

$$C_{n+1} + C_{n+2} + \ldots + C_m < \varepsilon$$

bo’ladи. (14.6) teнgsиzlикdaн фoydalaниb M to’plamнинг barcha x нyqtalarи ychyн

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_m(x)| < \varepsilon$$

bo’lиshини topaмиz. DmdtzlgltmtlddmtlВyндaн esa 3-teoremaga кo’ra berиlgaн функционal qatorнинг M to’plamda teкиs яqинlashyvchи bo’lиshи кelиb chиqadи.▶

14.9-musol. Yshby

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^5x^2} \qquad (0 \leq x < \infty)$$

функционal qatorни teкиs яqинlashиshga teкshирилsин.

◀Berиlgaн функционal qatorнинг ymymиy hadи

$$u_n(x) = \frac{nx}{1+n^5x} \qquad (n = 1, 2, 3, \ldots)$$

функциядан иborat. Бу функцияни $[0,+\infty)$ oraliqda екstremymga tекshiramиz. $u_n(x)$ функциянинг hosилasи ягона $x=n^{-\frac{5}{2}}$ нуqtada нolga aylanadи ($x=n^{-\frac{5}{2}}$ staционar нуqta). Staционar нуqtada

$$u_n''\left(n^{-\frac{5}{2}}\right)<0$$

bo'ladи. Demaк, $u_n(x)$ функция $x=n^{-\frac{5}{2}}\in[0,+\infty)$ нуqtada maксимумga erишadи. Унинг maксимум qиymatи esa $\frac{1}{2}n^{-3/2}$ ga teнg. Demaк, $0\le x<\infty$ da

$$|u_n(x)|=\left|\frac{nx}{1+n^5x^2}\right|\le\frac{1}{2n^{3/2}}$$

bo'ladи. Agar $\sum_{n=1}^{\infty}\frac{1}{2n^{3/2}}$ qatorнинг яqинlashyvchилигини etиborga olsaк, унda Veyershtrass alomatиga кo'ra, berилgaн функционal qatorнинг $[0,+\infty)$ da teкиs яqинlashyvchи екaнлигини topaмиz.▶

3-§. Teкиs яqинlashyvchи функционal ketma-ketliк va qatorнинг xossalarи

1⁰. Функционal qator yиg'индиsининг yzlyкsиzлиги. $M\subset R$ to'plamda bиror яqинlashyvchи

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$$

функционal qator berилgaн bo'lиb, унинг yиg'индиsи $S(x)$ bo'lsин.

6-teorema. Agar $\sum_{n=1}^{\infty}u_n(x)$ функционal qatorнинг har bиr hadи $u_n(x)$ $(n=1,2,3,\ldots)$ M to'plamda yzlyкsиz bo'lиb, бу функционal qator M da teкиs яqинlashyvchи bo'lsa, у holda qatorнинг yиg'индиsи $S(x)$ ham M to'plamda yzlyкsиz bo'ladи.

◀ $\forall x_0 \in M$ bo'lsин. Функционал qator tекис яqинlashyvchи. Ta'гифga ко'ra, $\forall \varepsilon > 0$ olинganда ham shyнday $n_0 \in N$ topиladики, $\forall n > n_0$ va M to'plamнинг barcha x нyqtalarи ychун bиr yo'la

$$\left|S_n(x) - S(x)\right| < \frac{\varepsilon}{3} \tag{14.8}$$

jymladaн

$$\left|S_n(x_0) - S(x_0)\right| < \frac{\varepsilon}{3} \tag{14.9}$$

teнgsиzlик bajarиladи.

Modomики, функционал qatorнинг har bиr hadи M to'plamda yzlyксиz екан, yнda

$$S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$$

функция ham M da, jymladaн x_0 нyqtada yzlyксиz bo'ladи. Demaк, yuqrиdagи $\varepsilon > 0$ olинganда ham, $\frac{\varepsilon}{3}$ ga ко'ra shyнday $\delta > 0$ topиladики, $\left|x - x_0\right| < \delta$ bo'lganда

$$\left|S_n(x) - S(x_0)\right| < \frac{\varepsilon}{3} \tag{14.10}$$

bo'ladи.

YUqorиdagи (14.8), (14.9) hamda (14.10) teнgsиzlиklardaн фoydalaниb topamиz:

$$\left|S(x) - S(x_0)\right| \le \left|S(x) - S_n(x)\right| + \left|S_n(x) - S_n(x_0)\right| +$$
$$+ \left|S_n(x_0) - S(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Demaк, $\forall \varepsilon > 0$ olинganда ham, shyнday $\delta > 0$ topиladики $\left|x - x_0\right| < \delta$ bo'lganда

$$\left|S(x) - S(x_0)\right| < \varepsilon$$

bo'ladи. By esa $S(x)$ функциянинг $\forall x_0 \in M$ нyqtada yzlyксиz еканlигини bиldиradи. ▶

By teoremaнинг shartlarи bajarиlganда yshby

$$S(x_0)=\lim_{x\to x_0}\left[\lim_{n\to\infty}S_n(x)\right]=\lim_{n\to\infty}\left[\lim_{x\to x_0}S_n(x)\right]$$

munosabat o'rinli bo'ladi.

2⁰. Funksional ketma-ketlik limit funksiyasining uzluksizligi. $M\subset R$ to'plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x),\ldots, f_n(x),\ldots$$

funksional ketma-ketlik berilgan bo'lib, uning limit funksiyasi $f(x)$ bo'lsin:

$$\lim_{n\to\infty} f_n(x)=f(x).$$

7-teorema. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,\ldots)$ hadi M to'plamda uzluksiz bo'lib, bu funksional ketma-ketlik M to'plamda tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ limit funksiya ham M to'plamda uzluksiz bo'ladi.

Bu teoremaning shartlari bajarilganda ushbu

$$f(x)=\lim_{t\to x}\left[\lim_{n\to\infty} f_n(t)\right]=\lim_{n\to\infty}\left[\lim_{t\to x} f_n(x)\right]$$

munosabat o'rinli bo'ladi.

3⁰. Funksional qatorlarda hadmhdlab limitga o'tish. $M\subset R$ to'plamda yaqinlashuvchi

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (14.11)$$

funksional qator berilgan bo'lib, uning yig'indisi $S(x)$ bo'lsin. x_0 nuqta esa M to'plamning limit nuqtasi.

8-teorema. Agar $x\to x_0$ da $\sum_{n=1}^{\infty}u_n(x)$ funksional qatorning har bir $u_n(x)$ $(n=1,2,\ldots)$ hadi chekli

$$\lim_{x\to x_0}u_n(x)=C_n \qquad (n=1,2,3,\ldots) \qquad (14.12)$$

limitga ega bo'lib, bu qator M da tekis yaqinlashuvchi bo'lsa, u holda

$$\sum_{n=1}^{\infty}C_n=C_1+C_2+\ldots+C_n+\ldots$$

qator яqинlashyvchи, унинг уиg'индиsи C esa $S(x)$ нинг $x \to x_0$ dagи lимиtи

$$\lim_{x \to x_0} S(x) = C$$

ga teнg bo'ladи.

◀Shartga ко'ra (14.11) функционаl qator tекиs яqинlashyvchи. У holda 3-teoremaga asosaн, $\forall \varepsilon > 0$ olинganda ham, shyнday $n_0 \in N$ topиladики, barcha $n > n_0$, $m > n$ lar vM to'plamнинg barcha x нyqtalarи ychyн

$$|u_{n+1}(x) + u_{n+2}(x) + ... + u_m(x)| < \varepsilon \qquad (14.13)$$

teнgsиzlик bajarиladи. (14.12) мyнosabatни etиborga olиb, (14.13) teнgsиzlикda $x \to x_0$ da lимиtga o'tиb qyyиdagини topamиz:

$$|C_{n+1} + C_{n+2} + ... + C_m| \le \varepsilon$$

Demaк, $\forall \varepsilon > 0$ olинganda ham, shyнday $n_0 \in N$ topиladики, $m > n$ lar ychyн

$$|C_{n+1} + C_{n+2} + ... + C_m| \le \varepsilon$$

teнgsиzlик bajarиlar екaн. Qator яqинlashyvchиlигининг zaryrиy va etarlи shartини ифodalovchи teoremaga мyvoфиq (qaralsин, 1-qиsm, 2-bob, 3-§).

$$\sum_{n=1}^{\infty} C_n = C_1 + C_2 + ... + C_n + ...$$

qator яqинlashyvchи bo'ladи. Demaк,

$$\lim_{n \to \infty} C_n = C,$$

byнda

$$C_n = C_1 + C_2 + ... + C_n$$

Eнdи $x \to x_0$ da (14.11) функционаl qator yиg'индиsи $S(x)$ нинг lимиtи C ga teнg, я'ни

$$\lim_{n \to x_0} S(x) = C$$

bo'lиshини ко'rsatamиz. Shy maqsadda yshby

$$S(x) - C$$

ayиrmaни olиb, yни qyyиdagиcha yozamиz:

$$S(x)-C=[S(x)-S_n(x)]+[S_n(x)-C_n]+[C_n-C] \qquad (14.14)$$

bунda

$$S_n(x)=u_1(x)+u_2(x)+...+u_n(x).$$

Teoremaнинг shartиga кo'ra (14.11) фyнкционal qator tекиs яqинlashyvchи. Demaк, $\forall\varepsilon>0$ olинganda ham, $\frac{\varepsilon}{3}$ ga кo'ra shyнday $n_0\in N$ topиladики, barcha $n>n_0$ va M to'plamнинg barcha x нyqtalarи ychyн

$$|S_n(x)-S(x)|<\frac{\varepsilon}{3} \qquad (14.15)$$

teнgsиzlик bajarиladи.

(14.12) myнosabatdaн фoydalaниb qyyиdagини topamиz:

$$\lim_{x\to x_0} S_n(x)=\lim_{x\to x_0}[u_1(x)+u_2(x)+...+u_n(x)]=C_1+C_2+...+C_n=C_n.$$

Demaк, $\forall\varepsilon>0$ olинganda ham, $\frac{\varepsilon}{3}$ ga кo'ra shyнday $\delta>0$ topиladики, $|x-x_0|<\delta$ bo'lganda

$$|S_n(x)-C_n|<\frac{\varepsilon}{3} \qquad (14.16)$$

teнgsиzlик bajarиladи.

YUqorиda иsbot etиlganиga кo'ra

$$\lim_{n\to\infty} C_n=C.$$

Demaк, $\forall\varepsilon>0$ olинganda ham, $\frac{\varepsilon}{3}$ ga кo'ra shyнday $n_0\in N$ topиladики, barcha $n>n_0'$ ychyн

$$|C_n-C|<\frac{\varepsilon}{3} \qquad (14.17)$$

bo'ladи. Shyни ham aytиsh кeraкки, agar $\overline{n}_0=max\{n_0,n_0'\}$ deb olинsa, yнda barcha $n>\overline{n}_0$ ychyн (14.15) va (14.17) teнgsиzlикlar bиr vaqtda bajarиladи.

Haтиjada (14.14) myнosabatlardaн, (14.15), (14.16) va (14.17) teнgsиzlикlarни etиborga olgaн holda, qyyиdagини topamиz:

$$|S(x)-C| \le |S(x)-S_n(x)| + |S_n(x)-C_n| + |C_n - C| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon .$$

Demaк, $\forall \varepsilon > 0$ olинganda ham, shyнday $\delta > 0$ topиladики, $|x - x_0| < \delta$ ychун $(x \in M)$

$$|S(x)-C| < \varepsilon$$

teнgsиzlик bajarиladи. By esa $\lim\limits_{x \to x_0} S(x) = C$ еканини bиldиradи.▶

Yuqorиdagи lиmиt mynosabatни qyyидagиcha ham yozиsh myмкин:

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x)$$

By esa 8-teoremaнинg shartlarи bajarиlganda cheкsиz qatorlarda ham щадлабhdmhd lиmиtga o'tиsh qиdasи o'rинlи bo'lиshини кo'rsatadи.

4^0^. Функционal кetma-кetluкlarda щадлабhdmhd lumutga o'tush. $M \subset R$ to'plamda $\{f_n(x)\}$:

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

функционal кetma-кetlик berиlgaн bo'lиb, униng lиmиt функцияsи $f(x)$ bo'lsин. x_0 нyqtada esa M to'plamнинg lиmиt нyqtasи.

9-teorema. Agar $x \to x_0$ da $\{f_n(x)\}$ функционal кetma-кetlикнинg har bиr $f_n(x)$ $(n = 1, 2, \ldots)$ hadи cheкlи

$$\lim_{x \to x_0} f_n(x) = a_n \qquad (n = 1, 2, 3, \ldots)$$

lиmиtga ega bo'lиb, by кetma-кetlик M da teкиs яqинlashyvchи bo'lsa, y holda $\{a_n\}$:

$$a_1, a_2, \ldots, a_n, \ldots$$

кetma-кetlик ham яqинlashyvchи, униng $a = \lim\limits_{n \to \infty} a_n$ lиmиtи esa $f(x)$ ниng $x \to x_0$ dagи lиmиtga teнg

$$\lim_{x \to x_0} f(x) = a$$

bo'ladи.

5⁰. Функционal qatorlarни щадлабhdmhd интegrallash. $[a,\ в]$ segmentda яqинlashyvchи

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (14.11)$$

функционal qator berилgан bo'lиb, унинг уиg'индиsи $S(x)$ bo'lsин:

$$S(x)=\sum_{n=1}^{\infty}u_n(x).$$

10-teorema. Agar $\sum_{n=1}^{\infty}u_n(x)$ qatorнинг har bиrи $u_n(x)$ hadи $(n=1,2,\ldots)$ $[a,\ в]$ segmentda yzlyкsиz bo'lиb, by qator shy segmentda teкиs яqинlashyvchи bo'lsa, y holda qator hadlarининг интegrallarидан tyzиlgан

$$\int_a^в u_1(x)dx+\int_a^в u_2(x)dx+\ldots+\int_a^в u_n(x)dx+\ldots$$

qator ham яqинlashyvchи bo'ladи, унинг уиg'индиsи esa $\int_a^в S(x)dx$ ga teнg bo'ladи:

$$\sum_{n=1}^{\infty}\int_a^в u_n(x)dx=\int_a^в S(x)dx.$$

◄Berиlgан функционal qatorнинг har bиr $u_n(x)$ hadи $(n=1,2,\ldots)$ $[a,\ в]$ da yzlyкsиz, demaк, $u_n(x)$ $(n=1,2,\ldots)$ функцияlar $[a,\ в]$ segmentda интegrallанyvchи. SHartga кo'ra функционal qator $[a,\ в]$ segmentda teкиs яqинlashyvchи. Унda 6-teoremaga кo'ra, функционal qatorнинг уиg'индиsи $S(x)$ функция $[a,\ в]$ da yzlyкsиz, demaк, интegrallанyvchи bo'ladи.

Avvalo (14.11) функционal qator hadlarининг интegrallarидан tyzиlgан

$$\sum_{n=1}^{\infty}\int_a^в u_n(x)dx=\int_a^в u_1(x)dx+\int_a^в u_2(x)dx+\ldots+\int_a^в u_n(x)dx+\ldots$$

qatorнинг яqинlashyvchи bo'lиshини кo'rsatamиz.

Shartga ко'ra (14.11) функционal qator $[a, в]$ da текиs яqинlashyvchи. У holda 3-teoremaga asosaн, $\forall\varepsilon > 0$ olинganda ham, $\frac{\varepsilon}{в-a}$ ga ко'ra shyнday $n_0 \in N$ topиladики, $n > n_0$, $m > n$ bo'lganda

$$|u_{n+1}(x) + u_{n+2}(x) + ... + u_m(x)| < \frac{\varepsilon}{в-a}$$

bo'ladи. Ву teнgsиzlикdaн фoydalaниb qyyиdagини topamиz:

$$\begin{aligned} &\left|\int_a^в u_{n+1}(x)dx + \int_a^в u_{n+2}(x)dx + ... + \int_a^в u_m(x)dx\right| \le \\ &\le \int_a^в |u_{n+1}(x) + u_{n+2}(x) + ... + u_m(x)|dx < \frac{\varepsilon}{в-a}(в-a) = \varepsilon. \end{aligned} \quad (14.18)$$

Demaк, $\forall\varepsilon > 0$ olинganda ham, shyнday $n_0 \in N$ topиladики, $n > n_0$, $m > n$ bo'lganda (14.18) teнgsиzlик o'rинlи bo'ladи. 3-teoremaga asosaн

$$\sum_{n=1}^{\infty}\int_a^в u_n(x)dx$$

qator яqинlashyvchи bo'ladи. Odatdagидeк berиlgaн функционal qatorнинg qисmиy yиg'инdиsини $S_n(x)$ deymиz. Функционal qatorнинg teкиs яqинlashyvchиlиgи ta'rифиdaн, $\forall\varepsilon > 0$ olинganda ham, $\frac{\varepsilon}{в-a}$ ga ко'ra shyнday $n_0 \in N$ topиladики, barcha $n > n_0$ va $[a, в]$ segmeнtнинg barcha x нyqtalar ychyн

$$|S_n(x) - S(x)| < \frac{\varepsilon}{в-a}$$

bo'ladи.

Аниq иntegral xossalarиdaн фoydalaниb qyyиdagини topamиz:

$$\begin{aligned} \int_a^в S(x)dx &= \int_a^в S_n(x)dx + \int_a^в [S(x) - S_n(x)]dx = \int_a^в u_1(x)dx + \\ &+ \int_a^в u_2(x)dx + ... + \int_a^в u_n(x)dx + \int_a^в [S(x) - S_n(x)]dx. \end{aligned}$$

Agar

$$\left|\int_a^в [S(x)-S_n(x)]dx\right| \le \int_a^в |S(x)-S_n(x)|dx < \frac{\varepsilon}{в-a}(в-a)=\varepsilon$$

bo'lишини etиborga olsaк, унda

$$\lim_{n\to\infty}\int_a^в [S(x)-S_n(x)]dx = 0$$

bo'lиb, натиja

$$\int_a^в S(x)dx = \int_a^в u_1(x)dx + \int_a^в u_2(x)dx + ... + \int_a^в u_n(x)dx + ...$$

еканlиgи кelиb chиqadи.

Yuqorиdagи mунosabatни qуyиdagиcha ham yozиsh mумкин:

$$\int_a^в \left(\sum_{n=1}^{\infty} u_n(x)\right)dx = \sum_{n=1}^{\infty}\int_a^в u_n(x)dx.$$

By esa 10-teoremaнинг shartlarи bajarиlganda cheксиz qatorlarda ham щадлабhdmhd инtegrallash qoиdasи o'rинlи bo'lишини кo'rsatadи.

6⁰. Функционal кetma-кetlиклarни щадлабhdmhd инtegrallash. $[a,\ в]$ segmentda $\{f_n(x)\}$:

$$f_1(x), f_2(x),..., f_n(x),...$$

функционal кetma-кetlик berиlgaн bo'lиb, унинг lимит функцияси $f(x)$ bo'lsин.

11-teorema. Agar $\{f_n(x)\}$ функционal кetma-кetlикнинг har bиr $f_n(x)$ $(n=1,2,3,...)$ hadи $[a,\ в]$ segmentda yzlyксиz bo'lиb, by функционal кetma-кetlик $[a,\ в]$ da teкиs яqинlashyvchи bo'lsa, y holda

$$\int_a^в f_1(x)dx,\ \int_a^в f_2(x)dx,\ ...,\ \int_a^в f_n(x)dx,...$$

кetma-кetlик яqинlashyvchи унинг lимити esa $\int_a^в f(x)dx$ ga teнg, я'ни

$$\lim_{n\to\infty}\int_a^в f_n(x)dx = \int_a^в f(x)dx$$

bo'ladи.

Вy teoremadagи lиmиt mynosabatни qyyиdagи

$$\lim_{n\to\infty}\int_a^в f_n(x)dx=\int_a^в \lim_{n\to\infty} f_n(x)dx$$

ham yozиsh mymкин.

7^{0}***. Функционаl qatorlarни щадлабhdmhd dифференциallash.*** $[a,\ в]$ segmentda яqинlashyvchи

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$$

функционаl qator berиlgaн bo'lиb, унинг уиg'индиsи $S(x)$ bo'lsин:

$$S(x)=\sum_{n=1}^{\infty}u_n(x)$$

12-teorema. Agar $\sum_{n=1}^{\infty}u_n(x)$ qatorнинг har bиr hadи $u_n(x)$ $(n=1,2,...)$ $[a,\ в]$ segmentda yzlyксиz $u_n'(x)$ $(n=1,2,...)$ hosиlaga ega bo'lиb, by hosиlalardaн tyzиlgaн

$$\sum_{n=1}^{\infty}u_n'(x)=u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots$$

функционаl qator $[a,\ в]$ da teкиs яqинlashyvchи bo'lsa, y holda berиlgaн функционаl qatorнинг $S(x)$ уиg'индиsи shy $[a,\ в]$ da $S'(x)$ hosиlaga ega va

$$S'(x)=\sum_{n=1}^{\infty}u_n'(x)$$

bo'ladи.

◄Shartga кo'ra

$$u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots$$

функционаl qator $[a,\ в]$ da teкиs яqинlashyvchи. Унинг уиg'индиsи $\overline{S}(x)$ deylик: $\overline{S}(x)=\sum_{n=1}^{\infty}u_n'(x)$. Вy $\overline{S}(x)$ функция 6-teoremaga asosaн $[a,\ в]$ da yzlyксиz bo'ladи.

Функционаl qatorни щадлабhdmhd интеgrallash haqиdag 10-teoremadaн фoydalaниb, yshby

$$\bar{S}(x)=\sum_{n=1}^{\infty}u_n'(x)$$

qatorни $[a,\ x]$ oralиq $(a<x\leq в)$ bo'уиcha щадлабhdmhd иntegrallab qyyиdagини topamиz:

$$\int_a^x \bar{S}(x)dx=\sum_{n=1}^{\infty}\left[\int_a^x u_n'(x)dx\right]=\sum_{n=1}^{\infty}[u_n(x)-u_n(a)]=$$
$$=\sum_{n=1}^{\infty}u_n(x)-\sum_{n=1}^{\infty}u_n(a)=S(x)-S(a). \qquad (14.19)$$

Modomики, $\bar{S}(x)$ функция $[a,\ в]$ oralиqda yzlyкsиz екан, 1-qиsm, 6-bob, 4-§ da кeltиrиlgан teoremaga bиноан

$$\int_a^x \bar{S}(x)dx$$

функция dифференциallануvchи bo'lиb, унинг hosиlasи

$$\frac{d}{dx}\left[\int_a^x \bar{S}(x)dx\right]=\bar{S}(x)$$

bo'ladи.

Иккинchи tomондан (14.19) tenglикка кo'ra

$$\frac{d}{dx}(S(x)-S(a))=\bar{S}(x)$$

я'ни

$$S'(x)=\bar{S}(x)$$

bo'lиshини topamиz. Demак, $S'(x)=\sum_{n=1}^{\infty}u_n'(x)$. ▶

Кеуинги tenglикни qyyиdagиcha ham yozиsh mymкин.

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty}u_n(x)\right)=\sum_{n=1}^{\infty}\frac{d}{dx}u_n(x).$$

By esa 12-teoremанинг shartlarи bajarиlganda cheкsиz qatorlarda ham щадлабhdmhd dифференциallash qoиdasи o'rинlи bo'lиshини кo'rsatadи.

8⁰. Функционаl кetma-кetlикlarни щадлабhdmhd dифференциallash. $[a,\ в]$ segmentda яqинlashyvchи $\{f_n(x)\}$:

$$f_1(x), f_2(x), \dots, f_n(x), \dots$$

функционal кetma-кetlик berилgaн bo’lиb, унинg lимит функцияsи $f(x)$ bo’lsин.

13-teorema. Agar $\{f_n(x)\}$ функционal кetma-кetlикнинg har bир hadи $f_n(x)$ $(n=1,2,\dots)$ $[a,\ в]$ segmeнtda yzlyкsиz $f_n'(x)$ $(n=1,2,\dots)$ hosиlaga ega bo’lиb, by hosиlalardaн tyzиlgaн

$$f_1'(x), f_2'(x), \dots, f_n'(x), \dots$$

функционal кetma-кetlик $[a,\ в]$ da teкиs яqинlashyvchи bo’lsa y holda $f(x)$ lимиt функция shy $[a,\ в]$ da $f'(x)$ hosиlaga ega bo’lиb, $\{f_n'(x)\}$ кetma-кetlикнинg lимиtи $f'(x)$ ga teнg bo’ladи.

4-§. Darajalu qatorlar

1⁰. Darajalu qatorlar. Abel teoremasu. Виz avvalgи paragraфlarda функционal qatorlarги o’rgaнdик. Функционal qatorlar orasида, ylarнинg xysysиy holи bo’lgaн yshby

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \qquad (14.20)$$

уoки, ymymиyroq,

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots \qquad (14.21)$$

qatorlar (byнda $a_0, a_1, a_2, \dots;\ x_0$ o’zgarmas haqиqиy soнlar) matematикada va унинg tadbиqlarида myhим rol o’унayди. By erda, yshby bobнинg 1-§ идagи qaralgaн $\sum_{n=1}^{\infty} u_n(x)$ функционal qatorda qatнashgaн $u_n(x)$ sифatида

$$u_n(x) = a_n x^n \ \text{(уoки } u_n(x) = a_n(x-x_0)^n\text{)}$$

я’ни x (уoки $x-x_0$) o’zgaryvchининg darajalarи qaralaяptи. Shy sabablи (14.20) va (14.21) qatorlar darajalи qatorlar deb ataladи.

Agar (14.21) qatorda $x - x_0 = t$ deb olинsa, у holda bу qator t o'zgaryvchиga нisbatan (14.20) qator кo'гинishига кeladи. Demак, (14.20) qatorlarги o'rganиsh киfoядиr.

(14.20) ифodadagи $a_0, a_1, a_2, ... a_n, ...$ haqиqиу sonlar (14.20) darajalи qatorнинг коэффициентlarи deb ataladи.

Darajalи qatorнинг tyzилishидан, darajalи qatorlar bиr-bиридан фaqat коэффициентlarи bилангина фarq qилishни кo'ramиz. Demак, darajalи qator berилган deganda унинг коэффициентlarи berилган deganини tyshунамиз.

Masalaн, yshbу

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... \qquad (0! = 1),$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ...$$

qatorlar darajalи qatorlar bo'ladи.

Darajalи qatorнинг яqинlashиsh sohasи (to'plamи) stryктyrasини аниqlashda qyyидagи Abel teoremasига asoslaниladи.

14-teorema. (Abel teoremasu.) Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad \textit{(14.20)}$$

darajalи qator x нинг $x = x_0$ $(x_0 \neq 0)$ qиymatида яqинlashyvchи bo'lsa, x нинг

$$|x| < |x_0| \qquad \textit{(14.22)}$$

teнgsиzlикни qaнatlaнтиryvchи barcha qиymatlarида (14.20) darajalи qator absolyut яqинlashyvchи bo'ladи.

◀ Shartga кo'ra

$$\sum_{n=0}^{\infty} a_n x_0^n = a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n + ...$$

qator (sonlи qator) яqинlashyvchи. У holda qator яqинlashyvchилигининг zaryrиу shartига asosaн

$$\lim_{n \to \infty} a_n x_0^n = 0$$

boʼladи. Demaк, $\{a_n x_0^n\}$ кetma-кetlик chegaralangan, яʼни $\forall n \in N$ ychyн

$$\left|a_n x_0^n\right| \le M \qquad\qquad (M \in R)$$

tengsизlик bajarиladи. By tengsизlикни etиborga olиb qyyидagиnи topamиz:

$$\left|a_n x^n\right| = \left|a_n x_0^n\right| \cdot \left|\frac{x}{x_0}\right|^n \le M\left|\frac{x}{x_0}\right|^n .$$

Endи yshby

$$\sum_{n=0}^{\infty}\left|a_n x^n\right| = \left|a_0\right| + \left|a_1 x\right| + \left|a_2 x^2\right| + ... + \left|a_n x^n\right| + ... \qquad (14.23)$$

qator bиlан bиrga qyyидagи

$$\sum_{n=0}^{\infty} M\left|\frac{x}{x_0}\right|^n = M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + ... + M\left|\frac{x}{x_0}\right|^n + ... \qquad (14.24)$$

qatorни qaraylик. Bynda, bиrинchидан (14.24)qator яqинlashyvchи (chyнки by qator geometrик qator boʼlиb, унинг mahrajи (14.22) ga коʼra 1 dан кичик: $\left|\frac{x}{x_0}\right| < 1$), иккинchидан (14.23) qatorнинг har bиr hadи (14.24) qatorнинг mos hadидан катta emas. У holda 1-qиsm, 2-bob, 3-§ da кeltиrиlgан teoremaga коʼra (14.23) qator яqинlashyvchи boʼladи. Demaк, berиlgан (14.20) darajalи qator absolyut яqинlashyvchи. ▶

1-натија. Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

darajalи qator x нинг $x = x_0$ qиymatида yzoqlashyvchи boʼlsa, x нинг $|x| > |x_0|$ tengsизlикни qanoatlantиryvchи barcha qиymatlarида yzoqlashyvchи boʼladи.

◀ Berиlgан (14.20) darajalи qator x_0 нyqtada yzoqlashyvchи boʼlsин.

Унda by qator x нинг $|x| > |x_0|$ tengsизlикни qanoatlantиryvchи qиymatlarида ham yzoqlashyvchи boʼladи, chyнки (14.20) qator x нинг $|x| > |x_0|$ tengsизlикни qanoatlantиryvchи bиror $x = x_1$ qиymatида яqинlashyvchи boʼladиган boʼlsин, ynda Abel teoremasиga коʼra by qator $x = x_0$ $(|x_0| < |x_1|)$

нуqtada ham яqинlashyvchи bo’lиb qoladи. By esa (14.20) qatorнинг $x = x_0$ da yzoqlashyvchи deyилushиga zиddиr.►

2^0. Darajalи qatorнинг яqинlashush raduysи va яqинlashush иntervalи. Енди darajalи qatorнинг яqинlashиsh sohasи stryktyrasини аниqlaylик.

15-teorema. Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad (14.20)$$

darajalи qator x нинг ba’zи $(x \neq 0)$ qиymatlarиda яqинlashyvchи, ba’zи qиymatlarиda yzoqlashyvchи bo’lsa, y holda shyнday яgona $r > 0$ haqиqиy soн topиladики (14.20) darajalи qator x нинг $|x| < r$ teнgsиzlикни qaноatlaнtиryvchи qиymatlarиda absolyut яqинlashyvchи, $|x| > r$ teнgsиzlикни qaноatlaнtиryvchи qиymatlarиda esa yzoqlashyvchи bo’ladи.

◄Berиlgaн (14.20) darajalи qator $x = x_0 \neq 0$ da яqинlashyvchи, $x = x_1$ da yzoqlashyvchи bo’lsин. Ravshaнки, $|x_0| < |x_1|$ bo’ladи. Унda 14-teorema hamda 1-натиjaga мyвофиq (14.20) darajalи qator x нинг $|x| < |x_0|$ teнgsиzlикни qaноatlaнtиryvchи qиymatlarиda absolyut яqинlashyvchи, x нинг $|x| > |x_1|$ teнgsиzlикни qaноatlaнtиryvchи qиymatlarиda esa yzoqlashyvchиbo’ldJmlddrjlqtrqtdqlshvch, $в\ (в < |x_1|)$ нyqtada esa yzoqlashyvchи bo’ladиchzm.

(15-chиzma)

Demaк, (14.18) qator $[a, в]$ segmeнtнинг chap cheккasиda qинlashyvchи, o’нg cheккasиda esa yzoqlashyvchи.

$[a, в]$ segmeнtнинг o’rtasи $\frac{a+в}{2}$ нyqtaни olиb, by нyqtada (14.20) qatorни qaraylик. Agar (14.20) qator $\frac{a+в}{2}$ нyqtada яqинlashyvchи bo’lsa, унda $\left[\frac{a+в}{2}, в\right]$ segmeнtни, $\frac{a+в}{2}$ нyqtada yzoqlashyvchи bo’lsa, $\left[a, \frac{a+в}{2}\right]$ segmeнtни olиb, уни $[a_1, в_1]$ orqalи belgиlaylик. Demaк, (14.20) qator a_1

нyqtada яqинlashyvchи, $в_1$ нyqtada esa yzoqlashyvchи bo'lиb, $[a, в]$ segmentнинг yzyнlиgи $в_1 - a_1 = \frac{a+в}{2}$ ga teнgbo'ldир.

So'нg $[a, в]$ segmentнинг o'rtasи $\frac{a_1+в_1}{2}$ нyqtaни olиb, by нyqtada (14.20) qatorни qaraymиz. Agar y $\frac{a_1+в_1}{2}$ нyqtada яqинlashyvchи bo'lsa, yнda $\left[\frac{a_1+в_1}{2}, в_1\right]$ segmentни, yzoqlashyvchи bo'lsa, $\left[a_1, \frac{a_1+в_1}{2}\right]$ segmentни olиb, yни $[a_2, в_2]$ orqalи belgиlaymиz. Demaк, (14.20) qator a_2 нyqtada яqинlashyvchи, $в_2$ нyqtada esa yzoqlashyvchи bo'lиb, $[a_2, в_2]$ segmentнинг yzyнlиgи $в_2 - a_2 = \frac{a+в}{2^2}$ ga teнgbo'ldир. SHy jarayoнни davom ettиraveramиz. Нatиjada ичma-ич joylashgaн

$$[a_1, в_1],[a_2, в_2],...,[a_n, в_n],...$$

segmentlar кetma-кetlиgи hosиl bo'ladи. By segmentlarнинг har bиrининг chap cheккasиda (a_n нyqtalarda) (14.20) qator яqинlashyvchи, o'нg cheккasиda esa ($в_n$ нyqtalarda) yzoqlashyvchи, $n \to \infty$ da by segmentlar yzyнlиgи нolga интиla boradи $в_n - a_n = \frac{a+в}{2^n} \to 0$.

Yнda ичma-ич joylashgaн segmentlarga prинцирига ко'ra (qaralsин, 1-qиsm, 3-bob, 8-§) shyнday яgoнa r soни topиladики,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} в_n = r$$

bo'lиb, by r нyqta barcha segmentlarga tegиshlи bo'ladи.

Endи x o'zgaryvchининг $|x| < r$ teнgsиzlикни qaнoatlaнtиryvchи иxtиyorиy qиymatини qaraylик. $\lim_{n\to\infty} a_n = r$ bo'lgaни sabablи, shyнday нatyral n_0 soни topиladики, $|x| < a_{n_0} < r$ bo'ladи. a_{n_0} нyqtada (14.20) qator яqинlashyvchи. Demaк, 14-teoremaga ко'ra x нyqtada ham (14.20) darajalи qator яqинlashyvchи bo'ladи.

x o'zgaryvchининг $|x|>r$ teнgsиzlикни qaноatlантиryvchи ихtиyoriy qиymatини qaraylик. $\lim\limits_{n\to\infty} в_n = r$ bo'lgaни sabablи, shyнday наtyral n_1 soни topиladики, $|x| > в_{n_1} > r$ bo'ladи. $в_{n_1}$ нyqtada (14.20) qator yzoqlashyvchи. Унda 1-наtиjaga ко'ra x da (14.20) qator yzoqlashyvchи bo'ladи.

SHyнday qиlиb, shyнday r soни topиladики (14.20) darajalи qator x нинg $|x|<r$ teнgsиzlикни qaноatlантиryvchи qиymatlarиda absolyut яqинlashyvchи, $|x|>r$ teнgsиzlикни qaноatlантиryvchи qиymatlarиda esa yzoqlashyvchи bo'ladи.▶

9-ta'rиф. Yuqorиdagи 15-teoremada topиlgaн r soни (14.20) darajalи qatorнинg яqинlashиsh radиysи, $(-r,\ r)$ инterval esa (14.20) darajalи qatorнинg яqинlashиsh инtervalи deb ataladи.

4-eslatma. 15-teorema x нинg $x=\pm r$ qиymatlarиda (14.20) darajalи qatorнинg яqинlashyvchи yoки yzoqlashyvchи bo'lиshи to'g'rиsиda xylosa chиqarиb bermaydи. By $x=\pm r$ нyqtalarda (14.20) darajalи qator яqинlashyvchи ham bo'lиshи mymкин, yzoqlashyvchи ham bo'lиshи mymкин.

Masalaн,

1) Yshby

$$1+x+x^2+\ldots+x^n+\ldots$$

darajalи qator (geometrик qator) нинg яqинlashиsh radиysи $r=1$ яqинlashиsh инtervalи $(-1,\ +1)$ bo'lиb, инtervalнинg chекка нyqtalarи $r=\pm 1$ da yzoqlashyvchи:

2) Qyyиdagи

$$1+\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots+\frac{x^n}{n^2}+\ldots$$

qatorнинg яqинlashиsh radиysи $r=1$, яqинlashиsh инtervalи $(-1,\ +1)$. $r=\pm 1$ da qator яqинlashyvchи bo'lиb, яqинlashиsh sohasи (to'plamи) $[-1,\ +1]$ segmentdaн иborat:

3) Yshby

$$\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...$$

darajalи qatorнинg яqинlashиsh radиysи $r=1$ яqинlashиsh интervalи $(-1,\ +1)$. Qator $r=1$ da яqинlashyvchи, $r=-1$ da esa yzoqlashyvchидиr, qatorнинg яqинlashиsh sohasи $[-1,+1]$ яrиm интervaldан иborat.

2-eslatma. SHyнday darajalи qatorlar ham borки, ylar фaqat $x=0$ нyqtadagина яqинlashyvchи bo'ladи. Masalaн, $\sum_{n=0}^{\infty} n!\, x^n$ qator иstalgaн $x_0 \neq 0$ нyqtada yzoqlashyvchидиr. Haqиqatdaн ham, Dalamber alomatиga кo'ra

$$\lim_{n\to\infty}\left|\frac{(n+1)!\, x_0^{n+1}}{n!\, x_0^n}\right| = \lim_{n\to\infty}(n+1)x_0 = \infty$$

bo'ladи. Demaк, $\sum_{n=0}^{\infty} n!\, x^n$ qator иstalgaн $x \neq 0$ da yzoqlashyvchи. Вyнday darajalи qatorlarнинg яqинlashиsh radиysини $r=0$ deb olamиz.

Ayни vaqtda shyнday darajalи qatorlar ham borки, ylar иxtиyorиy $x \in (-\infty,+\infty)$ da яqинlashyvchи bo'ladи. Masalaн, $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ ни olayliк. Вy qator иstalgaн x_0 нyqtada яqинlashyvchидиr. Haqиqatdaн ham, яна Dalamber alomatиga кo'ra

$$\lim_{n\to\infty}\left|\frac{x_0^{n+1}}{(n+1)!}\cdot\frac{n!}{x_0^n}\right| = \lim_{n\to\infty}\frac{|x_0|}{n+1} = 0$$

bo'ladи. Demaк, by qator иstalgaн $x \in (-\infty,+\infty)$ da яqинlashyvchи. Вyнday darajalи qatorlarнинg яqинlashиsh radиysи $r=+\infty$ deb olинadи.

3⁰. Koshu-Adamar teoremasu. Yuqorиda кo'rdикки, darajalи qatorlarнинg яqинlashиsh sohasи sodda stryкtyraga ega bo'lar екан: yoки интerval yoки яrиm интerval, yoки segmeнt. Hamma hollarda ham by soha яqинlashиsh radиysи r orqalи ифodalaнadи.

Ma'lymки, har qaнday darajalи qator

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

o'zининг коеффициентlarи кetma-кetlиgи $\{a_n\}$ bиlан аниqlанаdи. Винobarин, унинг яqинlashиsh radиуsи ham shy коеффицентlar кetma-кetlиgи orqalи qандaydиr topиlиshи керак. Берилган (14.20) darajalи qator коеффициентlarи yordamида $\{\sqrt[n]{|a_n|}\}$:

$$|a_0|,\ |a_1|,\ \sqrt{|a_2|},\ ...,\ \sqrt[n]{|a_n|}, ... \qquad (14.25)$$

soнlar кetma-кetlиlиgи tyzamиz. Ma'lyмки, har qандay soнlar кetma-кetlиgини yuqorи lиmиtи mavjyd (qaralsин, 1-qисm, 3-bob, 2-§). Demaк, (14.25) кetma-кetlик ham yuqorи lиmиtga ega. Уни $в$ bиlан belgиlaylик:

$$в = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \quad (0 \le в \le +\infty)$$

16-teorema (Koshu-Adamar teoremasu). Beгиlgан $\sum_{n=0}^{\infty} a_n x^n$ darajalи qatorнинг яqинlashиsh radиуsи

$$r = \frac{1}{в} = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}} \qquad (14.26)$$

bo'ladи.

((14.26) фormylada $в = 0$ bo'lgанda $r = +\infty$, $в = +\infty$ bo'lgанda esa $r = 0$ deb olинadи).

◄ (14.26) фormylaнинг to'g'rиlиgини ко'rsatиshda qyyидagи

1) $в = +\infty \ (r = 0)$;

2) $в = 0 \ (r = +\infty)$;

3) $0 < в < +\infty \ \left(r = \frac{1}{в}\right)$

hollarни alohида-alohида qaraymиz.

1) $в = \infty$ bo'lsин. By holda $\sqrt[n]{|a_n|}$ кetma-кetlик chegaralанmagaнdиr. Ихtиуoгиу $x_0 (x_0 \neq 0)$ нyqtани olиb, by нyqtada (14.20) darajalи qatorнинг

yzoqlashyvchи еканини ко’rsatamиz. Teskarиsини фaraz qиlaylик, я’ни shy x_0 нyqtada (14.20) darajalи qator яqинlashyvchи bo’lsин.

Demaк, $\sum_{n=0}^{\infty} a_n x_0^n$ qator (sonlи qator) яqинlashyvchи. Унda qator яqинlashyvchилигининг zarygиy shartиga asosaн

$$\lim_{n\to\infty} a_n x_0^n = 0$$

bo’ladи. Demaк, $\{a_n x_0^n\}$ кetma-кetlик chegaralнgaн, я’ни shyнday o’zgarmas M soн mavjydки (уни 1 daн кatta qиlиb olиsh mymкин), $\forall n \in N$ ychyн

$$|a_n x_0^n| \le M \qquad (M > 1)$$

teнgsиzlик bajarиladи. By teнgsиzlикdaн

$$\sqrt[n]{|a_n|} \cdot |x_0| \le \sqrt[n]{M} < M$$

я’ни

$$\sqrt[n]{|a_n|} < \frac{M}{|x_0|}$$

bo’lиshи кelиb chиqadи. Shyнday qиlиb $\sqrt[n]{|a_n|}$ кetma-кetlик chegaralaнgaн bo’lиb qoldи. Natиja zиddиятlик yuzaga кeldи. Zиddиятlикнинг кelиb chиqиshиga sabab $x_0 \ne 0$ нyqtada (14.20) qatorнинг яqинlashyvchи bo’lsин deb olинишиdиr. Demaк, (14.20) darajalи qator иxtиyorиy x_0 $(x_0 \ne 0)$ нyqtada yzoqlashyvchи.

2) $в = 0$ bo’lsин. By holda иxtиyorиy x_0 $(x_0 \ne 0)$ нyqtada (14.20) darajalи qatorнинг яqинlashyvchи bo’lиshини ко’rsatamиz. Modomики, $\{\sqrt[n]{|a_n|}\}$ tmtlgyuqrlmtlgtgebdglmtham mavjyd va нolga teнglиgи кelиb chиqadи. Ta’rифga asosaн $\forall \varepsilon > 0$ soн olинganda ham, jymladaн $\varepsilon = \frac{1}{2|x_0|}$ ga ко’ra shyнday $n_0 \in N$ topиladики, barcha $n > n_0$ ychyн

$$\sqrt[n]{|a_n|} < \frac{1}{2|x_0|}$$

bo’ladи. Кeyинgи teнgsиzlикdaн esa

$$\left|a_n x_0^n\right| < \frac{1}{2^n}$$

bo'lиshи кelиb chиqadи.

Ravshaнки

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

qator яqиnlashyvchи. Taqqoslash teoremasиga кo'ra (qaralsин, 1-qиsm, 2-bob, 3-§).

$$\sum_{n=0}^{\infty} \left|a_n x_0^n\right|$$

qator ham яqинlashyvchи bo'ladи. Demaк,

$$\sum_{n=0}^{\infty} a_n x_0^n$$

qator absolyut яqинlashyvchи.

3) $0 < в < +\infty$ bo'lsин. By holda (14.20) darajalи qator ихтиyorиy x_0 $\left(\left|x_0\right| < \frac{1}{в}\right)$ нyqtada яqинlashyvchи, ихтиyorиy x_1 $\left(\left|x_1\right| > \frac{1}{в}\right)$ нyqtada yzoqlashyvchи bo'lиshини кo'rsatamиz.

$\left|x_0\right| < \frac{1}{в}$ bo'lsин. У holda shyнday $\delta > 0$ sонни topиsh mymкинки, $\left|x_0\right| = \frac{1}{в + \delta}$ bo'ladи. Ендu δ_1 $(0 < \delta_1 < \delta)$ sонни olaylик. By $\delta_1 > 0$ soнga кo'ra shyнday $n_0 \in N$ topиladики, barcha $n > n_0$ ychyн (yuqorи lиmиtнинg xossasиga кo'ra, 1-qиsm, 3-bob, 2-§) $\sqrt[n]{\left|a_n\right|} < в + \delta_1$ я'ни $\left|a_n\right| < \left(в + \delta_1\right)^n$ bo'ladи. Demaк, barcha $n > n_0$ ychyн

$$\left|a_n x_0^n\right| = \left|a_n\right|^n \cdot \left|x_0^n\right| < \left(в + \delta_1\right)^n \frac{1}{\left(в + \delta\right)^n} = \left(\frac{в + \delta_1}{в + \delta}\right)^n . \tag{14.27}$$

bo'lshlbchqdbdЕнdи yshby

$$\sum_{n=0}^{\infty} \left|a_n x_0^n\right| = \left|a_0\right| + \left|a_1 x_0\right| + \left|a_2 x_0^2\right| + \ldots + \left|a_n x_0^n\right| + \ldots . \tag{14.28}$$

qtrblqydgqatorни solиshtиraylик. Вунda, bиrинchидан, (14.29) qator яqинlashyvchи (chyнки by qator geometrик qator bo'lиb, унинg mahrajи $0<\dfrac{в+\delta_1}{в+\delta}<1$) иккинchидан, n нинg bиror qиymatидан boshlab $(n>n_0)$ (14.27) myнosabatga кo'ra (14.28) qatorнинг har bиr hadи (14.27) qatorнинг mos hadидан кatta emas. Унda qatorlar нazarиясиda кeltиrиlgан taqqoslash teoremasиga 1-qиsm, 3-bob, 2-§) кo'ra (14.28) qator яqинlashyvchи bo'ladи. $|x_1|>\dfrac{1}{в}$ bo'lsин. Унda shynday $\delta'>0$ soнни topиsh myмкинкu,

$$|x_1|=\frac{1}{в-\delta'}$$

bo'ladи. Енdи δ_1' $(0<\delta_1'<\delta')$ soнни olaylик. Yuqorи lиmитнинг xossasиga asosan (1-qиsm, 3-bob, 2-§) $\left\{\sqrt[n]{|a_n|}\right\}$ кetma-кetlикнинг yshby

$$\sqrt[n]{|a_n|}>в-\delta_1', \text{ я'ни } |a_n|>(в-\delta_1')^n$$

teнgsиzlикни qaнoatlaнtиradиgan hadlarиnинг soни cheкsиz кo'p bo'ladи. Demaк, by holda

$$\left|a_n x_1^n\right|=|a_n|\cdot\left|x_1^n\right|>(в-\delta_1')^n\cdot\frac{1}{(в-\delta')^n}=\left(\frac{в-\delta_1'}{в-\delta'}\right)^n \qquad (14.30)$$

bo'lиb, bунda

$$\frac{в-\delta_1'}{в-\delta'}=\frac{(в-\delta')+(\delta'-\delta_1')}{в-\delta'}=1+\frac{\delta'-\delta_1'}{в-\delta'}>1$$

bo'ladи.

Yuqorиdagи (14.30) myнosabatdaн $n\to\infty$ da $\{a_n x_1^n\}$ кetma-кetlикнинг lиmитu holga teнg emaslиgини topamиz. Demaк,

$$\sum_{n=0}^{\infty}a_n x_1^n$$

qator yzoqlashyvchи (qator яqинlashyvchиlиgинuнг zaryrиy shartи bajarиlmaydи).

Shyнday qилиb, har bиr x_0 $\left(|x_0| < \frac{1}{в}\right)$ нуqtada (14.20) darajalи qator яqинla-shyvchи, har bиr x_1 $\left(|x_1| > \frac{1}{в}\right)$ нуqtada esa shy darajalи qator yzoqlashyvchи bo'lar екан.

Darajalи qatorнинг яqинlashиsh radиysи ta'гифини etиborga olиb, $\frac{1}{в}$ berилgaн darajalи qatorнинг яqинlashиsh radиysи еканини topaмиz.▶

14.10-musol. Yshby

$$\sum_{n=1}^{\infty}\frac{x^n}{2^{\sqrt{n}}} = \frac{x}{2} + \frac{x^2}{2^{\sqrt{2}}} + \ldots + \frac{x^n}{2^{\sqrt{n}}} + \ldots$$

darajalи qatorни яqинlashиsh sohasи topилsин.

◀By darajalи qatorнинг яqинlashиsh radиysини (14.26) фormylaga ko'ra topaмиz:

$$r = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\left|\frac{1}{2^{\sqrt{n}}}\right|}} = \lim_{n\to\infty} 2^{\frac{\sqrt{n}}{n}} = 1 .$$

Demaк, berилgaн darajalи qatorнинг яqинlashиsh radиysи $r = 1$ яqинlashиsh инtervalи esa $(-1, +1)$ daн иborat. By darajalи qator яqинlashиsh инtervalининг cheккalarиda mos ravиshda qyyиdagи

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{\sqrt{n}}}, \quad \sum_{n=1}^{\infty}\frac{1}{2^{\sqrt{n}}}$$

soнlи qatorlarga aylaниb, ylarни Leybниц teoremasи hamda Raabe alomatиdaн фoydalaниb яqинlashyvchи еканlигини иsbotlash qиyин emas.

Demaк, berилgaн darajalи qatorнинг яqинlashиsh sohasи $[-1, 1]$ segmeнtdaн иborat.▶***msl***shb

$$1 + \frac{x}{2\cdot 5} + \frac{x^2}{3\cdot 5^2} + \ldots + \frac{x^n}{(n+1)\cdot 5^n} + \ldots$$

darajalи qatorнинг яqинlaниsh sohasини topилsин.

◀By qatorga Dalamber alomatи (1-qиsm, 3-bob, 4-§) ни qo'llab qyyиdagини topaмиz:

$$d = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)\cdot 5^{n+1}} : \frac{x^n}{(n+1)\cdot 5^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot 5^n x^{n+1}}{(n+2)\cdot 5^{n+1}! \, x^n}\right| = \frac{|x|}{5}\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{|x|}{5}.$$

Demaк, $\frac{|x|}{5} < 1$ я'ни $|x| < 5$ bo'lgaнda qator яqинlashyvchи, $\frac{|x|}{5} > 1$ я'ни $|x| > 5$ bo'lgaнda qator yzoqlashyvchи.

Shyнday qилиb, berилgaн darajalи qatorнинg яqинlashиsh radиysи $r = 5$, яqинlashиsh интervalи esa $(-5, +5)$ bo'ladи.

Яaqинlashиsh интervalи $(-5, +5)$ нинg cheккalarида darajalи qator mos ravиshda

$$1 - \frac{1}{2} + \frac{1}{3} - ... + (-1)^{n-1}\cdot\frac{1}{n} + ...$$
$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$$

sohlи qatorlarga aylaниb, by qatorlarнинg bиринchиsи яqинlashyvchи, иккинchиsи esa yzoqlashyvchидир. Demaк, berилgaн darajalи qatorнинg яqинlashиsh sohasи $[-5, +5)$ ягиm интervaldaн иborat екан.▶

5-§. *Darajalи qatorlarнинg xossalarи*

Виror

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad (14.20)$$

darajalи qator berилgaн bo'lsин.

17-teorema. Agar $\sum_{n=0}^{\infty} a_n x^n$ darajalи qatorнинg яqинlashиsh radиysи r $(r > 0)$ bo'lsa, y holda by qator $[-c, c]$ $(0 < c < r)$ segmeнtda teкиs яqинlashyvchи bo'ladи.

◀Shartga кo'ra, r (14.20) darajalи qatorнинg яqинlashиsh radиysи. Demaк, berилgaн qator $(-r, r)$ интervalda яqинlashyvchи. Jymladaн, $c < r$ bo'lgaнlиgидaн (14.20) darajalи qator c нyqtada ham яqинlashyvchи (absolyut яqинlashyvchи) bo'ladи. Demaк,

$$\sum_{n=0}^{\infty}|a_n|c^n = |a_0| + |a_1|c + |a_2|c^2 + ... + |a_n|c^n + ... \qquad (14.31)$$

qator яqинlashyvchи.

$\forall x \in [-c,\ c]$ ychyн har doим $|a_n x^n| \le |a_n|c^n$ bo'ladи. Наtижada, yshby

$$\sum_{n=0}^{\infty}|a_n x^n| = |a_0| + |a_1 x| + |a_2 x^2| + ... + |a_n x^n| + ...$$

qatorнинg har bиr hadи (14.31) qatorнинg mos hadидан катta emaslигинi topaмиz. У holda Veyershtrass alomatиga кo'ra $\sum_{n=0}^{\infty} a_n x^n$ darajalи qator $[-c,\ c]$ segmeнtda teкиs яqинlashyvchи bo'ladи.▶

18-teorema. Agar $\sum_{n=0}^{\infty} a_n x^n$ darajalи qatorнинg яqинlashиsh radиysи $r > 0$ bo'lsa, y holda by qatorнинg $S(x) = \sum_{n=0}^{\infty} a_n x^n$ yиg'индиsи $(-r,\ r)$ oralиqda yzlyкsиz функция bo'ladи.

◀(14.20) darajalи qatorнинg яqинlashиsh иntervalи $(-r,\ r)$ dan ихtиyorиy x_0 $(x_0 \in (-r,\ r))$ нyqtaни olaмиz. Ravshaнкi, $|x_0| < r$ bo'ladи. Уshby $|x_0| < c < r$ teнgsиzlиklarни qaноatlantиryvchи c sоnни olaylик. (14.20) darajalи qator yuqorида кeltиrиlgaн 17-teoremaga кo'ra $[-c,\ c]$ segmeнtda teкиs яqинlashyvchи bo'ladи. Униda yshby bobнинg 3-§ иdagи 6-teoremaga asosaн, berиlgaн (14.20) darajalи qatorнинg yиg'индиsи $S(x)$ функция $[-c,\ c]$ da, va deмaк, x_0 нyqtada yzlyкsиz bo'ladи. Deмaк, (14.20) qatorнинg yиg'индиsи $S(x)$ функция $(-r,\ r)$ иntervalda yzlyкsиzdиr.▶

19-teorema. Agar $\sum_{n=0}^{\infty} a_n x^n$ darajalи qatorнинg яqинlashиsh radиysи r $(r > 0)$ bo'lsa, by qatorни $[a,\ в]$ $([a,\ в] \subset (-r,\ r))$ oralиqda щадлабhdmhd иntegrallash myмкин.

◀Shyнday c $(0 < c < r)$ topa olaмизкi, $[a,\ в] \subset [-c,\ c] \subset (-r,\ r)$ bo'ladи.

Beruлgaн darajalи qator $[-c,\ c]$ da teкиs яqинlashyvchи bo'ladи. Demaк, $[a,\ в]$ da (14.20) darajalи qator teкиs яqинlashyvchи. Унda (14.20) qatorнинg уиg'инdиsи

$$S(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+...$$

yzlyкsиzlик bo'lиb, yshby bobнинg 5-§ da кeltиrиlgaн teoremaga кo'ra by qatorни щадлабhdmhd интegrallash mymкин:

$$\int_a^в S(x)dx=\int_a^в\sum_{n=1}^{\infty}a_nx^ndx=\sum_{n=0}^{\infty}a_n\int_a^в x^ndx=\sum_{n=0}^{\infty}a_n\frac{в^{n+1}-a^{n+1}}{n+1}.\ ►$$

Xysysaн, $a=0$, $в=x$ $(|x|<r)$ bo'lgaнda

$$\int_0^x S(x)dx=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}=a_0x+\frac{a_1}{2}x^2+...+\frac{a_{n-1}}{n}x^n+...$$

bo'ladи. By qatorнинg яqинlashиsh radиysи ham r ga teнg. Haqиqatdaн ham, Кoshи-Adamar teoremasидан фoydalaниb qyyиdagини topamиz:

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\left|\frac{a_n}{n+1}\right|}=\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|a_n|}}{\sqrt[n]{n+1}}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n]{n+1}}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=r.$$

20-teorema. $\sum_{n=0}^{\infty}a_nx^n$ darajalи qatorнинg яqинlashиsh radиysи r bo'lsa, $(-r,\ r)$ da by qatorни щадлабhdmhd dиффereнциallash mymкин.

◄Avvalo beruлgaн (14.20) darajalи qator hadlarининg hosиlalarидан tyzиlgaн yshby

$$\sum_{n=0}^{\infty}na_nx^{n-1}=a_1+2a_2x+3a_3x^2+...+na_nx^{n-1}+... \qquad (14.32)$$

qatorнинg $|x_0|<r$ teнgsиzlикни qaнoatlaнtиryvchи иxtиyorиy нyqtada яqинlashyvchи bo'lиshини кo'rsatamиz. Qyyиdagи $|x_0|<c<r$ teнgsиzlикlarни qaнoatlaнtиryvchи c soнни olaylик. Унda $\frac{1}{c}|x_0|=q<1$ bo'lиb,

$$|na_nx_0^{n-1}|=nq^{n-1}\cdot\frac{1}{c}|a_nc^n|$$

bo’ladи. Ravshanки, $\sum_{n=1}^{\infty} nq^{n-1}$ $(q<1)$ qator яqинlashyvchи (уни Dalamber alomatиga ко’ra ко’rsatиsh qиyин emas). Унda

$$\lim_{n\to\infty} nq^{n-1}=0$$

bo’ladи. Demaк, n нинg bиror, n_0 qиymatиdaн boshlab, $(n>n_0$ ychyн) $nq^{n-1}<c$ bo’lиb, natиjada $\forall n>n_0$ ychyн yshby

$$\left|na_n x_0^{n-1}\right| \le \left|a_n c^n\right| \qquad (14.33)$$

teнgsиzlиккa кelamиz.

$c\in(-r,\, r)$ bo’lgaнlиgи sabablи $\sum_{n=0}^{\infty} a_n c^n$ qator absolyut яqинlashyvchи. Унda (14.33) mynosabatни hиsobga olиb, Veyershtrass alomatиdaн фoydalaниb, $\sum_{n=0}^{\infty} na_n x^{n-1}$ qatorнинg $(-r,\, r)$ da яqинlashyvchи bo’lиshини topamиz. Demaк, by qator $[-c,\, c]$ da teкиs яqинlashyvchи bo’ladи.

Shyнday qиlиb, berиlgaн (14.20) darajalи qator hadlarининg hosиlalarиdaн tyzиlgaн (14.32) qator teкиs яqинlashyvchи. У holda yshby bobнинg 6-§ da кeltиrиlgaн 12-teoremaga ко’ra

$$S'(x)=\left(\sum_{n=0}^{\infty} a_n x^n\right)^1=\sum_{n=0}^{\infty} a\left(_n x^n\right)^1=\sum_{n=0}^{\infty} na_n x^{n-1}$$

bo’ladи. ▶

Shyни ham aytиsh кeraкки, (14.20) va (14.32) qatorlarнинg яqинlashиsh radиyslarи bиr xиl bo’ladи. Haqиqatdaн ham Кoshи-Adamar teoremasиdaн фoydalaниb qyyиdagини topamиz:

$$\overline{\lim_{n\to\infty}}\sqrt[n]{n|a_n|}=\overline{\lim_{n\to\infty}}\left(\sqrt[n]{n}\,\sqrt[n]{|a_n|}\right)=\overline{\lim_{n\to\infty}}\sqrt[n]{n}\cdot\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}\,.$$

Demaк,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{n|a_n|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}\,.$$

2-натиja. Agar (14.20) darajalи qatorнинg яqинlashиsh radиysи r bo'lsa, by qatorни $(-r,\ r)$ da иstalgaн marta dиффегенциallash myмкин.

SHyнday qилиb, яqинlashиsh radиysи $r>0$ bo'lgaн $\sum_{n=0}^{\infty} a_n x^n$ darajalи qatorни щадлабhdmhd инtegrallash va щадлабhdmhd (иstalgaн marta) dиф-фегенциallash myмкин va hosиl bo'lgaн darajalи qatorlarнинg яqинlashиsh radиysи ham r ga teнg bo'ladи.

10-ta'rиф. Agar $f(x)$ функция $(-r,\ r)$ da яqинlashyvchи darajalи qatorнинg yиg'инdиsи bo'lsa, $f(x)$ функция $(-r,\ r)$ da aнalиtик deb ataladи.

21-teorema. Иккиta

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad \textit{(14.20)}$$

va

$$\sum_{n=0}^{\infty} в_n x^n = в_0 + в_1 x + в_2 x^2 + ... + в_n x^n + ... \qquad \textit{(14.34)}$$

darajalи qatorlar berиlgaн bo'lиb, (14.20) darajalи qatorнинg яqинlashиsh radиysи $r_1>0$ yиg'инdиsи esa $S_1(x)$ (14.34) darajalи qatorнинg яqинlashиsh radиysи $r_2>0$ yиg'инdиsи $S_2(x)$ bo'lsин.

Agar $\forall x \in (-r,\ r)$ $(r = min(r_1, r_2))$ da

$$S_1(x) = S_2(x) \qquad \textit{(14.35)}$$

bo'lsa, y holda $\forall n \in N$ ychyн

$$a_n = в_n$$

я'нИ (14.20) va (14.32) darajalи qatorlar bиr xиl bo'ladи.

◄РавшинкиRvsh, (14.20) va (14.32) darajalи qatorlar $(-r,\ r)$ da яqинlashyvchи va ylarнинg yиg'инdиlarи $S_1(x)$ va $S_2(x)$ функцияlar shy инtervalda yzlyксиz bo'ladи. Demaк,

$$\lim_{x\to 0} S_1(x) = S_1(0), \quad \lim_{x\to 0} S_2(x) = S_2(0).$$

Yuqorиdagи (14.35) shartga ко'ra $S_1(0) = S_2(0)$ bo'ladи. Вунdaн esa $a_0 = в_0$ екaнlиgи кelиb chиqadи. Винobarин, $\forall x \in (-r,\ r)$ ychyн

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} в_n x^n .$$

Agar $x \neq 0$ desaк, by tенglикdан barcha $x \in (-r,\ 0) \cup (0,\ r)$ ychyн

$$\sum_{n=1}^{\infty} a_n x^{n-1} = \sum_{n=1}^{\infty} в_n x^{n-1}$$

ga ega bo'lamиz. By darajalи qatorlarнинг har bиrи ham $(-r,\ r)$ da яqинlashyvchи bo'ladи va demaк, ylarнинг yиg'индиlarи shy интervalda yzlyксиz функция bo'ladи. Shy xysysиятdан фoydalaнsaк, $x \to 0$ da

$$\lim_{x \to 0} \sum_{n=1}^{\infty} a_n x^{n-1} = a_1 , \quad \lim_{x \to 0} \sum_{n=1}^{\infty} в_n x^{n-1} = в_1$$

bo'lиshини va demaк, $a_1 = в_1$ еканини topamиz. By jarayoнни davom ettиra borиb, barcha $n \in N$ ychyн $a_n = в_n$ bo'lиshи topиladи. Demaк, (14.20) va (14.34) darajalи qatorlar bиr xиl.▶

$(-r,\ r)$ $(r > 0)$ oralиqda $f(x)$ функция berиlgaн va yzlyксиz bo'lsин. Yuqorиdagи teorema, $f(x)$ ни darajalи qator yиg'индиsи sифatида ифodalay olaдиsган былсак, bунday ифodalash ягона bo'lиshини bиldиradи.

6-§. Teylor qatorи.

Биz yuqorиda, har qaнday darajalи

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

qator o'zининг яqинlashиsh интervalи $(-r,\ r)$ da yzlyксиz $S(x)$ функцияни (darajalи qator yиg'индиsини) ифodalab, by функция shy oralиqda иstalgaн tartиbdagи hosиlaga ega bo'lиshини ко'rdик.

Endи bиror oralиqda иstalgaн tartиbdagи hosиlaga ega bo'lgaн функцияни darajalи qatorga yoyиsh masalasини qaraymиz.

1⁰. Функцияlarни Teylor qatorиga yoyиsh. $f(x)$ функция $x = x_0$ нуqtaнинг bиror

$$U_\delta(x_0) = \{x \in R :\ x_0 - \delta < x < x_0 + \delta\}\ (\delta > 0)$$

atrофida berилgaн bo'lиb, shy atrофda функция иstalgaн tartиbdagи hosиlaga ega bo'lsин. Ravshaнкi, by функциянинг 1-qисm, 6-bob, 7-§ da bataфsиl o'rgaнилgaн Teylor формylasи

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x)$$

ни yozиsh mymкин, byнda $r_n(x)$ qoldиq had.

Berилgaн $f(x)$ функциянинг x_0 нyqtada иstalgaн tartиbdagи hosиlaga eg bo'lиshи Teylor формylasиdagи hadlarнинг soнини har qaнcha кatta olиsh иmкoнини beradи. Виноbarин, tabииy ravиshda yshby

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots \qquad (14.36)$$

qator yuzaga кeladи. By maxsys darajalи qator bo'lиb, униng коеффициентlarи $f(x)$ функция va униng hosиlalarининг x_0 нyqtadagи qиymatlarи orqalи ифodalaнadи.

Odatda (14.36) darajalи qator $f(x)$ функциянинг Teylor qatorи deb ataladи.

Xysysaн, $x_0=0$ da qyyиdagиcha bo'ladи:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots. \qquad (14.37)$$

Darajalи qatorlar deb нomlaнgaн 8-§ нинг boshlaнишиda $\sum_{n=0}^{\infty}a_nx^n$ кo'ринишdagи darajalи qatorlarни o'rgaнишни кelиshиb olинgaн edи. Shyни etиborga olиb, $f(x)$ функциянинг (14.37) кo'ринишdagи Teylor qatorиni o'rgaнamиz.

Яна bиr bor ta'кидlaymиzкi, (14.36) qator $f(x)$ функция bилан o'zининг коеффициентlarи orqalи bog'laнgaн bo'lиb, by (14.36) qator яqинlashyvchи bo'ladими, яqинlashyvchи bo'lgaн holda униng yиg'инdиси $f(x)$ ga teнg bo'ladими, bунdaн qat'иy нazar, уни $f(x)$ функциянинг Teylor qatorи deb atadик.

Tabиuy ravиshda qyyиdagи savol tyg'иladи: qachoн bиror $U_\delta(0)$ oralиqda berиlgaн, иstalgaн tartиbdagи hosиlaga ega bo'lgaн $f(x)$ функциянинг Teylor qatorи

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...$$

shy oralиqda xyddи shy $f(x)$ ga яqинlashadи.

22-teorema. $f(x)$ функция bиror $(-r,\ r)$ $(r>0)$ oralиqda иstalgaн tartиbdagи hosиlaga ega bo'lиb, унинг $x=0$ нyqtadagи Teylor qatorи

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...$$

bo'lsин.

Вy qator $(-r,\ r)$ oralиqda $f(x)$ ga яqинlashиshи ychyн $f(x)$ функция Teylor фormylasи

$$f(x)= f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+r_n(x) \qquad (14.38)$$

нинг qoldиq hadи barcha $x\in(-r,\ r)$ da holga интилиshи $\left(\lim\limits_{n\to\infty} r_n(x)=0\right)$ zaryr va etarlи.

◄***Zaryrlugu.*** Avvalo (14.37) qatorнинг коеффициентlarи bиlaн (14.38) Teylor фormylasидагu коеффициентlarнинг bиr xиl еканlиgини ta'кидlaymиz.

(14.37) qator яqинlashyvchи bo'lиb, унинг уиg'индиsи $f(x)$ ga teнg bo'lsин. У holda by qatorнинг qиsmиy уиg'индиsи

$$S_n(x)= f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n$$

ychyн

$$\lim_{n\to\infty} S_n(x)= f(x) \quad (\forall x\in(-r,\ r))$$

bo'ladи. Уndaн esa $(\forall x\in(-r,\ r))$ ychyн

$$\lim_{n\to\infty}[f(x)-S_n(x)]=\lim_{n\to\infty} r_n(x)=0$$

bo'lиshи кelиb chиqadи.

Etarlulugu. $(\forall x \in (-r, r))$ da $\lim_{n\to\infty} r_n(x) = 0$ bo'lsин. У holda qyyидagисha $\lim_{n\to\infty}[f(x) - S_n(x)] = 0$ bo'lиb, ундан esa

$$\lim_{n\to\infty} S_n(x) = f(x)$$

bo'lиshи кelиb chиqadи. By esa (14.37) qator $(-r,\ r)$ da яqинlashyvchи bo'lиb, унинг уиg'индиsи $f(x)$ ga teнg bo'lиshини, я'ни

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

еканlиgини bиldиradи.▶

Odatda кеуинги mунosabat o'rинlи bo'lsa, $f(x)$ функция Teylor qatorиga yoyиlган былсаdbtld:***trm***grrlqddrjlqtrgyoylgbo'ls

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \qquad (14.39)$$

by qator $f(x)$ функциянинг Teylor qatorи bo'ladи.

◀20-teorema va унинг натиjasиga кo'ra (14.39) darajalи qator $(-r,\ r)$ oralиqda иstalgaн marta (щадлабhdmhd) дифференциallанyvchи bo'lиb,

$$f'(x) = 1\cdot a_1 + 2\cdot a_2x + 3\cdot a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

$$f''(x) = 1\cdot 2\cdot a_2 + 2\cdot 3\cdot a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots$$

$$f'''(x) = 1\cdot 2\cdot 3\cdot a_3 + \ldots + n(n-1)(n-2)a_nx^{n-3} + \ldots$$

....

$$f^{(n)}(x) = 1\cdot 2\cdot 3\cdot\ldots(n-1)na_n + \ldots$$

....

bo'ladи. Кеуинги teнglиклarda $x = 0$ deb qyyидagиlarни topaмиз:

$$a_0 = f(0),\ a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!}, \ldots,\ a_n = \frac{f^{(n)}(0)}{n!}, \ldots$$

tjdqtrgo'rshqydgchbo'ld

QydgTylrqtrgyoylshgtrlshrtdlvchtrmltrmz

trmbrrrlqdstlgtrtbdghslgegbo'lsgrshdysmvjdbo'lsbrchhmdbrchchtgszlbjrlshldr lqdTylrqtrgyoyld'bo'ldchTylrrmlsyozbgLgrjo'rshdgqldqhdlylhldbo'ldgrbo'lshetbr glshldelgqlymzBesmsbtgo’rlbo’lshbldrd***ElmtrlrgTylrqtrlrgTylrqtr***M’lmgtyorychlrl qdgTylrrmlsbo’lbgqldqhdesLgrjo’rshdqydgchbo’ldqrgqsmbbHrbrdbo’lshetbrglsd

elglbchqdvdlgtldDmtyorychldbo’ld***gTylrqtr***M’lm-gtyorychlrlqdgTylrrmlsbo’ldBrmlqldqhdgLgrjo’rshdydlbqrlsqsmbbchbo’lshtpmzd

bo’lshlbchqdDmchbo’ld***gTylrqtr***BgTylrrm-lsqldqhdgLgrjo’rshdydlbqrlsqsmbbchbo’lshtpmzdbo’lshlbchqdDmch***gTylrqtr***M’l mbgTylrrmlsqydgchbo’ldBrmlddqldqhdLgrjo’rshdqydgchyozbgchbo’lshbo’lgdesq ldqhdsho’rshdqydgchyozbgchbo’lsho’rgedqsmbbvmsbtlrdbo’lshtpmzDmdbo’ldSH t’dlshlzmrlqdbrlgbo’lshmbgTylrqtrrmtrvldo’rldr***gTylrqtr***BgTylrrmlsbo’lbqrlsqsmb bgqldqhdsho’rshdqydgchbo’ldshbo’rshdyozblmzytylbo’lsdbrchdchbqlshvchqtrgm myhdbqtrgqlshvchlgDlmbrlmtgo’ro’rstldchdvhtchchdbo’lglgdbo’lshlbchqdDmdbo ’ld***Mshqlr***shbltmtlgdlmtstplsvgtsqlshshsbtlsgrvltmtllrto’plmdmsrvshdvlrgtsqlshvch bo’lsltmtlto’plmdgtsqlshshsbtlsshbdbrlgltmtlglmtsgdtsqlshsho’rstlsgrshblqtrto’plm dtsqlshvchbo’lsltmtlto’plmdgtsqlshshsbtlsshblqtryg’dstplsytylslrtmtlgchlmtmvjdbo ’lshlddrjlqtrgqlshshrdsbo’lshsbtls

15-BOB

Xosmas integrallar

Mazkur kursning 9, 10- boblarida funksiyaning aniq integrali (Riman integrali) tushunchasi kiritilib, u batafsil o’rganildi. Integral bayonida integrallash oralig’ining chekliligi va funksiyaning chegaralanganligi bevosita ishtirok etdi.

Endi aniq integral tushunchasini:

1) cheksiz oraliqda aniqlangan funksiyalarga;

2) chegaralanmagan funksiyalarga

nisbatan umumlashtirilishini qaraymiz. Odatda, bunday integrallar xosmas interallar deyiladi.

1-§. Cheksiz oraliq bo'yicha xosmas integrallar

1⁰. Cheksiz oraliq bo'yicha xosmas integral tushunchasi. $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lib, uning istalgan $[a,t]$ qismida integrallanuvchi bo'lsin $(a\in R,\ t\in R,\ t\geq a)$. Ravshanki,

$$\int_a^t f(x)dx$$

integral t o'zgaruvchiga bog'liq bo'ladi:

$$F(t)=\int_a^t f(x)dx.$$

1-ta'rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud bo'lsa, bu limit $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo'yicha xosmas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx \qquad (15.1)$$

kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx=\lim_{t\to+\infty} f(t)=\lim_{t\to+\infty}\int_a^t f(x)dx.$$

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo'lsa, (15.1) xosmas integral yaqinlashuvchi deyiladi.

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki $F(t)$ funksiyaning limiti mavjud bo'lmasa, (15.1) xosmas integral uzoqlashuvchi deyiladi.

Masalan, ushbu

$$\int_0^{+\infty} e^{-x}dx$$

xosmas integral yaqinlashuvchi bo'ladi, chunki

$$\lim_{t\to+\infty}\int_0^t e^{-x}dx=\lim_{t\to+\infty}\left(-e^{-t}+1\right)=1$$

va demak,

$$\int_0^{+\infty} e^{-x}dx=1.$$

Funksiyaning $(-\infty,a]$ va $(-\infty,+\infty)$ oraliqlar bo'yicha xosmas integrallari va ularning yaqinlashuvchiligi (uzoqlashuvchiligi) yuqoridagi kabi ta'riflanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_{t}^{a} f(x)dx,$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{t\to-\infty \\ s\to+\infty}}\int_{t}^{s} f(x)dx.$$

Masalan, ushbu

$$\int_{-\infty}^{0}\frac{dx}{1+x^2}$$

xosmas integral yaqinlashuvchi bo'ladi, chunki

$$\lim_{t\to-\infty}\int_{t}^{0}\frac{1}{1+x^2}dx = \lim_{t\to-\infty}(-arctgt) = \frac{\pi}{2}$$

va demak,

$$\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

Shunday qilib, xosmas integrallar avval o'rganilgan integraldan limitga o'tish amali orqali yuzaga kelar ekan.

15.1-misol. Ushbu

$$\int_{a}^{+\infty}\frac{dx}{x^{\alpha}} \qquad (a>0,\ \alpha>0)$$

xosmas integral yaqinlashuvchilikka tekshirilsin.

◀ Ta'rifga ko'ra

$$\int_{a}^{+\infty}\frac{dx}{x^{\alpha}} = \lim_{t\to+\infty}\int_{a}^{t}\frac{dx}{x^{\alpha}}.$$

Aytaylik, $\alpha<1$ va $\alpha=1$ bo'lsin. Bu holda, mos ravishda

$$\lim_{t\to+\infty}\int_{a}^{t}\frac{dx}{x^{\alpha}} = \lim_{t\to+\infty}\frac{1}{1-\alpha}\left(t^{1-\alpha}-a^{1-\alpha}\right) = +\infty,$$

$$\lim_{t\to+\infty}\int_{a}^{t}\frac{dx}{x} = \lim_{t\to+\infty}(\ln t-\ln a) = +\infty$$

bo'ladi.

Aytaylik, $\alpha>1$ bo'lsin. Bu holda

$$\lim_{t\to+\infty}\int_{a}^{t}\frac{dx}{x} = \lim_{t\to+\infty}\left[\frac{t^{-\alpha+1}}{-\alpha+1}-\frac{a^{-\alpha+1}}{-\alpha+1}\right] = \frac{a^{1-\alpha}}{\alpha-1}$$

bo'ladi.

Shunday qilib

$$\int_{a}^{+\infty}\frac{dx}{x^{\alpha}} \qquad (a>0,\ \alpha>0)$$

xosmas integral $\alpha>1$ bo'lganda yaqinlashuvchi, $\alpha\le1$ bo'lganda uzoqlashuvchi bo'ladi.▶

Biz quyida, asosan, $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo'yicha $\int\limits_a^{+\infty} f(x)dx$ xosmas integralini o'rganamiz. $(-\infty,a]$ va $(-\infty,+\infty)$ oraliqlar bo'yicha xosmas integrallar tegishlicha bayon etilishi mumkin.

2^0. Xosmas integrallarning yaqinlashuvchiligi. Integralning absalyut yaqinlashuvchiligi. Xosmas integrallarning yaqinlashuvchiligi shartlarini keltiramiz.

Faraz qilaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lib, $\forall x\in[a,+\infty)$ da

$$f(x)\geq 0$$

bo'lsin. U holda $\forall t_1,t_2\in(a,+\infty)$ uchun $t_1<t_2$ bo'lganda

$$F(t_2)=\int\limits_a^{t_2} f(x)dx=\int\limits_a^{t_1} f(x)dx+\int\limits_{t_1}^{t_2} f(x)dx=F(t_1)+\int\limits_{t_1}^{t_2} f(x)dx\geq F(t_1).$$

Demak,

$$F(t)=\int\limits_a^{t} f(x)dx$$

funksiya $[a,+\infty)$ da o'suvchi bo'ladi.

1-teorema. $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lib, $\forall x\in[a,+\infty)$ da

$$f(x)\geq 0$$

bo'lsin. Bu funksiyaning $[a,+\infty)$ oraliq bo'yicha xosmas integrali

$$\int\limits_a^{+\infty} f(x)dx$$

ning yaqinlashuvchi bo'lishi uchun,

$$F(t)=\int\limits_a^{t} f(x)dx$$

funksiyaning yuqoridan chegaralangan, ya'ni

$$\exists C\in R,\qquad \forall t\in[a,\ +\infty):\quad \int\limits_a^{t} f(x)dx\leq C$$

bo'lishi zarur va etarli.

◄***Zarurligi.*** Aytaylik, xosmas integral

$$\int\limits_a^{+\infty} f(x)dx$$

yaqinlashuvchi bo'lsin. U holda

$$\lim_{t\to+\infty} F(t)=\lim_{t\to+\infty}\int\limits_a^{t} f(x)dx=J$$

mavjud va chekli bo'lib,

$$J=\sup_{a\leq x<+\infty} F(t)$$

bo'ladi. Aniq yuqori chegaraning ta'rifiga ko'ra, $\forall t\in[a,+\infty)$ da

$$F(t)=\int_a^t f(x)dx \le \int_a^{+\infty} f(x)dx$$

ya'ni

$$F(t)=\int_a^t f(x)dx \le C$$

bo'ladi.

Etarliligi. Aytaylik,

$$\exists C\in R, \quad \forall t\in [a,\ +\infty): \quad \int_a^t f(x)dx \le C$$

bo'lsin. Unda monoton funksiyaning limiti haqidagi teoremaga ko'ra ushbu

$$\lim_{t\to\infty} F(t)$$

limit mavjud va chekli bo'ladi. Demak, $\int_a^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi. ▶

Eslatma. Agar $\forall x\in [a,+\infty)$ da $f(x)\ge 0$ bo'lib,

$$F(t)=\int_a^t f(x)dx$$

funksiya yuqoridan chegaralanmagan bo'lsa, $\int_a^{+\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'ladi.

2-teorema. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalari $[a,+\infty)$ oraliqda berilgan bo'lib, $\forall x\in [a,+\infty)$ da

$$0\le f(x)\le g(x)$$

bo'lsin.

Agar $\int_a^{+\infty} g(x)dx$ xosmas integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ xosmas integral ham yaqinlashuvchi bo'ladi.

Agar $\int_a^{+\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'lsa, $\int_a^{+\infty} g(x)dx$ ham uzoqlashuvchi bo'ladi.

◀ Aytaylik, $\int_a^{+\infty} g(x)dx$ xosmas integral yaqinlashuvchi bo'lsin. Ravshanki,

$$\int_a^t f(x)dx \le \int_a^t g(x)dx \le \int_a^{+\infty} g(x)dx$$

bo'ladi. Bundan $\int_a^t f(x)dx$ ning yuqoridan chegaralanganligi kelib chiqadi. 1-teoremaga ko'ra

$$\int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi bo'ladi.

Aytaylik, $\int_a^{+\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'lsin. U holda

$$F(t)=\int_a^t f(x)dx$$

funksiya yuqoridan chegaralanmagan bo'lib,

$$\int_a^t f(x)dx \le \int_a^t g(x)dx$$

tengsizlikka ko'ra

$$\int_a^t g(x)dx$$

funksiya ham yuqoridan chegaralanmagan bo'ladi. Yuqorida keltirilgan eslatmaga binoan $\int_a^{+\infty} g(x)dx$ xosmas integral uzoqlashuvchi bo'ladi. ▶

15.2-misol. Ushbu

$$\int_1^{+\infty} \frac{cos^4 3x}{\sqrt[5]{1+x^6}}dx$$

xosmas integral yaqinlashuvchilikka tekshirilsin.

◀ Ravshanki, integral ostidagi funksiya

$$f(x)=\frac{cos^4 3x}{\sqrt[5]{1+x^6}} \ge 0$$

bo'ladi. Ayni paytda $x \ge 1$ bo'lganda

$$f(x)=\frac{cos^4 3x}{\sqrt[5]{1+x^6}} \le \frac{1}{x^{6/5}}$$

tengsizlik bajariladi. Quyidagi

$$\int_1^{+\infty} \frac{dx}{x^{6/5}}$$

xosmas integral yaqinlashuvchi (qaralsin, 15.1-misol) bo'lganligi uchun 2-teoremaga ko'ra berilgan xosmas integral yaqinlashuvchi bo'ladi. ▶

Endi $[a,+\infty)$ oraliqda berilgan ixtiyoriy $f(x)$ funksiya xosmas integrali

$$\int_a^{+\infty} f(x)dx$$

ning yaqinlashuvchiligi haqidagi teoremani keltiramiz.

3-teorema. (Koshi teoremasi). Ushbu

$$\int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi bo’lishi uchun

$$\forall\varepsilon>0,\ \exists t_0>a,\ \forall t'>t_0,\ \forall t''>t_0$$

bo’lganda

$$\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

◀Ma’lumki,

$$\int_a^{+\infty} f(x)dx$$

xosmas integralning yaqinlashuvchiligi $t\to+\infty$ da

$$F(t)=\int_a^t f(x)dx$$

funksiyaning chekli limitga ega bo’lishidan iborat.

Funksiyaning chekli limitga ega bo’lishi haqidagi Koshi teoremasiga (qaralsin, 4-bob, 6-§) binoan,

$$\forall\varepsilon>0,\ \exists t_0>a,\ \forall t'>t_0,\ \forall t''>t_0:\left|F(t'')-F(t')\right|<\varepsilon$$

ya’ni

$$\left|F(t'')-F(t')\right|=\left|\int_a^{t''} f(x)dx-\int_a^{t'} f(x)dx\right|=\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

bo’lishi zarur va etarli edi.▶

Bu nazariy ahamiyatga ega bo’lgan muhim teorema bo’lib, undan xosmas integrallarning yaqinlashuvchanligini aniqlashda foydalanish ko’pincha qiyin bo’ladi.

Xosmas integrallarning yaqinlashuvchanligini aniqlashda ko’p qo’llaniladigan alomatlardan birini keltiramiz.

4-teorema. (Dirixle alomati). $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo’lib, ular quyidagi shartlarni bajarsin:

1) $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz va uning shu oraliqdagi boshlang’ichi $F(x)$ $(F'(x)=f(x))$ funksiyasi chegaralangan;

2) $g(x)$ funksiya $[a,+\infty)$ oraliqda $g'(x)$ hosilaga ega va u uzluksiz funksiya;

3) $g(x)$ funksiya $[a,+\infty)$ oraliqda kamayuvchi;

4) $\lim\limits_{x\leftrightarrow+\infty} g(x)=0$. U holda $\int_a^{+\infty} f(x)g(x)dx$ integral yaqinlashuvchi bo’ladi.

◀Uzluksiz $f(x)$ va $g(x)$ funksiyalarning ko’paytmasi $f(x)\,g(x)$ funksiya ham $[a,+\infty)$ oraliqda uzluksiz bo’lgani uchun, bu $f(x)\,g(x)$ funksiya istalgan $[a, t]$ oraliqda integrallanuvchi bo’ladi, ya’ni

$$\varphi(t)=\int_a^t f(x)g(x)dx \qquad (15.2)$$

integral mavjud.

$t \to +\infty$ da $\varphi(t)$ funksiyaning chekli limitga ega bo'lishini ko'rsatamiz. Teoremaning 1- va 2- shartlaridan foydalanib, (15.2) integralni bo'laklab hisoblaymiz.

$$\int_a^t f(x)g(x)dx = \int_a^t g(x)dF(x) = g(x)dF(x)\Big|_a^t - \int_a^t F(x)g'(x)d(x). \qquad (15.3)$$

O'ng tomondagi birinchi qo'shiluvchi uchun ushbu

$$|g(t)F(t)| \le Mg(t) \qquad \left(M = sup|F(t)| < +\infty\right)$$

tengsizlikka ega bo'lamiz. Undan, $t \to +\infty$ da $g(t) \to 0$ bo'lishini e'tiborga olsak,

$$\lim_{t \to +\infty} g(t)F(t) = 0$$

bo'lishi kelib chiqadi.

Endi o'ng tomondagi ikkinchi $\int_a^t F(x)g'(x)dx$ hadni qaraymiz. Modomiki, $g(x)$ funksiya $[\,a,+\infty)$ oraliqda uzluksiz differensiallanuvchi hamda shu oraliqda kamayuvchi ekan, unda $\forall x \in [\,a,+\infty)$ da $g'(x) \le 0$ bo'lib,

$$\int_a^t F(x)g'(x)dx \le M\int_a^t |g'(x)|dx = -M\int_a^t g'(x)dx = M[g(a) - g(t)] \le Mg(a)$$

$$(g(t) \ge 0)$$

bo'ladi. Shunday qilib, t o'zgaruvchining barcha $t > a$ qiymatlarida

$$\int_a^t |F(x)g'(x)|dx$$

integral (t o'zgaruvchining funksiyasi) yuqoridan chegaralangan. U holda 1-teoremaga ko'ra $\int_a^{+\infty} F(x)g'(x)dx$ integral yaqinlashuvchi bo'ladi. Demak,

$$\lim_{t \to +\infty} \int_a^t F(x)g'(x)dx$$

limit mavjud va chekli.

Yuqoridagi (15.3) tenglikda $t \to +\infty$ da limitga o'tib, Ushbu

$$\lim_{t \to +\infty} \int_a^t f(x)g(x)dx$$

limitning mavjud hamda chekli bo'lishini topamiz. Bu esa $\int_a^{+\infty} f(x)g(x)dx$ integralning yaqinlashuvchiligini bildiradi.

15.3-misol. Ushbu

$$\int_a^{+\infty} \frac{sin\, x}{x^{\alpha}} dx \qquad (\alpha > 0)$$

integral yaqinlashuvchilikka tekshirilsin.

◀Bu integraldagi $f(x)=\sin x,\ g(x)=\frac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiyalar yuqorida keltirilgan teoremaning barcha shartlarini qanoatlantiradi:

1) $f(x)=\sin x$ funksiya $[1,+\infty)$ oraliqda uzluksiz va boshlang'ich funksiyasi $F(x)=$-cosx chegaralangan;

2) $g(x)=\frac{1}{x^{\alpha}}$ funksiya $[1,+\infty)$ oraliqda $g'(x)=-\frac{\alpha}{x^{1+\alpha}}$ hosilaga ega va u uzluksiz;

3) $g(x)=\frac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiya $[1,+\infty)$ oraliqda kamayuvchi;

4) $\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\frac{1}{x^{\alpha}}=0$ $(\alpha>0)$ bo'ladi. Demak, Dirixle alomatiga ko'ra berilgan integral yaqinlashuvchi.▶

$f(x)$ funksiyaning xosmas integrali $\int_a^{+\infty} f(x)dx$ bilan bir qatorda

$$\int_a^{+\infty}|f(x)|dx$$

xosmas integralni qaraymiz.

5-teorema. Agar $\int_a^{+\infty}|f(x)|dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^{+\infty} f(x)dx$ integrali ham yaqinlashuvchi bo'ladi.

◀Shartga ko'ra $\int_a^{+\infty}|f(x)|dx$ integral yaqinlashuvchi. 4-teoremaga asosan, $\forall\varepsilon>0$ olinganda ham, shunday t_0 $(t_0>a)$ topiladiki, $t'>t_0$, $t''>t_0$ bo'lganda $\int_{t'}^{t''}|f(x)|dx<\varepsilon$ tengsizlik bajariladi.

Agar

$$\left|\int_{t'}^{t''} f(x)dx\right|\le\int_{t'}^{t''}|f(x)|dx$$

tengsizlikni e'tiborga olsak, u holda

$$\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

bo'lishini topamiz.

Shunday qilib, $\forall\varepsilon>0$ son olinganda ham, shunday t_0 $(t_0>a)$ topiladiki, $t'>t_0$, $t''>t_0$ bo'lganda

$$\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

bo'ladi. Bundan 4-teoremaga asosan $\int_a^{+\infty} f(x)dx$ integralning yaqinlashuvchiligini topamiz.▶

2-ta'rif. Agar $\int_a^{+\infty} |f(x)|dx$ integral yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ absolyut yaqinlashuvchi integral deb ataladi, $f(x)$ funksiya esa $[a,+\infty)$ oraliqda absolyut integrallanuvchi funksiya deyiladi.

3-ta'rif. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lib, $\int_a^{+\infty} |f(x)|dx$ integral uzoqlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ shartli yaqinlashuvchi integral deyiladi.

Shunday qilib, $\int_a^{+\infty} f(x)dx$ xosmas integralni yaqinlashuvchilikka tekshirish quyidagi tartibda olib borilishi mumkin:

$\forall x \in [a,+\infty)$ da $f(x) \geq 0$ bo'lsin. Bu holda $\int_a^{+\infty} f(x)dx$ integralning yaqinlashuvchi (uzoqlashuvchi) ligini yuqorida keltirilgan alomatlardan foydalanib topish mumkin. Boshqa hollarda $f(x)$ funksiyaning $|f(x)|$ absolyut qiymatining $[a,+\infty)$ oraliq bo'yicha $\int_a^{+\infty} |f(x)|dx$ integralini qaraymiz. Ravshanki, keyingi integralga nisbatan yana yuqoridagi alomatlarni qo'llash mumkin. Agar biror alomatga ko'ra $\int_a^{+\infty} |f(x)|dx$ integralining yaqinlashuvchiligi topilsa, unda 5-teoremaga ko'ra berilgan $\int_a^{+\infty} f(x)dx$ integralning ham yaqinlashuvchiligi (hatto absolyut yaqinlashuvchiligi) topilgan bo'ladi.

Agar biror alomatga ko'ra $\int_a^{+\infty} |f(x)|dx$ integralining uzoqlashuvchiligini aniqlasak, aytish mumkinki, $\int_a^{+\infty} f(x)dx$ yoki uzoqlashuvchi bo'ladi yoki shartli yaqinlashuvchi bo'ladi va buni aniqlash qo'shimcha tahlil qilishni talab etadi.

3⁰. Yaqinlashuvchi xosmas integrallarning xossalari. Riman integralini umumlashtirishdan hosil qilingan yaqinlashuvchi xosmas integrallar ham shu Riman integrali xossalari singari xossalarga ega.

$f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lsin.

1) Agar $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo'yicha $\int_a^{+\infty} f(x)dx$ integrali yaqinlashuvchi bo'lsa, bu funksiyaning $[в,+\infty)$ $(a<в)$ oraliq bo'yicha $\int_a^{+\infty} f(x)dx$ integrali ham yaqinlashuvchi bo'ladi va aksincha. Bunda

$$\int_a^{+\infty} f(x)dx = \int_a^{в} f(x)dx + \int_в^{+\infty} f(x)dx \qquad (15.4)$$

bo'ladi.

◀ Aniq integral xossasiga ko'ra

$$\int_a^{t} f(x)dx = \int_a^{в} f(x)dx + \int_в^{t} f(x)dx \qquad (a<t<\infty) \qquad (15.5)$$

bo'ladi.

$\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi, ya'ni

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty}\int_a^{t} f(x)dx$$

limit mavjud va chekli bo'lsin. Yuqoridagi (15.5) munosabatni ushbu

$$\int_в^{t} f(x)dx = \int_a^{t} f(x)dx - \int_a^{в} f(x)dx$$

ko'rinishda yozib, $t\to+\infty$ da limitga o'tib quyidagini topamiz:

$$\lim_{t\to+\infty}\int_a^{t} f(x)dx = \lim_{t\to+\infty}\int_a^{t} f(x)dx - \int_a^{в} f(x)dx = \int_a^{+\infty} f(x)dx - \int_a^{в} f(x)dx.$$

Bundan esa $\int_в^{+\infty} f(x)dx$ integralning yaqinlashuvchi va

$$\int_в^{+\infty} f(x)dx = \int_a^{+\infty} f(x)dx - \int_a^{в} f(x)dx$$

ya'ni

$$\int_a^{+\infty} f(x)dx = \int_a^{в} f(x)dx + \int_в^{+\infty} f(x)dx$$

ekanligi kelib chiqadi.

Xuddi shunga o'xshash $\int_в^{+\infty} f(x)dx$ integralning yaqinlashuvchi bo'lishidan $\int_a^{+\infty} f(x)dx$ integralning ham yaqinlashuvchi hamda (15.4) formulaning o'rinli bo'lishi ko'rsatiladi. ▶

2) Agar $\int_{a}^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_{a}^{+\infty} cf(x)dx$ integral ham yaqinlashuvchi bo'lib,

$$\int_{a}^{+\infty} cf(x)dx = c\int_{a}^{+\infty} f(x)dx$$

bo'ladi, bunda $c = const$.

3) Agar $\forall x \in [a, +\infty)$ da $f(x) \geq 0$ bo'lsa, bu funksiyaning xosmas integrali

$$\int_{a}^{+\infty} f(x)dx \geq 0$$

bo'ladi.

Endi $f(x)$ funksiya bilan bir qatorda $g(x)$ funksiya ham $[a, +\infty)$ oraliqda berilgan bo'lsin.

4) Agar $\int_{a}^{+\infty} f(x)dx$ da $\int_{a}^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo'lsa, u holda $\int_{a}^{+\infty} [f(x) \pm g(x)]dx$ integral ham yaqinlashuvchi bo'lib,

$$\int_{a}^{+\infty} [f(x) \pm g(x)]dx = \int_{a}^{+\infty} f(x)dx \pm \int_{a}^{+\infty} g(x)dx$$

bo'ladi.

1-natija. Agar $f_1(x), f_2(x), ..., f_n(x)$ funksiyalarning har biri $[a, +\infty)$ oraliqda berilgan bo'lib, $\int_{a}^{+\infty} f_k(x)dx \quad (k = 1, 2, ..., n)$ integrallar yaqinlashuvchi bo'lsa, u holda

$$\int_{a}^{+\infty} [c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x)]dx$$

integral yaqinlashuvchi bo'lib,

$$\int_{a}^{+\infty} [c_1 f_1(x) + ... + c_n f_n(x)]dx = c_1 \int_{a}^{+\infty} f_1(x)dx + ... + c_n \int_{a}^{+\infty} f_n(x)dx$$

bo'ladi.

5) Agar $\forall x \in [a, +\infty)$ uchun $f(x) \leq g(x)$ tengsizlik o'rinli bo'lib, $\int_{a}^{+\infty} f(x)dx$ va $\int_{a}^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo'lsa, u holda

$$\int_{a}^{+\infty} f(x)dx \leq \int_{a}^{+\infty} g(x)dx$$

bo'ladi.

Yuqorida keltirilgan 2- 5- xossalarning isboti xosmas integral va uning yaqinlashuvchiligi ta'riflaridan bevosita kelib chiqadi.

O’rta qiymat haqidagi teorema. $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo’lsin. $f(x)$ funksiya shu oraliqda chegaralangan, ya’ni shunday m va M o’zgarmas sonlar mavjudki, $\forall x\in[a,+\infty)$ uchun

$$m\le f(x)\le M$$

bo’lib, $g(x)$ funksiya esa $[a,+\infty)$ da o’z ishorasini o’zgartirmasin, ya’ni $\forall x\in[a,+\infty)$ uchun har doim $g(x)\ge 0$ yoki $g(x)\le 0$ bo’lsin.

6) Agar $\int\limits_a^{+\infty} f(x)g(x)dx$ va $\int\limits_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo’lsa, u holda shunday o’zgarmas μ $(m\le\mu\le M)$ son topildiki,

$$\int\limits_a^{+\infty} f(x)g(x)dx=\mu\int\limits_a^{+\infty} g(x)dx \qquad (15.6)$$

tenglik o’rinli bo’ladi.

◄Yuqorida keltirilgan $g(x)$ funksiya $[a,+\infty)$ oraliqda manfiy bo’lmasin: $g(x)\ge 0$ $(\forall x\in[a,+\infty))$. U holda

$$mg(x)\le f(x)g(x)\le Mg(x)$$

bo’lib, unda esa (Riman integralining tegishli xossasiga ko’ra) $m\int\limits_a^{t} g(x)dx\le\int\limits_a^{t} f(x)g(x)dx\le M\int\limits_a^{t} g(x)dx$ bo’lishini topamiz. Keyingi, tengsizliklarda $t\to+\infty$ da limitga o’tsak,

$$m\int\limits_a^{+\infty} g(x)dx\le\int\limits_a^{+\infty} f(x)g(x)dx\le M\int\limits_a^{+\infty} g(x)dx \qquad (15.7)$$

ekanligi kelib chiqadi.

Ikki holni qaraylik:

a) $\int\limits_a^{+\infty} g(x)dx=0$ bo’lsin. U holda $\int\limits_a^{+\infty} f(x)g(x)dx=0$ bo’lib, bunda μ deb $m\le\mu\le M$ tengsizliklarni qanoatlantiruvchi ixtiyoriy sonni olish mumkin.

b) $\int\limits_a^{+\infty} g(x)dx>0$ bo’lsin. Bu holda (15.7) tengsizliklardan

$$m\le\frac{\int\limits_a^{+\infty} f(x)g(x)dx}{\int\limits_a^{+\infty} g(x)dx}\le M$$

bo’lishi kelib chiqadi. Agar

$$\mu=\frac{\int\limits_a^{+\infty} f(x)g(x)dx}{\int\limits_a^{+\infty} g(x)dx}$$

deb olsak, unda

$$\int_a^{+\infty} f(x)g(x)dx = \mu \int_a^{+\infty} g(x)dx$$

bo'ladi.

$[a,+\infty)$ oraliqda $g(x) \le 0$ bo'lganda (15.6) formula xuddi shunga o'xshash isbotlanadi. ▶

Bu 6- xossa o'rta qiymat haqidagi teorema deb ham yuritiladi.

4^0. Xosmas integrallarni hisoblash. Ushbu

$$J = \int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi bo'lsin. uni hisoblash masalasini qaraymiz.

1) Nyuton-Leybnits formulasi. Faraz qilaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz bo'lsin. Ma'lumki, bu holda $f(x)$ funksiya shu oraliqda $\phi(x)$ $(\phi'(x) = f(x),\ x \in [a,+\infty))$ boshlang'ich funksiyaga ega bo'ladi. $x \to +\infty$ da $\phi(x)$ funksiyaning limiti mavjud va chekli bo'lsa, bu limitni $\phi(x)$ boshlang'ich funksiyaning $+\infty$ dagi qiymati deb qabul qilamiz, ya'ni

$$\lim_{x \to +\infty} \phi(x) = \phi(+\infty).$$

Xosmas integral ta'rifi hamda Nyuton-Leybnits formulasidan foydalanib quyidagini topamiz:

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^{t} f(x)dx = \lim_{t \to +\infty} \int_a^{t} [\phi(t) - \phi(a)] = \tag{15.8}$$

$$= \phi(+\infty) - \phi(a) = \phi(x)\Big|_a^{+\infty}$$

Bu esa yuqoridagi kelishuvga ko'ra boshlang'ich funksiyaga ega bo'lgan $f(x)$ funksiya xosmas integrali uchun Nyuton-Leybnits formulasi o'rinli bo'lishini ko'rsatadi.

15.4-misol. Ushbu

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2} \sin\frac{1}{x} dx$$

xosmas integral hisoblansin.

◀Ravshanki, $f(x) = \frac{1}{x^2} \sin\frac{1}{x}$ funksiya $\left[\frac{2}{\pi},+\infty\right)$ oraliqda uzluksiz bo'lib, uning boshlang'ich funksiyasi $\phi(x) = \cos\frac{1}{x}$ bo'ladi. Demak, (15.8) formulaga ko'ra

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2} \sin\frac{1}{x} dx = \cos\frac{1}{x}\Bigg|_{\frac{2}{\pi}}^{+\infty} = 1. ▶$$

Ba'zan berilgan $\int_a^{+\infty} f(x)dx$ xosmas integral o'zgaruvchilarni almashtirib yoki bo'laklab integrallash natijasida hisoblanadi.

2) Bo'laklab integrallash usuli. $u(x)$ va $v(x)$ funksiyalarning har biri $[a,+\infty)$ oraliqda berilgan hamda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin.

Agar $\int_a^{+\infty} v(x)du(x)$ integral yaqinlashuvchi hamda ushbu

$$\lim_{t\to+\infty} u(t)=u(+\infty), \quad \lim_{t\to+\infty} v(t)=v(+\infty)$$

limitlar mavjud va chekli bo'lsa, u holda $\int_a^{+\infty} u(x)dv(x)$ integral yaqinlashuvchi bo'lib,

$$\int_a^{+\infty} u(x)dv(x)=u(x)v(x)\Big|_a^{+\infty}-\int_a^{+\infty} v(x)du(x) \qquad \textit{(15.9)}$$

bo'ladi.

Haqiqatdan ham 1-qism, 9-bob, 10-§ da keltirilgan formulaga ko'ra

$$\int_a^{t} u(x)dv(x)=u(x)v(x)\Big|_a^{t}-\int_a^{t} v(x)du(x)=[u(t)v(t)-u(a)v(a)]- \\ -\int_a^{t} v(x)du(x) \qquad \textit{(15.10)}$$

bo'lib, bu tenglikda $t\to+\infty$ da limitga o'tib, quyidagini topamiz:

$$\lim_{t\to+\infty}\int_a^{t} u(x)dv(x)=\lim_{t\to+\infty}[u(t)v(t)-u(a)v(a)]-\lim_{t\to+\infty}\int_a^{t} v(x)du(x).$$

Shartga ko'ra $\int_a^{+\infty} v(x)du(x)$ integral yaqinlashuvchi hamda $\lim\limits_{t\to+\infty}[u(t)v(t)-u(a)v(a)]$ limit mavjud va chekli ekanligini e'tiborga olsak, unda (15.10) munosabatdan $\int_a^{+\infty} u(x)dv(x)$ integralning yaqinlashuvchiligi hamda (15.9) formulaning o'rinli bo'lishi kelib chiqadi.

15.5-misol. Qo'yidagi

$$\int_0^{+\infty} xe^{-x}dx$$

integralni hisoblansin.

◄Agar $u(x)=x$, $dv(x)=e^{-x}dx$ deyilsa, unda $u(x)v(x)\Big|_0^{+\infty}=x\left(e^{-x}\right)\Big|_0^{+\infty}=$ $=\lim\limits_{x\to+\infty}\left(-xe^{-x}\right)=0$, $\int_a^{+\infty} v(x)du(x)=-\int_0^{+\infty} e^{-x}dx=-1$ bo'lib, (15.9) formulaga ko'ra

$$\int_a^{+\infty} u(x)dv(x) = \int_0^{+\infty} xe^{-x}dx = -xe^{-x}\Big|_0^{+\infty} - \int_0^{+\infty}\left(-e^{-x}\right)dx = 1$$

bo'ladi. Demak,

$$\int_0^{+\infty} xe^{-x}dx = 1. \blacktriangleright$$

2-eslatma. Yuqoridagi (15.9) formulani keltirib chiqarishda $\int_a^{+\infty} v(x)du(x)$ integralning yaqinlashuvchiligi hamda $\lim_{t\to+\infty} u(t)v(t)$ limitning mavjud va chekli bo'lishi talab etiladi.

Agar $\int_a^{+\infty} u(x)dv(x)$, $\int_a^{+\infty} v(x)du(x)$ integrallarning yaqinlashuvchiligi hamda $\lim_{t\to+\infty} u(t)v(t)$ limitning mavjud va chekli bo'lishi kabi uchta faktdan istalgan ikkitasi o'rinli bo'lsa, u holda ularning uchinchisi hamda (15.9) formula o'rinli bo'ladi.

3) O'zgaruvchilarni almashtirish usuli. Quyidagi

$$J = \int_a^{+\infty} f(x)dx$$

integralni qaraylik. Bu integralda $x = \varphi(z)$ deylik, bunda $\varphi(z)$ funksiya quyidagi shartlarni bajarsin:

a) $\varphi(z)$ funksiya $[\alpha, +\infty)$ oraliqda berilgan, $\varphi'(z)$ hosilaga ega va bu hosila uzluksiz;

b) $\varphi(z)$ funksiya $[\alpha, +\infty)$ oraliqda qat'iy o'suvchi;

v) $\varphi(\alpha) = a$, $\varphi(+\infty) = \lim_{z\to+\infty} \varphi(z) = +\infty$ bo'lsin.

U holda $\int_a^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$ integral yaqinlashuvchi bo'lsa, unda $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi va

$$\int_a^{+\infty} f(x)dx = \int_\alpha^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz \qquad (15.11)$$

bo'ladi.

◀ Ixtiyoriy z $(\alpha < z < +\infty)$ nuqtani olib, unga mos $\varphi(z) = t$ nuqtani topamiz. $[a,\ t]$ oraliqda 1-qism, 9-bob, 2-§ da keltirilgan formulaga ko'ra

$$\int_a^t f(x)dx = \int_\alpha^z f(\varphi(z))\cdot\varphi'(z)dz$$

bo'ladi. Bu munosabatda $t \to +\infty$ da (bunga $z = \varphi^{-1}(t) \to +\infty$) limitga o'tib quyidagini topamiz:

$$\lim_{t\to+\infty}\int_a^t f(x)dx = \int_\alpha^z f(\varphi(z))\cdot\varphi'(z)dz.$$

Bu esa $\int\limits_a^{+\infty} f(x)dx$ integralning yaqinlashuvchiligini hamda (15.11) formulaning o'rinli bo'lishini ko'rsatadi. ▶

3-eslatma. $\int\limits_a^{+\infty} f(x)dx$ yaqinlashuvchi bo'lsin. bu integralda

$$x=\varphi(z)$$

bo'lib, bu funksiya yuqoridagi shartlarni bajarsin. U holda

$$\int\limits_a^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$$

integral ham yaqinlashuvchi bo'lib,

$$\int\limits_a^{+\infty} f(x)dx=\int\limits_\alpha^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$$

bo'ladi.

15.6-misol. Ushbu

$$J=\int\limits_0^{+\infty}\frac{dx}{1+x^4} \qquad \textit{(15.12)}$$

integral hisoblansin.

◀ Ravshanki, bu integral yaqinlashuvchi. Uni hisoblaylik. Avvalo bu integral $x=\frac{1}{z}$ almashish qilamiz. Natijada

$$J=\int\limits_{+\infty}^{0}\frac{1}{1+\frac{1}{z^4}}\left(-\frac{1}{z^2}\right)dz=\int\limits_0^{+\infty}\frac{z^2}{1+z^4}dz \qquad \textit{(15.13)}$$

bo'lib, (15.12) va (15.13) tengliklardan

$$J=\frac{1}{2}\int\limits_0^{+\infty}\frac{1+x^2}{1+x^4}dx$$

bo'lishi kelib chiqadi. Keyingi integralda

$$x=\frac{y+\sqrt{y^2+4}}{2} \qquad \left(x-\frac{1}{x}=y\right)$$

almashtirishni bajarib, quyidagini topamiz:

$$J=\frac{1}{2}\int\limits_{-\infty}^{+\infty}\frac{dy}{2+y^2}=\frac{1}{2\sqrt{2}}arctg\frac{y}{\sqrt{2}}\Bigg|_{-\infty}^{+\infty}=\frac{\pi}{2\sqrt{2}}.$$

Demak,

$$\int\limits_0^{+\infty}\frac{dx}{1+x^2}=\frac{\pi}{2\sqrt{2}}. \blacktriangleright$$

2-§. Chegaralanmagan funksiyaning xosmas integrallari

1^0. Funksiyaning maxsus nuqtasi. $X\subset R$ to'plamda berilgan $f(x)$ funksiya va $x_0\subset R$ nuqtaning ushbu

$$\overset{\bullet}{U}_{\delta}(x_0)=\{x\in R:\ x_0-\delta<x<x_0+\delta;\ x\neq x_0\}\ (\delta>0)$$

atrofini qaraylik.

4-ta'rif. Agar $f(x)$ funksiya $\overset{\bullet}{U}_{\delta}(x_0)\cap X\neq\theta$ to'plamda chegaralanmagan bo'lsa, x_0 nuqta $f(x)$ funksiyaning maxsus nuqtasi deyiladi.

Masalan,

1) $f(x)=\dfrac{1}{в-x}$ funksiya $(a\le x<в)$ uchun $x=в$ maxsus nuqta;

2) $f(x)=\dfrac{1}{x-a}$ funksiya $(a<x\le в)$ uchun $x=a$ maxsus nuqta;

3) $f(x)=\dfrac{1}{x(x^2-1)}$ funksiya $(x\in R\setminus\{-1,\ 0,\ 1\})$ uchun $x=-1$, $x=0$, $x=1$ maxsus nuqtalar bo'ladi.

2^0. Chegaralanmagan funksiyaning xosmas integrali tushunchasi. $f(x)$ funksiya $[a,\ в)$ yarim integralda berilgan bo'lib, $x=в$ nuqta uning maxsus nuqtasi bo'lsin. Bu funksiya $[a,\ в)$ yarim integralning istalgan $[a,\ t]$ qismida $(a<t<в)$ integrallanuvchi bo'lsin. Ravshanki,

$$\int_a^t f(x)dx$$

integral t o'zgaruvchiga bog'liq bo'ladi:

$$F(t)=\int_a^t f(x)dx.$$

5-ta'rif. Agar $t\to в-o$ da $F(x)$ funksiyaning limiti mavjud bo'lsa, bu limit chegaralanmagan $f(x)$ funksiyaning $[a,\ в)$ oraliq bo'yicha xosmas integrali deyiladi va

$$\int_a^в f(x)dx$$

kabi belgilanadi:

$$\int_a^в f(x)dx=\lim_{t\to в-o}F(x)=\lim_{t\to в-o}\int_a^t f(x)dx. \qquad (15.14)$$

Agar $t\to в-o$ da $F(x)$ funksiyaning limiti mavjud va chekli bo'lsa, (15.14) xosmas integral yaqinlashuvchi deyiladi.

Agar $t\to в-o$ da $F(x)$ funksiyaning limiti cheksiz yoki $F(x)$ funksiyaning limiti mavjud bo'lmasa, (15.14) xosmas integral uzoqlashuvchi deyiladi.

Masalan, ushbu

$$\int_0^1\frac{dx}{\sqrt{1-x^2}}$$

xosmas integral ($x=1$ maxsus nuqta) yaqinlashuvchi bo’ladi, chunki

$$\lim_{t\to 1-0}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1-0}\left(arcsin\,t\right)\Big|_0^t = \lim_{t\to 1-0} arcsin\,t = \frac{\pi}{2}$$

va demak,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

$(a,\ в]$ da berilgan $f(x)$ funksiyaning ($x=a$ maxsus nuqta), $(a,\ в)$ da berilgan $f(x)$ funksiyaning ($x=a$, $x=в$ maxsus nuqtalar) xosmas integrallari va ularning yaqinlashuvchiligi (uzoqlashuvchiligi) yuqoridagi kabi ta’riflanadi:

$$\int_a^в f(x)dx = \lim_{t\to a+o}\phi(t) = \lim_{t\to a+o}\int_t^в f(x)dx,$$

$$\int_a^в f(x)dx = \lim_{\substack{s\to в-o\\ t\to a+o}}\varphi(t,s) = \lim_{\substack{s\to в-o\\ t\to a+o}}\int_t^s f(x)dx.$$

Masalan, ushbu

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

xosmas integral ($x=o$ maxsus nuqta) yaqinlashuvchi bo’ladi, chunki

$$\lim_{t\to +o}\int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to +o} 2\left(1-\sqrt{t}\right) = 2$$

va demak,

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\,.$$

15.7-misol. Ushbu

$$J_1 = \int_a^в \frac{dx}{(x-a)^\alpha},\ J_2 = \int_a^в \frac{dx}{(в-x)^\alpha} \qquad (\alpha>0)$$

xosmas integrallar yaqinlashuvchilikka tekshirilsin.

◄Ta’rifga ko’ra

$$J_1 = \int_a^в \frac{dx}{(x-a)^\alpha} = \lim_{t\to a+o}\int_t^в \frac{dx}{(x-a)^\alpha}\,.$$

Aytaylik, $\alpha=1$ bo’lsin. Bu holda

$$\lim_{t\to a+o}\int_t^в \frac{dx}{x-a} = \lim_{t\to a+o}\left[ln(x-a)\right]_t^в = \infty$$

bo’ladi.

Aytaylik, $\alpha\neq 1$ bo’lsin. Bu holda

$$\lim_{t\to a+o}\int_t^{в}\frac{dx}{(x-a)^{\alpha}}=\lim_{t\to a+o}\left[\frac{(x-a)^{1-\alpha}}{1-\alpha}\right]_t^{в}=$$

$$=\lim_{t\to a+o}\frac{1}{1-\alpha}\left[(в-a)^{1-\alpha}-(t-x)^{1-\alpha}\right]$$

bo'ladi. Bu limit $0<\alpha<1$ bo'lganda chekli, $\alpha>1$ bo'lganda cheksiz bo'ladi.

Shunday qilib,

$$J_1=\int_a^{в}\frac{dx}{(x-a)^{\alpha}}, \qquad (\alpha>0)$$

xosmas integral $\alpha<1$ bo'lganda yaqinlashuvchi, $\alpha\geq 1$ bo'lganda uzoqlashuvchi bo'ladi.

Xuddi shunga o'xshash ko'rsatish mumkinki,

$$J_2=\int_a^{в}\frac{dx}{(в-x)^{\alpha}} \qquad (\alpha>0)$$

xosmas integral $\alpha<1$ bo'lganda yaqinlashuvchi, $\alpha\geq 1$ bo'lganda uzoqlashuvchi bo'ladi.

Chegaralanmagan funksiyaning xosmas integrali haqidagi bu tushunchalarni 1-§ da keltirilgan cheksiz oraliq bo'yicha xosmas integral tushunchalari bilan solishtirib, ularning o'xshashligini va bu xosmas integrallarni bitta nuqtai nazaridan, ya'ni

$$\int_a^{в} f(x)dx$$

integralda:

1) $f(x)$ funksiyaning $[a,\ в)$ oraliq berilgan bo'lib, bunda a - chekli nuqta, $в$ - chekli yoki $+\infty$;

2) $f(x)$ funksiya ixtiyori $[a,\ t]$ da integrallanuvchi, bunda $t\in[a,\ в)$ deb qarash mumkinligini ko'ramiz. Bu hol chegaralanmagan funksiyaning xosmas integrali haqidagi keyingi tushuncha va tasdiqlarni keltirish bilan kifoyalanish imkonini beradi.

3^0. Xosmas integrallarning yaqinlashuvchiligi. Integralning absolyut yaqinlashuvchiligi. Aytaylik, $f(x)$ funksiya $[a,\ в)$ oraliqda berilgan bo'lib, $в$ nuqta $f(x)$ funksiyaning maxsus nuqtasi bo'lsin.

6-teorema. Faraz qilaylik, $\forall x\in[a,\ в)$ da $f(x)\geq 0$ bo'lsin. Bu funksiyaning $[a,\ в)$ oraliq bo'yicha xosmas integrali

$$\int_a^{в} f(x)dx$$

ning yaqinlashuvchi bo'lishi uchun,

$$F(t)=\int_a^{t} f(x)dx \qquad (a\leq t<в)$$

funksiyaning yuqoridan chegaralangan, ya'ni

$$\exists C\in R,\quad \forall t\in[a,в):\quad \int_a^t f(x)dx\le C$$

bo’lishi zarur va etarli.

7-teorema. $f(x)$ va $g(x)$ funksiyalar $[a,\ в)$ oraliqda ($в$ maxsus nuqta) bo’lib, $\forall x\in[a,\ в)$ da

$$0\le f(x)\le g(x)$$

bo’lsin.

Agar $\int_a^в g(x)dx$ yaqinlashuvchi bo’lsa, $\int_a^в f(x)dx$ ham yaqinlashuvchi, agar $\int_a^в f(x)dx$ uzoqlashuvchi bo’lsa, $\int_a^в g(x)dx$ ham uzoqlashuvchi bo’ladi.

15.8-misol. Ushbu

$$\int_0^1\frac{\cos x}{\sqrt[4]{1-x}}dx$$

xosmas integral yaqinlashuvchilikka tekshirilsin.

◀Ravshanki, integral ostidagi funksiya

$$f(x)=\frac{\cos x}{\sqrt[4]{1-x}}\ge 0$$

bo’ladi. Ayni paytda, $\forall x\in[0,\ 1)$ da

$$f(x)=\frac{\cos x}{\sqrt[4]{1-x}}\le\frac{1}{\sqrt[4]{1-x}}=\frac{1}{(1-x)^{\frac{1}{4}}}$$

tengsizlik bajariladi. Ma’lumki, $g(x)=\frac{1}{(1-x)^{\frac{1}{4}}}$ funksiyaning integrali

$$\int_0^1\frac{1}{(1-x)^{\frac{1}{4}}}dx$$

yaqinlashuvchi. Unda 7-teoremaga ko’ra berilgan xosmas integral yaqinlashuvchi bo’ladi.▶

8-teorema. (Koshi teoremasi). Ushbu

$$\int_a^в f(x)dx\qquad (в \text{ maxsus nuqta})$$

xosmas integralning yaqinlashuvchi bo’lishi uchun

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall t',t'',\ в-\delta<t'<в,\ в-\delta<t''<в:$$

$$\left|\int_{t'}^{t''} f(x)dx\right|<\varepsilon$$

bo’lishi zarur va etarli.

Aytaylik, $f(x)$ funksiya $[a,\ в)$ oraliqda berilgan, $в$ nuqta funksiyaning maxsus nuqtasi bo’lib,

$$\int_a^в f(x)dx$$

uning xosmas integrali bo'lsin. Bu integral bilan bir qatorda

$$\int_a^в |f(x)|dx$$

xosmas integralni qaraymiz.

9-teorema. Agar

$$\int_a^в |f(x)|dx$$

integral yaqinlashuvchi bo'lsa, u holda

$$\int_a^в f(x)dx$$

integral ham yaqinlashuvchi bo'ladi.

6-ta'rif. Agar

$$\int_a^в |f(x)|dx$$

integral yaqinlashuvchi bo'lsa, $\int_a^в f(x)dx$ absolyut yaqinlashuvchi integral deyiladi.

Agar $\int_a^в f(x)dx$ yaqinlashuvchi bo'lib, $\int_a^в |f(x)|dx$ uzoqlashuvchi bo'lsa, $\int_a^в f(x)dx$ shartli yaqinlashuvchi integral deyiladi.

4^0. Yaqinlashuvchi xosmas integrallarning xossalari. Chegaralanmagan funksiyaning xosmas integrali ham cheksiz oraliq bo'yicha integrali xossalari kabi xossalarga ega. Ularni keltirishni hamda isbotlashni o'quvchiga havola etamiz.

5^0. Xosmas integrallarni hisoblash. $f(x)$ funksiya $[a,\, в)$ da berilgan, $в$ esa shu funksiyaning maxsus nuqtasi bo'lsin. Bu funksiyaning xosmas integrali

$$J = \int_a^в f(x)dx$$

yaqinlashuvchi, uni hisoblash talab etilsin.

1) ***Nyuton-Leybnits formulasi.*** Faraz qilaylik, $f(x)$ funksiya $[a,\, в)$ da uzluksiz bo'lsin. Ma'lumki, bu holda $f(x)$ funksiya shu oraliqda $\phi(x)$ $(\phi'(x) = f(x),\ x \in [\,a,\, в\,))$ boshlang'ich funksiyaga ega bo'ladi, $t \to в - o$ da $\phi(x)$ funksiyaning limiti mavjud va chekli bo'lsa, bu limitni $\phi(x)$ boshlang'ich funksiyaning $в$ nuqtasidagi qiymati deb qabul qilamiz:

$$\lim_{x \to в - o} \phi(x) = \phi(в).$$

Xosmas integral ta'rifi hamda Nyuton-Leybnits formulasidan foydalanib quyidagini topamiz:

$$\int_a^в f(x)dx = \lim_{t\to в-o}\int_a^t f(x)dx = \lim_{t\to в-o}(\phi(t)-\phi(a)) = \phi(в)-\phi(a) = \phi(x)\Big|_a^в .$$

Bu esa, yuqoridagi kelishuvga asosida, boshlang'ich funksiyaga ega bo'lgan funksiya xosmas integrali uchun Nyuton-Leybnits formulasi o'rinli bo'lishini ko'rsatadi.

Berilgan xosmas integral o'zgaruvchilarni almashtirib yoki bo'laklab integrallash natijasida hisoblanishi mumkin.

2) ***Bo'laklab integrallash usuli.*** $u(x)$ va $v(x)$ funksiyalarning har biri $[a,\ в)$ da berilgan bo'lib, shu oraliqda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. $в$ nuqta esa $v(x)\cdot u'(x)$ hamda $u(x)\cdot v'(x)$ funksiyalarning maxsus nuqtalari.

Agar $\int_a^в v(x)du(x)$ integral yaqinlashuvchi hamda ushbu

$$\lim_{t\to в-o} u(t)v(t)$$

limit mavjud va chekli bo'lsa, u holda $\int_a^в u(x)dv(x)$ integral yaqinlashuvchi bo'lib,

$$\int_a^в u(x)dv(x) = u(x)v(x)\Big|_a^в - \int_a^в v(x)du(x) \qquad (15.15)$$

bo'ladi, bunda

$$u(в)v(в) = \lim_{t\to в-o} u(t)v(t)$$

15.9-misol. Ushbu

$$\int_0^1 \frac{(x+1)dx}{\sqrt[3]{(x-1)^2}}$$

integralni qaraylik. Agar $u(x) = x+1,\ dv(x) = \dfrac{1}{\sqrt[3]{(x-1)^2}}dx$ deb olsak, unda

$$u(x)\cdot v(x)\Big|_0^1 = (x+1)3(x-1)^{\frac{1}{3}}\Big|_0^1 = 3,$$

$$\int_0^1 v(x)du(x) = \int_0^1 3(x-1)^{\frac{1}{3}}dx = \frac{9}{4}(x-1)^{\frac{4}{3}}\Big|_0^1 = -\frac{9}{4}$$

bo'lib, (15.15) formulaga ko'ra

$$\int_0^1 v(x)du(x) = \int_0^1 \frac{(x+1)dx}{\sqrt[3]{(x-1)^2}} = 3 - \left(-\frac{9}{4}\right) = \frac{21}{4}$$

bo'ladi. Demak,

$$\int_0^1 \frac{(x+1)dx}{\sqrt[3]{(x-1)^2}} = \frac{21}{4}$$

3-eslatma. Yuqoridagi (15.15) formulani keltirib chiqarishda $\int_{a}^{в} v(x)du(x)$ integralning yaqinlashuvchiligi hamda $\lim\limits_{t\to в-o}[u(t)\cdot v(t)]$ limitning mavjud va chekli bo'lishi talab etiladi.

Agar $\int_{a}^{в} u(x)dv(x)$, $\int_{a}^{в} v(x)du(x)$ integrallarning yaqinlashuvchiligi hamda $\lim\limits_{t\to в-o}[u(t)\cdot v(t)]$ limitning mavjud va chekli bo'lishi kabi uchta faktdan istalgan ikkitasi o'rinli bo'lsa, unda ularning uchinchisi hamda (15.15) formula o'rinli bo'ladi.

3) ***O'zgaruvchilarni almashtirish usuli***. $f(x)$ funksiya $[a,\, в)$ da berilgan, $в$ esa shu funksiyaning maxsus nuqtasi bo'lsin. Quyidagi

$$\int_{a}^{в} f(x)dx$$

xosmas integralni qaraylik. Bu integralda $x=\varphi(z)$ deylik, bunda $\varphi(z)$ funksiya $[\alpha,\, \beta)$ oraliqda $\varphi'(z)>0$ hosilaga ega va u uzluksiz hamda $\varphi(\alpha)=a$, $\varphi(\beta)=в$, $\left(\varphi(\beta)=\lim\limits_{z\to в-o}\varphi(z)\right)$. Agar $\int_{\alpha}^{\beta} f(\varphi(z))\cdot\varphi'(z)dz$ integral yaqinlashuvchi bo'lsa, u holda $\int_{a}^{в} f(x)dx$ integral ham yaqinlashuvchi bo'lib,

$$\int_{a}^{в} f(x)dx=\int_{\alpha}^{\beta} f(\varphi(z))\cdot\varphi'(z)dz$$

bo'ladi.

4-eslatma. Aytaylik, $\int_{a}^{в} f(x)dx$ integral yaqinlashuvchi bo'lsin. Bu integralda $x=\varphi(x)$ bo'lib, u yuqoridagi shartlarni bajarsin. U holda $\int_{\alpha}^{\beta} f(\varphi(z))\cdot\varphi'(z)dz$ integral ham yaqinlashuvchi bo'lib,

$$\int_{a}^{в} f(x)dx=\int_{\alpha}^{\beta} f(\varphi(z))\cdot\varphi'(z)dz$$

bo'ladi.

15.10-misol. Ushbu

$$J=\int_{0}^{1}\frac{dx}{(1+x)\sqrt{x}}$$

integralda $x=\varphi(z)=z^2$ almashtirish bajaramiz. Ravshanki, bu $x=z^2$ funksiya $(0,\ 1]$ oraliqda $x'=2z>0$ hosilaga ega va u uzluksiz hamda $\varphi(0)=0$, $\varphi(1)=1$. Integralni hisoblaymiz:

$$J=\int_0^1\frac{2dz}{1+z^2}=2arctgz\Big|_0^1=2\cdot\frac{\pi}{4}=\frac{\pi}{2}.$$

3-§. Muhim misollar

Ushbu paragrafda ikkita xosmas integrallarning yaqinlashuvchilikka tekshiramiz. Bu integrallar kelgusida juda ahamiyatli bo'lib, ulardan ko'p masalalarni echishda foydalaniladi.

15.11-misol. Ushbu

$$J_1=\int_0^{+\infty}x^{a-1}e^{-x}dx$$

xosmas integral yaqinlashuvchilikka tekshirilsin.

◀Ravshanki, J_1 cheksiz oraliq bo'yicha xosmas integral. Ayni paytda, $a<1$ bo'lganda $x=0$ nuqta integral ostidagi funksiyaning maxsus nuqtasi bo'lgani sabali J_1 chegaralanmagan funksiyaning xosmas integrali ham bo'ladi.

Bu integralni quyidagicha

$$\int_0^{+\infty}x^{a-1}e^{-x}dx=\int_0^1 x^{a-1}e^{-x}dx+\int_1^{+\infty}x^{a-1}e^{-x}dx$$

yozib olamiz. So'ng tenglikning o'ng tomonidagi integrallarning har birini alohida-alohida yaqinlashuvchilikka tekshiramiz. Integrallarning birinchisi

$$\int_0^1 x^{a-1}e^{-x}dx$$

da integral ostidagi funksiya uchun

$$\frac{1}{e}\,\frac{1}{x^{1-a}}\le x^{a-1}e^{-x}\le\frac{1}{x^{1-a}}\quad(0<x\le 1)$$

tengsizliklar o'rinli bo'ladi. Ma'lumki,

$$\int_0^1\frac{dx}{x^{1-a}}$$

integral $1-a<a$ ya'ni $a>0$ da yaqinlashuvchi bo'ladi. Unda 7-teoremadan foydalanib

$$\int_0^1 x^{a-1}e^{-x}dx$$

ning $a>0$ da yaqinlashuvchi bo'lishini topamiz.

Ravshanki,

$$\lim_{x\to+\infty}\frac{x^{a-1}e^{-x}}{\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{x^{a+1}}{e^x}=0.$$

Bu holda

$$\int_1^{+\infty}x^{a-1}e^{-x}dx,\quad\int_1^{+\infty}\frac{dx}{x^2}$$

xosmas integrallar bir vaqtda yo yaqinlashuvchi yoki uzoqlashuvchi bo'ladi. Ma'lumki,

$$\int_{1}^{+\infty} \frac{dx}{x^2}$$

yaqinlashuvchi. Demak,

$$\int_{1}^{+\infty} x^{a-1} e^{-x} dx$$

xosmas integral ixtiyoriy a jumladan $a > 0$ da yaqinlashuvi bo'ladi.

Shunday qilib, berilgan

$$\int_{0}^{+\infty} x^{a-1} e^{-x} dx$$

xosmas integral $a > 0$ bo'lganda yaqinlashuvchi bo'ladi. ▶

15.12-misol. Ushbu

$$J_2 = \int_{0}^{1} x^{a-1} (1-x)^{в-1} dx$$

xosmas integral yaqinlashuchilikka tekshirilsin.

◀ Integral ostidagi funksiya uchun

1) $a < 1$, $в \geq 1$ bo'lganda $x = 0$ maxsus nuqta;

2) $a \geq 1$, $в < 1$ bo'lganda $x = 1$ maxsus nuqta;

3) $a < 1$, $в < 1$ bo'lganda $x = 0$, $x = 1$ maxsus nuqtalar bo'ladi.

Berilgan xosmas integralni yaqinlashuvchilikka tekshirish uchun uni quyidagicha yozib olamiz:

$$\int_{0}^{1} x^{a-1} (1-x)^{в-1} dx = \int_{0}^{\frac{1}{2}} x^{a-1} (1-x)^{в-1} dx + \int_{\frac{1}{2}}^{1} x^{a-1} (1-x)^{в-1} dx .$$

Ravshanki,

$$\lim_{x \to 0} (1-x)^{в-1} = 1, \quad \lim_{x \to 1} x^{a-1} = 1.$$

U holda

$$\lim_{x \to 0} \frac{x^{a-1} (1-x)^{в-1}}{x^{a-1}} = 1, \quad \lim_{x \to 1} \frac{x^{a-1} (1-x)^{в-1}}{(1-x)^{в-1}} = 1$$

bo'lib, 9-teoremaga ko'ra

$$\int_{0}^{1/2} x^{a-1} (1-x)^{в-1} dx \text{ bilan } \int_{0}^{1/2} x^{a-1} dx ,$$

hamda

$$\int_{1/2}^{1} x^{a-1} (1-x)^{в-1} dx \text{ bilan } \int_{1/2}^{1} (1-x)^{в-1} dx$$

xosmas integral lar bir vaqtda yaqinlashuvchi bo'ladi, yoki uzoqlashuvchi bo'ladi.

Ma'lumki, $a > 0$ bo'lganda

$$\int_0^{1/2} x^{a-1}dx$$

xosmas integral yaqinlashuvchi, $в > 0$ bo'lganda

$$\int_{1/2}^{1} (1-x)^{в-1}dx$$

xosmas integral yaqinlashuvchi. Demak,

$a > 0$ bo'lganda $\int_0^{1/2} x^{a-1}(1-x)^{в-1}dx$ integral,

$в > 0$ bo'lganda $\int_{1/2}^{1} x^{a-1}(1-x)^{в-1}dx$ integral yaqinlashuvchi bo'ladi.

Shunday qilib,

$$\int_0^1 x^{a-1}(1-x)^{в-1}dx$$

xosmas integral $a > 0$, $в > 0$ bo'lganda yaqinlashuvchi.

Mashqlar

15.13. Ushbu

$$J = \int_{-\infty}^{+\infty} \frac{dx}{x^2 + x + 1}$$

xosmas integralning yaqinlashuvchiligi ko'rsatilsin, qiymati topilsin.

15.14. Ushbu

$$\int_{-\infty}^{+\infty} f(x)dx$$

integral uchun yaqinlashuvchilik teoremalari keltirilsin.

15.15. Aytaylik, $\forall x \in [a, +\infty)$ da $f(x) \geq 0$, $g(x) \geq 0$ funksiyalar uchun $\lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = k$ bo'lsin. Agar:

1) $k < +\infty$ va $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi;

2) $k > 0$ va $\int_a^{+\infty} g(x)dx$ uzoqlashuvchi bo'lsa, $\int_a^{+\infty} f(x)dx$ ham uzoqlashuvchi bo'lishi isbotlansin.

15.16. Yuqoridagi 15.15-masala shartida $0<k<+\infty$ bo'lganda $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ xosmas integrallar bir vaqtda yoki yaqinlashuvchi yoki uzoqlashuvchi bo'lishi isbotlansin.

15.17. Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lib, uning ixtiyoriy $[a,\ t]$ qismida $(a<t<+\infty)$ integrallanuvchi bo'lsin. Agar $x\to+\infty$ da

$$f(x)\sim\frac{A}{x^{\alpha}}\qquad (A\neq 0)$$

bo'lsa, u holda

$$\int_a^{+\infty} f(x)dx$$

xosmas integral $\alpha>1$ bo'lganda yaqinlashuvchi, $\alpha\le 1$ bo'lganda uzoqlashuvchi bo'lishi ko'rsatilsin.

15.18. Ushbu

$$\int_1^{+\infty}\frac{\sin x}{x^{\alpha}}dx$$

xosmas integral

a) $\alpha>1$ bo'lganda absolyut yaqinlashuvchi;

b) $0<\alpha\le 1$ bo'lganda shartli yaqinlashuvchi;

v) $\alpha\le 0$ bo'lganda uzoqlashuvchi bo'lishi isbotlansin.

15.19. Agar $f(x)$ funksiya $(-\infty,+\infty)$ oraliqda berilgan bo'lib,

$$\lim_{t\to+\infty}\int_{-t}^{t} f(x)dx\qquad (*)$$

mavjud bo'lsa, u holda $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integralning yaqinlashuvchi ham bo'lishi, uzoqlashuvchi ham bo'lishi ko'rsatilsin.

(Odatda (*) limit chekli bo'lganda $\int_{-\infty}^{+\infty} f(x)dx$ xosmas integral bosh qiymat ma'nosida yaqinlashuvchi deyilib, $V.\rho.\int_{-\infty}^{+\infty} f(x)dx$ kabi belgilanadi).

16-BOB

Parametrga bog'liq integrallar

Ushbu bobda ko'p o'zgaruvchili funksiyaning bitta o'zgaruvchisi bo'yicha integralini qaraymiz.

$f(x_1, x_2, ..., x_m)$ funksiya biror $M \subset R^m$ to'plamda berilgan bo'lsin. Bu funksiyaning bitta x_k $(k = 1, 2, ..., m)$ o'zgaruvchisidan boshqa barcha o'zgaruvchilarini o'zgarmas deb hisoblasak, $f(x_1, x_2, ..., x_m)$ funksiya bitta x_k o'zgaruvchiga bog'liq bo'lgan funksiyaga aylanadi. Uning shu o'zgaruvchi bo'yicha integrali (agar u mavjud bo'lsa), ravshanki, $x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_m$ larga bog'liq bo'ladi. Bunday integrallar parametrga bog'liq integrallar tushunchasiga olib keladi.

Soddalik uchun ikki o'zgaruvchili $f(x,y)$ funksiyaning bitta o'zgaruvchi bo'yicha integralini o'rganamiz. Bunda $f(x,y)$ funksiyaning y o'zgaruvchisi bo'yicha limiti va unga intilishi xarakteri muhim rol o'ynaydi.

1-§. Limit funksiya. Tekis yaqinlashish. Limit funksiyaning uzluksizligi

$f(x,y)$ funksiya $M = \{(x,y) \in R^2 : a \le x \le в,\ y \in E \subset R\}$ to'plamda berilgan, y_0 esa $E \subset R$ to'plamning limit nuqtasi bo'lsin.

x o'zgaruvchining $[a, в]$ oraliqdan olingan har bir tayin qiymatida $f(x,y)$ funksiya y ninggina funksiyasiga aylanadi. Agar $y \to y_0$ da bu funksiyaning limiti mavjud bo'lsa, ravshanki, y limit x o'zgaruvchining $[a, в]$ oraliqdan olingan qiymatiga bog'liq bo'ladi:

$$\lim_{y \to y_0} f(x,y) = \varphi(x, y_0) = \varphi(x).$$

Bu $\varphi(x)$ funksiya $f(x,y)$ funksiyaning $y \to y_0$ dagi limit funksiyasi deyiladi. Bu quyidagini anglatadi: $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, x) > 0$, $|y - y_0| < \delta$ bo'lgan $\forall y \in E :\ |f(x,y) - \varphi(x)| < \varepsilon$.

1-ta'rif. M to'plamda berilgan $f(x,y)$ funksiyaning $y \to y_0$ dagi limit funksiyasi $\varphi(x)$ bo'lsin. $\forall \varepsilon > 0$ olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ topilsaki, $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $\forall y \in E$ va $\forall x \in [a, в]$ uchun

$$|f(x,y) - \varphi(x)| < \varepsilon$$

bo'lsa, $f(x,y)$ funksiya limit funksiya $\varphi(x)$ ga $[a, в]$ da tekis yaqinlashadi deyiladi.

Aks holda, ya'ni $\forall \delta > 0$ olinganda ham shunday $\varepsilon_0 > 0$, $x_0 \in [a, в]$ va $|y_1 - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $y_1 \in E$ topilsaki, ushbu

$$|f(x_0, y_1) - \varphi(x_0)| \ge \varepsilon_0$$

tengsizlik o'rinli bo'lsa, $f(x,y)$ funksiya $\varphi(x)$ ga notekis yaqinlashadi deyiladi.

16.1-misol. $M = \{(x,y) \in R^2 :\ 0 \le x \le 1,\ 0 < y \le \pi\}$ to'plamda berilgan ushbu

$$f(x,y) = x \sin y$$

funksiyaning $y_0 = \dfrac{\pi}{3}$ nuqtada limit funksiyasi topilsin va unga tekis yaqinlashish ko'rsatilsin.

◀Ravshanki, $y \to y_0 = \frac{\pi}{3}$ bo'lganda $f(x,y) = x\sin y$ funksiyaning limiti $\frac{\sqrt{3}}{2}x$ ga teng bo'ladi. Demak, $\varphi(x) = \frac{\sqrt{3}}{2}x$.

$\forall \varepsilon > 0$ sonni olaylik. Agar $\delta = \varepsilon$ desak, u holda $\left| y - \frac{\pi}{3} \right| < \delta$ tengsizlikni qanoatlantirgan $\forall y$ va $\forall x \in [0,1]$ uchun

$$|f(x,y) - \varphi(x)| = \left| x\sin y - \frac{\sqrt{3}}{2}x \right| = |x| \left| \sin y - \frac{\sqrt{3}}{2} \right| = |x| \left| \sin y - \sin\frac{\pi}{3} \right| \le \left| y - \frac{\pi}{3} \right| < \varepsilon$$

tengsizliklar bajariladi. 1-ta'rifga ko'ra $y \to \frac{\pi}{3}$ da berilgan $f(x,y) = x\sin y$ funksiya limit funksiya $\varphi(x) = \frac{\sqrt{3}}{2}x$ ga tekis yaqinlashadi.▶

Endi $f(x,y)$ funksiyaning limit funksiyaga ega bo'lishi va unga tekis yaqinlashishi haqidagi teoremani keltiramiz.

$f(x,y)$ funksiya $M = \{(x,y) \in R^2 : a \le x \le в,\ y \in E\}$ to'plamda berilgan bo'lib, y_0 esa $E \subset R$ to'plamning limit nuqtasi bo'lsin.

1-teorema. $f(x,y)$ funksiya $y \to y_0$ da limit funksiya $\varphi(x)$ ga ega bo'lishi va unga tekis yaqinlashishi uchun $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi $\forall y, y' \in E$ hamda $\forall x \in [a, в]$ uchun

$$|f(x,y) - f(x,y')| < \varepsilon \qquad (16.1)$$

tengsizlikning bajarilishi zarur va etarli.

◀***Zarurligi.*** $f(x,y)$ funksiya $y \to y_0$ da $\varphi(x)$ limit funksiyaga ega bo'lib, unga $[a, в]$ da tekis yaqinlashsin. Ta'rifga ko'ra, $\forall \varepsilon > 0$ olinganda ham, $\frac{\varepsilon}{2}$ ga ko'ra shunday $\delta = \delta(\varepsilon) > 0$ topilpdiki, $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi $\forall y \in E$ hamda $\forall x \in [a, в]$ uchun $|f(x,y) - \varphi(x)| < \frac{\varepsilon}{2}$ bo'ladi. Jumladan $|y' - y_0| < \delta \Rightarrow |f(x,y') - \varphi(x)| < \frac{\varepsilon}{2}$ bo'ladi. Natijada

$$|f(x,y) - f(x,y')| \le |f(x,y) - \varphi(x)| + |f(x,y') - \varphi(x)| < \varepsilon$$

bo'lib, undan (16.1) shartning bajarilishini topamiz.

Etarliligi. Teoremadagi (16.1) shart bajarilsin. U holda x o'zgaruvchining $[a, в]$ oraliqda olingan har bir tayin qiymatida $f(x,y)$ funksiya y o'zgaruvchininggina funksiyasi bo'lib, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi $\forall y, y' \in E$ uchun

$$|f(x,y)-f(x,y')|<\varepsilon \qquad (16.2)$$

bo'ladi. Funksiya limitining mavjudligi haqidagi Koshi teoremasiga asosan (qaralsin, 1-qism, 4-bob, 6-§) $y\to y_0$ da $f(x,y)$ funksiya limitga ega bo'ladi. Ravshanki, bu limit tayinlangan x $(x\in[a,в])$ ga bog'liq. Demak,

$$\lim_{y\to y_0} f(x,y)=\varphi(x).$$

Shu bilan $y\to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ limit funksiyaga ega bo'lishi ko'rsatildi.

Endi y o'zgaruvchini $|y-y_0|<\delta$ tengsizlikni qanoatlantiradigan qiymatda tayinlab, (16.2) tengsizlikda $y'\to y_0$ da limitga o'tsak, u holda

$$|f(x,y)-\varphi(x)|\le\varepsilon$$

hosil bo'ladi. Bu esa $y\to y_0$ da $f(x,y)$ funksiyaning $\varphi(x)$ limit funksiyaga $[a,в]$ da tekis yaqinlashishini bildiradi.▶

Endi limit funksiyaning uzluksizligi haqidagi teoremani keltiraylik. Bu teoremadan kelgusida ko'p foydalanamiz.

2-teorema. Agar $f(x,y)$ funksiya y o'zgaruvchining E to'plamdan olingan har bir qiymatida, x o'zgaruvchining funksiyasi sifatida, $[a,в]$ oraliqda uzluksiz bo'lsa va $y\to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ limit funksiyaga $[a,в]$ da tekis yaqinlashsa, u holda $\varphi(x)$ funksiya $[a,в]$ da uzluksiz bo'ladi.

◀ y_0 ga intiladigan $\{y_n\}$ ketma-ketlikni olaylik $(y_n\in E,\ n=1,2,...)$. Shartga ko'ra har bir y_n, $(n=1,2,...)$ da $f(x,y_n)$ funksiya x o'zgaruvchining $[a,в]$ oraliqdagi uzluksiz funksiyasi bo'ladi. Demak, $\{f(x,y_n)\}$ funksional ketma-ketlikning har bir hadi $[a,в]$ oraliqda uzluksiz.

Teoremaning ikkinchi shartiga ko'ra $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $\forall x\in[a,в]$ uchun

$$|y-y_0|<\delta\Rightarrow|f(x,y)-\varphi(x)|<\varepsilon \quad (y\in E) \qquad (16.3)$$

bo'ladi.

$y_n\to y_0$ dan yuqorida olingan $\delta=\delta(\varepsilon)>0$ ga ko'ra shunday $n_0\in N$ topiladiki, $\forall n>n_0$ uchun $|y_n-y_0|<\delta$ bo'ladi. U holda, (16.3) ga asosan, $\forall\varepsilon>0$ olinganda ham shunday $n_0\in N$ topiladiki, $\forall n>n_0$ va $\forall x\in[a,в]$ uchun

$$|f(x,y_n)-\varphi(x)|<\varepsilon$$

bo'ladi. Bu esa $\{f(x,y_n)\}$ funksional ketma-ketlik $\varphi(x)$ ga $[a,в]$ da tekis yaqinlashuvchiligini bildiradi. 14-bob, 3-§ da keltirilgan 6-teoremaga asosan $\varphi(x)$ funksiya $[a,в]$ oraliqda uzluksizdir.▶

2-§. Parametrga bog'liq integrallar

$f(x,y)$ funksiya

$$M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in E\subset R\}$$

to'plamda berilgan bo'lib, y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida $f(x,y)$ x o'zgaruvchining funksiyasi sifatida $[a,в]$ oraliqda integrallanuvchi, ya'ni

$$\int_a^в f(x,y)dx$$

integral mavjud bo'lsin. Ravshanki, bu integralning qiymati olingan y ga bog'liq bo'ladi:

$$\Phi(y)=\int_a^в f(x,y)dx\,. \qquad (16.4)$$

Odatda (16.4) parametrga bog'liq integral, u o'zgaruvchi esa parametr deyiladi.

Ushbu paragrafda parametrga bog'liq (16.4) integralning ($\Phi(y)$-funksiyaning) funksional xossalarini o'rganamiz.

1^0. Integral belgisi ostida limitga o'tish. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 :$ $: x\in[a,в],\ y\in E\subset R\}$to'plamda berilgan bo'lib, y_0 nuqta E to'plamning limit nuqtasi bo'lsin.

3-teorema. $f(x,y)$ funksiya y ning E to'plamdan olingan har bir tayin qiymatida x ning funksiyasi sifatida $[a,в]$ oraliqda uzluksiz bo'lsin. Agar $f(x,y)$ funksiya $y\to y_0$ da $\varphi(x)$ limit funksiyaga ega bo'lsa va unga tekis yaqinlashsa, u holda

$$\lim_{y\to y_0}\int_a^в f(x,y)dx=\int_a^в \varphi(x)dx \qquad (16.5)$$

bo'ladi.

◄Shartga ko'ra $f(x,y)$ funksiya $y\to y_0$ da $\varphi(x)$ limit funksiyaga ega va unga tekis yaqinlashadi. Demak, $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $|y-y_0|<\delta$ ni qanoatlantiruvchi $\forall y\in E$ va $\forall x\in[a,в]$ uchun

$$|f(x,y)-\varphi(x)|<\frac{\varepsilon}{в-a}$$

bo'ladi.

Ikkinchi tomondan, 2-teoremaga asosan, $\varphi(x)$ funksiya $[a,в]$ oraliqda uzluksiz bo'ladi. Demak, bu funksiyaning integrali $\int_a^в \varphi(x)dx$ mavjud.

Natijada

$$\left|\int_a^в f(x,y)dx-\int_a^в \varphi(x)dx\right|\le\int_a^в |f(x,y)-\varphi(x)|dx<\frac{\varepsilon}{в-a}\int_a^в dx=\varepsilon$$

bo'lib, undan

$$\lim_{y\to y_0}\int_a^в f(x,y)dx=\int_a^в \varphi(x)dx$$

ekanligi kelib chiqadi.►

(16.5) munosabatni quyidagicha

$$\lim_{y\to y_0}\int_a^в f(x,y)dx=\int_a^в\left(\lim_{y\to y_0} f(x,y)\right)dx$$

ham yozish mumkin. Bu esa integral belgisi ostida limitga o'tish mumkinligini ko'rsatadi.

2^0. Integralning parametr bo'yicha uzluksizligi.

4-teorema. Agar $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in[c,d]\}$$

to'plamda uzluksiz bo'lsa, u holda

$$\Phi(y)=\int_a^в f(x,y)dx$$

funksiya $[c,d]$ oraliqda uzluksiz bo'ladi.

◀Ixtiyoriy $y_0\in[c,d]$ nuqtani olaylik. Shartga ko'ra $f(x,y)$ funksiya M to'plamda (to'g'ri to'rtburchakda) uzluksiz. Kantor teoremasiga ko'ra bu funksiya M to'plamda tekis uzluksiz bo'ladi. Unda $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki,

$$\rho((x,y),(x,y_0))=|y-y_0|<\delta$$

tengsizlikni qanoatlantiruvchi $\forall(x,y)\in M$, $\forall(x,y_0)\in M$ uchun

$$|f(x,y)-f(x,y_0)|<\varepsilon$$

bo'ladi. Bu esa $f(x,y)$ funksiya $y\to y_0$ da $f(x,y_0)$ limit funksiyaga tekis yaqinlashishini bildiradi. U holda 3-teoremaga asosan

$$\lim_{y\to y_0}\Phi(y)=\lim_{y\to y_0}\int_a^в f(x,y)dx=\int_a^в\left(\lim_{y\to y_0} f(x,y)\right)dx=\int_a^в f(x,y_0)dx=\Phi(y_0)$$
$$(\forall y_0\in[c,d])$$

bo'ladi. Demak, $\Phi(y)$ funksiya y_0 nuqtada uzluksiz.▶

3^0. Integralni parametr bo'yicha differensiallash. Endi parametrga bog'liq integralni parametr bo'yicha differensiallashni qaraymiz.

5-teorema. $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in[c,d]\}$$

to'plamda berilgan va y o'zgaruvchining $[c,d]$ oraliqdan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a,в]$ oraliqda uzluksiz bo'lsin. Agar $f(x,y)$ funksiya M to'plamda $f'_y(x,y)$ xususiy hosilaga ega bo'lib, u uzluksiz bo'lsa, u holda $\Phi(y)$ funksiya ham $[c,d]$ oraliqda $\Phi'(y)$ hosilaga ega va ushbu

$$\Phi'(y)=\int_a^в f'_y(x,y)dx$$

munosabat o'rinlidir.

◀Shartga ko'ra $f(x,y)$ funksiya x o'zgaruvchisi bo'yicha $[a,в]$ oraliqda uzluksiz. Binobarin,

$$\Phi(y)=\int_a^{в} f(x,y)dx$$

integral mavjud.

Endi $\forall y_0\in[c,d]$ nuqtani olib, unga shunday $\Delta y(\Delta y\geq 0)$ orttirma beraylikki, $y_0+\Delta y\in[c,d]$ bo'lsin. $\Phi(y)$ funksiyaning y_0 nuqtadagi orttirmasini topib, ushbu

$$\frac{\Phi(y_0+\Delta y)-\Phi(y_0)}{\Delta y}=\int_a^{в}\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}dx$$

tenglikni hosil qilamiz. Lagranj teoremasi (1-qism, 6-bob, 6-§) ga ko'ra

$$\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}=f'_y(x,y_0+\theta\Delta y)$$

bo'ladi, bunda $0<\theta<1$.

Natijada

$$\frac{\Phi(y_0+\Delta y)-\Phi(y_0)}{\Delta y}=\int_a^{в} f'_y(x,y_0+\theta\Delta y)dx=\int_a^{в} f'_y(x,y_0)dx+$$

$$+\int_a^{в}\left[f'_y(x,y_0+\theta\Delta y)-f'_y(x,y_0)\right]dx$$

bo'lib, undan esa

$$\left|\frac{\Phi(y_0+\Delta y)-\Phi(y_0)}{\Delta y}-\int_a^{в} f'_y(x,y_0)dx\right|\leq\int_a^{в}\left|f'_y(x,y_0+\theta\Delta y)-f'_y(x,y_0)\right|dx\leq$$

$$\leq\int_a^{в}\omega(f'_y,\Delta y)dx=\omega(f'_y,\Delta y)\cdot(в-a) \qquad (16.7)$$

bo'lishini topamiz, bunda $\omega(f'_y,\Delta y)-f'_y(x,y)$ funksiyaning uzluksizlik moduli.

Modomiki, $f'_y(x,y)$ funksiya M to'plamda uzluksiz ekan, unda Kantor teoremasiga ko'ra bu funksiya shu to'plamda tekis uzluksiz bo'ladi. U holda mazkur kursning 12-bob, 4-§ida keltirilgan teoremaga asosan

$$\lim_{\Delta y\to 0}\omega(f'_y,\Delta y)=0$$

bo'ladi.

(16.7) munosabatdan

$$\lim_{\Delta y\to 0}\frac{\Phi(y_0+\Delta y)-\Phi(y_0)}{\Delta y}=\int_a^{в} f'_y(x,y_0)dx$$

bo'lishi kelib chiqadi. Demak,

$$\Phi'(y_0)=\int_a^{в} f'_y(x,y_0)dx.$$

Qaralayotgan y_0 nuqta $[c,d]$ oraliqning ixtiyoriy nuqtasi bo'lganligini e'tiborga olsak, unda keyingi tenglik teoremaning isbotlanganligini ko'rsatadi. ►

(16.6) munosabatni quyidagicha ham yozish mumkin:

$$\frac{d}{dy}\int_a^в f(x,y)dx=\int_a^в\left(\frac{d}{dy}f(x,y)\right)dx$$

Bu esa differensiallash amalini integral belgisi ostida o'tkazish mumkinligini ko'rsatadi.

Isbot etilgan bu 5-teorema Leybnits qoidasi deb ataladi.

4^0. Integralni parametr bo'yicha integrallash. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:$ $: x\in[a,в],\ y\in[c,d]\}$ to'plamda berilgan va shu to'plamda uzluksiz bo'lsin. U holda 4-teoremaga ko'ra

$$\Phi(y)=\int_a^в f(x,y)dx$$

funksiya $[c,d]$ oraliqda uzluksiz bo'ladi. Binobarin bu funksiyaning $[c,d]$ oraliq bo'yicha integrali mavjud.

Demak, $f(x,y)$ funksiya M to'plamda uzluksiz bo'lsa, u holda parametrga bog'liq integralni parametr bo'yicha $[c,d]$ oraliqda integrallash mumkin:

$$\int_c^d \Phi(y)dy=\int_c^d\left(\int_a^в f(x,y)dx\right)dy\,.$$

Bu tenglikning o'ng tomonida $f(x,y)$ funksiyani avval x o'zgaruvchi bo'yicha $[a,в]$ oraliqda integrallab (bunda y ni o'zgarmas hisoblab), so'ng natijani $[c,d]$ oraliqda integrallanadi.

Ba'zan $f(x,y)$ funksiya M to'plamda uzluksiz bo'lgan holda bu funksiyani avval y o'zgaruvchisi bo'yicha $[c,d]$ oraliqda integrallab (bunda x ni o'zgarmas hisoblab), so'ng hosil bo'lgan x o'zgaruvchining funksiyasini $[a,в]$ oraliqda integrallash qulay bo'ladi. Natijada ushbu

$$\int_c^d\left(\int_a^в f(x,y)dx\right)dy,\quad \int_a^в\left(\int_c^d f(x,y)dy\right)dx$$

integrallar hosil bo'ladi. Bu integrallar bir-biriga teng bo'ladimi degan savol tug'iladi. Bu savolga quyidagi teorema javob beradi.

6-teorema. Agar $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in[c,d]\}$ to'plamda uzluksiz bo'lsa, u holda

$$\int_c^d\left(\int_a^в f(x,y)dx\right)dy=\int_a^в\left(\int_c^d f(x,y)dy\right)dx$$

bo'ladi.

◄ $\forall t\in[c,d]$ nuqtani olib, quyidagi

$$\varphi(t)=\int_c^t\left(\int_a^в f(x,y)dx\right)dy,\quad \psi(t)=\int_a^в\left(\int_c^t f(x,y)dy\right)dx$$

integrallarni qaraylik. Bu $\varphi(t)$, $\psi(t)$ funksiyalarning hosilalarini hisoblaymiz.

$\Phi(y)=\int\limits_a^{в} f(x,y)dx$ funksiya $[c,d]$ oraliqda uzluksiz bo'lgani sababli 1-qism, 9-bob, 6-§ da keltirilgan 9-teoremaga asosan

$$\varphi'(t)=\left(\int\limits_a^t \Phi(y)dy\right)'=\Phi(t)=\int\limits_a^{в} f(x,t)dx \qquad (16.8)$$

bo'ladi.

$f(x,y)$ funksiya M to'plamda uzluksiz. Yana usha 1-qism, 9-bob, 6-§ dagi teoremaga ko'ra

$$\left(\int\limits_c^t f(x,y)dy\right)'_t = f(x,t) \quad (x \text{ o'zgarmas})$$

bo'ladi. Demak, $\int\limits_c^t f(x,y)dy$ funksiyaning $M=\{(x,t)\in R^2:\ x\in[a,в],\ t\in[c,d]\}$ to'p-lamdagi t bo'yicha xususiy hosilasi $f(x,t)$ ga teng va demak, uzluksiz. U holda 5-teoremaga muvofiq

$$\psi'(t)=\left[\int\limits_a^{в}\left(\int\limits_c^t f(x,y)dy\right)dx\right]'_t=\int\limits_a^{в}\left(\int\limits_c^t f(x,y)dy\right)'_t dx=\int\limits_a^{в} f(x,t)dx \qquad (16.9)$$

bo'ladi.

(16.8) va (16.9) munosabatlardan

$$\varphi'(t)=\psi'(t)=\int\limits_a^{в} f(x,t)dt$$

bo'lishi kelib chiqdi. Demak,

$$\varphi(t)\equiv\psi(t)+C, \qquad (C=const).$$

Ayni paytda $t=c$ bo'lganda $\varphi(c)=\psi(c)=0$ bo'lib, undan $C=0$ bo'lishini topamiz. Demak, $\varphi(t)=\psi(t)$ bo'ladi. Xususan, $t=d$ bo'lganda $\varphi(d)=\psi(d)=0$ bo'lib, u teoremani isbotlaydi. ▶

16.2-misol. Parametrga bog'liq integralni parametr bo'yicha integrallashdan foydalanib, ushbu

$$A=\int\limits_0^1 \frac{x^{в}-x^{a}}{ln\, x}dx \quad (0<a<в)$$

integral hisoblansin.

◀ Ravshanki, $(x>0)$

$$\int\limits_a^{в} x^y dy=\frac{x^{в}-x^{a}}{ln\, x}$$

bo'ladi. Demak,

$$A=\int\limits_0^1 \frac{x^{в}-x^{a}}{ln\, x}dx=\int\limits_0^1\left(\int\limits_a^{в} x^y dy\right)dx.$$

Integral ostidagi $f(x,y)=x^y$ funksiya $M=\{(x,y)\in R^2:\ x\in[0,1],\ \ y\in[a,в]\}$ to'plamda uzluksizdir. U holda 6-teoremaga ko'ra

$$A=\int_a^в\left(\int_0^1 x^y dx\right)dy$$

bo'ladi. Ravshanki,

$$\int_0^1 x^y dx=\frac{1}{y+1}.$$

Unda $A=\int_a^в \frac{dy}{y+1}=ln\frac{в+1}{a+1}$ bo'ladi. Demak,

$$\int_0^1 \frac{x^в-x^a}{ln\,x}dx=ln\frac{в+1}{a+1}. ▶$$

5⁰. Chegaralari ham parametrga bog'liq integrallar. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in[c,d]\}$ to'plamda berilgan. y o'zgaruvchining $[c,d]$ oraliqdan olingan har bir tayin qiymatida $f(x,y)$ funksiya x o'zgaruvchining funksiyasi sifatida $[a,в]$ oraliqda integrallanuvchi bo'lsin.

$x=\alpha(y),\ x=\beta(y)$ funksiyalarning har biri $[c,d]$ da berilgan va $\forall y\in[c,d]$ uchun ushbu

$$a\le\alpha(y)\le\beta(y)\le в \qquad (16.9)$$

tengsizlikni qanoatlantirsin.

Ravshanki, ushbu

$$\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

integral mavjud, y o'zgaruvchi (parametr)ga bog'liqdir:

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx \qquad (16.10)$$

7-teorema. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,в],\ y\in[c,d]\}$ to'plamda uzluksiz, $\alpha(y)$ va $\beta(y)$ funksiyalarning har biri $[c,d]$ da uzluksiz va ular (16.9) shartni qanoatlantirsin. U holda

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya ham $[c,d]$ oraliqda uzluksiz bo'ladi.

◀ $\forall y_0\in[c,d]$ nuqtani olib, unga shunday Δy $(\Delta y\ge 0)$ orttirma beraylikki, $y_0+\Delta y\in[c,d]$ bo'lsin. U holda

$$F(y_0+\Delta y)-F(y_0)=\int_{\alpha(y_0+\Delta y)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx-\int_{\alpha(y_0)}^{\beta(y_0)} f(x,y_0)dx=$$

$$= \int_{\alpha(y_0)}^{\beta(y_0)} [f(x,y_0+\Delta y) - f(x,y_0)]dx + \int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx - \int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx \quad (16.11)$$

bo'ladi. Bu tenglikning o'ng tomonidagi qo'shiluvchilarni baholaymiz.

$f(x,y)$ funksiya M to'plamda uzluksiz, demak, Kantor teoremasiga asosan, tekis uzluksiz bo'ladi. U holda $\Delta y \to 0$ da $f(x,y_0+\Delta y)$ funksiya o'z limit funksiyasi $f(x,y_0)$ ga tekis yaqinlashadi (qaralsin, 16§-5-bet 2^0 -bet) va 16.3-teoremaga ko'ra

$$\lim_{\Delta y \to 0} \int_{\alpha(y_0)}^{\beta(y_0)} [f(x,y_0+\Delta y) - f(x,y_0)]dx = \int_{\alpha(y_0)}^{\beta(y_0)} \lim_{\Delta y \to 0} [f(x,y_0+\Delta y) - f(x,y_0)]dx = 0 \quad (16.12)$$

bo'ladi.

(16.11) munosabatdagi

$$\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx, \quad \int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx$$

integrallar uchun quyidagi bahoga egamiz:

$$\left| \int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx \right| \le M_0 |\beta(y_0+\Delta y) - \beta(y_0)|, \quad (16.13)$$

$$\left| \int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx \right| \le M_0 |\alpha(y_0+\Delta y) - \alpha(y_0)|,$$

bunda $M_0 = sup\{|f(x,y)|, (x,y) \in M\}$.

Shartga ko'ra $\alpha(y)$, $\beta(y)$ funksiyalarning har biri $[c,d]$ da uzluksiz. Demak,

$$\lim_{\Delta y \to 0} [\alpha(y_0+\Delta y) - \alpha(y_0)] = 0,$$
$$\lim_{\Delta y \to 0} [\beta(y_0+\Delta y) - \beta(y_0)] = 0. \quad (16.14)$$

Yuqoridagi (16.12), (16.13) va (16.14) munosabatlarni e'tiborga olib, (16.11) tenglikda $\Delta y \to 0$ da limitga o'tsak, unda

$$\lim_{\Delta y \to 0} [F(y_0+\Delta y) - F(y_0)] = 0$$

bo'lishi kelib chiqadi. Demak, $F(y)$ funksiya $\forall y_0 \in [c,d]$ da uzluksiz.►

8-teorema. $f(x,y)$ funksiya $M = \{(x,y) \in R^2 : x \in [a,в], y \in [c,d]\}$ to'plamda uzluksiz, $f'_y(x,y)$ xususiy hosilaga ega va u uzluksiz, $\alpha(y)$, $\beta(y)$ funksiyalar esa $\alpha'(y)$, $\beta'(y)$ hosilalarga ega hamda ular (16.9) shartni qanoatlantirsin. U holda

$$F(y) = \int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c,d]$ oraliqda $F'(y)$ hosilaga ega va

$$F'(y) = \int_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx + \beta'(y) \cdot f(\beta(y),y) - \alpha'(y) \cdot f(\alpha(y),y)$$

bo'ladi.

◀ $\forall y_0 \in [c,d]$ nuqtani olib, unga shunday Δy $(\Delta y \leq 0)$ orttirma beraylikki, $y_0 + \forall y \in [c,d]$ bo'lsin.

(16.11) munosabatdan foydalanib quyidagini topamiz:

$$\frac{F(y_0+\Delta y)-F(y_0)}{\Delta y} = \int_{\alpha(y_0)}^{\beta(y_0)} \frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}dx + \\ +\frac{1}{\Delta y}\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx - \frac{1}{\Delta y}\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx. \tag{16.15}$$

$\Delta y \to 0$ da

$$\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}$$

funksiya o'z limit funksiyasi $f'_y(x,y_0)$ ga $[a,в]$ oraliqda tekis yaqinlashadi (qaralsin, 16-§ 5-bet). Unda

$$\lim_{\Delta y\to 0}\int_{\alpha(y_0)}^{\beta(y_0)} \frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}dx = \int_{\alpha(y_0)}^{\beta(y_0)} f'_y(x,y_0)dx \tag{16.16}$$

bo'ladi.

Endi

$$\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx, \quad \int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx$$

integrallarga o'rta qiymat haqidagi teoremani qo'llab (qaralsin, 1-qism, 9-bob, 5-§), ushbu

$$\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx = f(x',y_0+\Delta y)\cdot[\beta(y_0+\Delta y)-\beta(y_0)],$$

$$\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx = f(x'',y_0+\Delta y)\cdot[\alpha(y_0+\Delta y)-\alpha(y_0)]$$

tengliklarni hosil qilamiz, bunda x' nuqta $\beta(y_0),\beta(y_0+\Delta y)$ nuqtalar orasida, x'' esa $\alpha(y_0),\alpha(y_0+\Delta y)$ nuqtalar orasida joylashgan.

$f(x,y)$ funksiyaning M to'plamda uzluksizligini, $\alpha(y)$ av $\beta(y)$ funksiyalarning esa $[c,d]$ oraliqda hosilaga ega bo'lishini e'tiborga olsak, u holda

$$\lim_{\Delta y\to 0}\frac{1}{\Delta y}\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx = \lim_{\Delta y\to 0}\left[f'(x',y_0+\Delta y)\cdot\frac{\beta(y_0+\Delta y)-\beta(y_0)}{\Delta y}\right] = \\ = f(\beta(y_0),y_0)\cdot\beta'(y_0). \tag{16.17}$$

$$\lim_{\Delta y\to 0}\frac{1}{\Delta y}\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx = \lim_{\Delta y\to 0}\left[f'(x'',y_0+\Delta y)\cdot\frac{\alpha(y_0+\Delta y)-\alpha(y_0)}{\Delta y}\right] = \\ = f(\alpha(y_0),y_0)\cdot\alpha'(y_0).$$

ekanligi kelib chiqadi.

Yuqoridagi (16.15) munosabatda, $\Delta y \to 0$ da limitga o’tib, (16.16) va (16.17) tengliklarni e’tiborga olib ushbuni topamiz:

$$\lim_{\Delta y\to 0}\frac{F(y_0+\Delta y)-F(y_0)}{\Delta y}=\int\limits_{\alpha(y_0)}^{\beta(y_0)} f'_y(x,y_0)dx+f(\beta(y_0),y_o)\cdot\beta'(y_0)-$$
$$-f(\alpha(y_0),y_0)\cdot\alpha'(y_0).$$

Demak,

$$F'(y_0)=\int\limits_{\alpha(y_0)}^{\beta(y_0)} f'_y(x,y_0)dx+f(\beta(y_0),y_0)\cdot\beta'(y_0)-f(\alpha(y_0),y_0)\cdot\alpha'(y_0).$$

Modomiki, y_0 nuqta $[c,d]$ oraliqdagi ixtiyoriy nuqta ekan, u holda $\forall y\in[c,d]$ uchun

$$F'(y)=\int\limits_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx+f(\beta(y),y)\cdot\beta'(y)-f(\alpha(y),y)\cdot\alpha'(y)$$

bo’lishi ravshandir.►

3-§. Parametrga bog’liq xosmas integrallar. Integrallarning tekis yaqinlashishi

1⁰. Parametrga bog’liq xosmas integral tushunchasi. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,+\infty),\ y\in E\subset R\}$ to’plamda berilgan. So’ng y o’zgaruvchining E to’plamdan olingan har bir tayin qiymatida $f(x,y)$ x o’zgaruvchining funksiyasi sifatida $[a,+\infty)$ oraliq bo’yicha integrallanuvchi, ya’ni

$$\int\limits_a^{+\infty} f(x,y)dx \quad (y\in E\subset R)$$

xosmas integral mavjud va chekli bo’lsin. Bu integral y ning qiymatiga bog’liqdir:

$$J(y)=\int\limits_a^{+\infty} f(x,y)dx. \qquad (16.18)$$

(16.18) integral parametrga bog’liq cheksiz oraliq bo’yicha xosmas integral deb ataladi.

$f(x,y)$ funksiya $M_1=\{(x,y)\in R^2:\ x\in[a,в),\ y\in E\subset R\}$ to’plamda berilgan. So’ng y o’zgaruvchining E to’plamdan olingan har bir tayin qiymatida $f(x,y)$ ni x o’zgaruvchining funksiyasi sifatida qaralganda uning uchun $x=в$ maxsus nuqta bo’lsin va bu funksiya $[a,в)$ oraliqda integrallanuvchi, ya’ni,

$$\int\limits_a^{в} f(x,y)dx \quad (y\in E\subset R)$$

xosmas integral mavjud bo’lsin. Ravshanki, bu integral y ning qiymatiga bog’liq:

$$J_1(y)=\int\limits_a^{в} f(x,y)dx. \qquad (16.19)$$

(16.19) integral parametrga bog’liq, chegaralanmagan funksiyaning xosmas integrali deb ataladi.

Masalan, 15-bobning 1-§ ida qaralgan

$$J(\alpha)=\int_a^{+\infty}\frac{dx}{x^\alpha}\quad (a>0,\ \alpha>0)$$

integral, shu bobning 5-§ ida qaralgan

$$\int_a^{в}\frac{dx}{(x-a)^\alpha},\ \int_a^{в}\frac{dx}{(в-x)^\alpha}\quad (\alpha>0)$$

integrallar, 15-bobning 9-§ da qaralgan

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx$$

integrallar parametrga bog'liq xosmas integrallardir.

Bu erda ham asosiy masalalardan biri - $f(x,y)$ funksiyaning funksional xossalariga ko'ra, (16.18), (16.19) parametrlariga bog'liq xosmas integrallarning funksional xossalarini o'rganishdir.

Parametrlarga bog'liq xosmas integrallarni o'rganishda integralning tekis yaqinlashishi tushunchasi muhim rol o'ynaydi.

2^0. *Integralning tekis yaqinlashishi.* $f(x,y)$ funksiya $M=\{(x,y)\in R^2:$ $: x\in[a,+\infty),\ y\in E\subset R\}$ to'plamda berilgan. y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida $f(x,y)$ x o'zgaruvchining funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi bo'lsin.

Cheksiz oraliq bo'yicha xosmas integral ta'rifiga ko'ra ixtiyoriy $[a,t]$ da $(a<t<+\infty)$

$$F(t,y)=\int_a^t f(x,y)dx \qquad (16.20)$$

integral mavjud va

$$J(y)=\int_a^{+\infty} f(x,y)dx=\lim_{t\to+\infty}F(t,y). \qquad (16.21)$$

Shunday qilib, (16.20) va (16.21) integrallar bilan aniqlangan $F(t,y)$ va $J(y)$ funksiyalarga ega bo'lamiz va $J(y)$ funksiya $F(t,y)$ funksiyaning $t\to+\infty$ dagi limit funksiyasi bo'ladi.

5-ta'rif. Agar $t\to+\infty$ da $F(t,y)$ funksiya o'z limit funksiyasi $J(y)$ ga E to'plamda tekis yaqinlashsa,

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to'plamda tekis yaqinlashuvchi deb ataladi.

6-ta'rif. Agar $t\to+\infty$ da $F(t,y)$ funksiya o'z limit funksiya $J(y)$ ga E da notekis yaqinlashsa,

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to'plamda notekis yaqinlashuvchi deb ataladi.

Ravshanki, $\int\limits_a^{+\infty} f(x,y)dx$ integral E to'plamda tekis yaqinlashuvchi bo'lsa, u shu to'plamda yaqinlashuvchi bo'ladi.

Shunday qilib,

$$\int\limits_a^{+\infty} f(x,y)dx$$

integralning E to'plamda tekis yaqinlashuvchi bo'lishi quyidagini anglatadi:

1) $\int\limits_a^{+\infty} f(x,y)dx$ xosmas integral y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida yaqinlashuvchi;

2) $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $\forall t > \delta$ va $\forall y \in E$ uchun

$$\left| \int\limits_t^{+\infty} f(x,y)dx \right| < \varepsilon$$

bo'ladi.

$\int\limits_a^{+\infty} f(x,y)dx$ integral E to'plamda yaqinlashuvchi, ammo u shu to'plamda notekis yaqinlashuvchi degani quyidagini anglatadi:

1) $\int\limits_a^{+\infty} f(x,y)dx$ xosmas integral y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida yaqinlashuvchi;

2) $\forall \delta > 0$ olinganda ham, shunday $\varepsilon_0 > 0$, $y_0 \in E$ va $t_1 > \delta$ tengsizlikni qanoatlantiruvchi $t_1 \in [a, +\infty)$ topiladiki,

$$\left| \int\limits_{t_1}^{+\infty} f(x,y_0)dx \right| \geq \varepsilon_0$$

bo'ladi.

16.3-misol. Ushbu

$$J(y) = \int\limits_0^{+\infty} ye^{-xy}dx \quad (y \in E = (0, +\infty))$$

integralni tekis yaqinlashuvchilikka tekshirilsin.

◄Bu holda

$$F(t,y) = \int\limits_0^t ye^{-xy}dx = 1 - e^{-ty} \quad (0 \leq t < +\infty)$$

bo'lib, y o'zgaruvchining $E = (0, +\infty)$ to'plamdan olingan har bir tayin qiymatida

$$\lim_{t \to +\infty} F(t,y) = \lim_{t \to +\infty} \left(1 - e^{-ty}\right) = 1$$

bo'ladi. Demak, berilgan xosmas integral yaqinlashuvchi va

$$J(y)=\int_0^{+\infty} ye^{-xy}dx=1$$

bo'ladi.

Endi berilgan integralni tekis yaqinlashuvchilikka tekshiramiz.

$y\in E=(0,+\infty)$ bo'lsin. Ixtiyoriy katta musbat δ sonni olaylik. Agar $\varepsilon=\frac{1}{3}$, $t>\delta$ tengsizlikni qanoatlantiradigan ixtiyoriy t_0 va $y_0=\frac{1}{t_0}$ deb olsak, u holda

$$\left|\int_{t_0}^{+\infty} y_0 e^{-xy_0}dx\right|=e^{-t_0y_0}=e^{-1}>\frac{1}{3}=\varepsilon_0$$

bo'ladi. Bu esa

$$J(y)=\int_0^{+\infty} ye^{-xy}dx$$

integral $E=(0,+\infty)$ da notekis yaqinlashuvchi ekanini bildiradi.

Endi $y\in E'=[c,+\infty)\subset E$ bo'lsin, bunda c- ixtiyoriy musbat son. Unda $\forall\varepsilon>0$ olinganda ham $(0<\varepsilon<1)$ $\delta=\frac{1}{c}ln\frac{1}{\varepsilon}$ deyilsa, $\forall t>\delta$ va $\forall y\in[c,+\infty)$ uchun

$$\left|\int_t^{+\infty} ye^{-xy}dx\right|=e^{-ty}=e^{-c\frac{1}{\varepsilon}ln\frac{1}{\varepsilon}}=\varepsilon$$

bo'ladi. Demak,

$$J(y)=\int_0^{+\infty} ye^{-xy}dx$$

integral $E'=[c,+\infty)$ da $(c>0)$ tekis yaqinlashuvchi.▶

Biz ko'rdikki, parametrga bog'liq xosmas integral

$$J(y)=\int_a^{+\infty} f(x,y)dx \qquad (16.18)$$

ning E to'plamda tekis yaqinlashuvchi bo'lishi, $t\to+\infty$ da $F(t,y)$ funksiyani limit funksiya $J(y)$ ga $(y\in E)$ tekis yaqinlashishidan iborat.

Ushbu bobning 1-§ ida $y\to y_0$ da $f(x,y)$ funksiya limit funksiya $\varphi(x)$ ga tekis yaqinlashishining zaruriy va etarli shartini ifodalovchi 1-teoremani keltirdik. Bu teoremadan foydalanib, (16.18) integralning tekis yaqinlashuvchi bo'lishining zaruriy va etarli sharti keltiriladi.

$f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,+\infty),\ y\in E\subset R\}$ to'plamda berilgan. y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida $f(x,y)$ x o'zgaruvchining funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi, ya'ni

$$J(y)=\int_a^{+\infty} f(x,y)dx \qquad (16.18)$$

xosmas integral mavjud bo'lsin.

7-ta'rif. Agar $\forall\varepsilon>0$ olinganda ham y ga bog'liq bo'lmagan shunday $\delta=\delta(\varepsilon)>0$ topilsaki, $t'>\delta$, $t''>\delta$ ni qanoatlantiruvchi $\forall t',t''$ va $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

tengsizlik bajarilsa, (16.18) xosmas integral E to'plamda fundamental integral deb ataladi.

10-teorema. (Koshi teoremasi). Ushbu $J(y)\int_a^{+\infty} f(x,y)dx$ integralning E to'p-lamda tekis yaqinlashuvchi bo'lishi uchun uning E to'plamda fundamental bo'lishi zarur va etarli.

Quyida biz integralning tekis yaqinlashuvchiligini ta'minlaydigan, ko'pincha qo'llaniladigan alomatlarni keltiramiz.

Veyershtrass alomati. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a,+\infty), y\in E\subset R\}$ to'plamda berilgan, y o'zgaruvchining E to'plamdan olingan har bir tayin qiymatida $f(x,y)$ funksiya x o'zgaruvchining funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi bo'lsin. Agar shunday $\varphi(x)$ funksiya $(\forall x\in[a,+\infty))$ topilsaki,

1) $\forall x\in[a,+\infty)$ va $\forall y\in E$ uchun $|f(x,y)|\le\varphi(x)$ bo'lsa;

2) $\int_a^{+\infty}\varphi(x)dx$ xosmas integral yaqinlashuvchi bo'lsa, u holda

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to'plamda tekis yaqinlashuvchi bo'ladi.

◄Shartga ko'ra $\int_a^{+\infty}\varphi(x)dx$ yaqinlashuvchi. Unda 15-bobning 2-§ ida keltirilgan 4-teoremaga asosan, $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $\forall t'>\delta$, $\forall t''>\delta$ bo'lganda $\left|\int_{t'}^{t''}\varphi(x)dx\right|<\varepsilon$ bo'ladi. Ikkinchi tomondan, 1) shartdan foydalanib quyidagini topamiz:

$$\left|\int_{t'}^{t''} f(x,y)dx\right|\le\int_{t'}^{t''}|f(x,y)|dx\le\int_{t'}^{t''}\varphi(x)dx. \qquad ((t'<t''))$$

Demak,

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

Bu esa $\int_a^{+\infty} f(x,y)dx$ xosmas integralning E to’plamda fundamental ekanini bildiradi. Yuqoridagi 10-teoremaga asosan $\int_a^{+\infty} f(x,y)dx$ integral E to’plamda tekis yaqinlashuvchi bo’ladi.

16.4-misol. Ushbu

$$\int_0^{+\infty} \frac{\cos xy}{1+x^2}dx \qquad (y\in E=(-\infty,+\infty))$$

integralni tekis yaqinlashuvligi ko’rsatilsin.

◀ Agar $\varphi(x)$ funksiya sifatida $\varphi(x)=\dfrac{1}{1+x^2}$ olinsa, u holda

1) $\forall x\in[0,+\infty)$ va $\forall y\in(-\infty,+\infty)$ uchun

$$|f(x,y)|=\left|\frac{\cos xy}{1+x^2}\right|\le\frac{1}{1+x^2}=\varphi(x);$$

2) $\int_0^{+\infty}\varphi(x)dx=\int_0^{+\infty}\dfrac{dx}{1+x^2}$ integral yaqinlashuvchi (qaralsin, 15-bob, 1-§) bo’ladi. Demak, Veyershtrass alomatiga ko’ra berilgan integral $E=(-\infty,+\infty)$ da tekis yaqinlashuvchi bo’ladi. ▶

Integralning tekis yaqinlashuvchiligini aniqlashda qo’l keladigan alomatlardan -Abel va Dirixle alomatlarini isbotsiz keltiramiz.

Abel alomati. $f(x,y)$ va $g(x,y)$ funksiyalar $M=\{(x,y)\in R^2 : x\in[a,+\infty), ,y\in E\subset R\}$ to’plamda berilgan. y o’zgaruvchining E to’plamdan olingan har bir tayin qiymatida $g(x,y)$ funksiya x ning funksiyasi sifatida $[a,+\infty)$ da monoton funksiya bo’lsin.

Agar

$$\int_a^{+\infty} f(x,y)dx$$

integral E to’plamda tekis yaqinlashuvchi va $\forall(x,y)\in M$ uchun $|g(x,y)|\le C$ $(C=const)$ bo’lsa, u holda

$$\int_a^{+\infty} f(x,y)g(x,y)dx$$

integral E to’plamda tekis yaqinlashuvchi bo’ladi.

16.5-misol. Ushbu

$$\int_0^{+\infty}\frac{\sin x}{x}e^{-xy}dx \qquad (y\in E=[0,+\infty))$$

integralni tekis yaqinlashuvchiligi ko’rsatilsin.

◀ Agar

$$f(x,y)=\frac{\sin x}{x},\quad g(x,y)=e^{-xy}$$

deb olinsa, Abel alomati shartlari bajariladi. Haqiqatdan ham, $\int_0^{+\infty} f(x,y)dx$ tekis yaqinlashuvchi:

$$\int_0^{+\infty} f(x,y)dx = \int_0^{+\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

(15-bob, 2-§ va 15-bob, 8-§), $g(x,y) = e^{-xy}$ esa y ning $E = [a,+\infty)$ dan olingan har bir tayin qiymatida x ning kamayuvchi funksiyasi va $\forall x \in [0,+\infty)$, $\forall y \in E[= 0,+\infty)$ uchun $|g(x,y)| \le 1$ bo'ladi. Demak, berilgan integral Abel alomatiga ko'ra $E = [0,+\infty)$ da tekis yaqinlashuvchi.▶

Dirixle alomati. $f(x,y)$ va $g(x,y)$ funksiyalar M to'plamda berilgan. Agar $\forall t \ge a$ hamda $\forall y \in E$ uchun

$$\left|\int_a^t f(x,y)dx\right| \le c \quad (c = const)$$

bo'lsa va y o'zgaruvchining E dan olingan har bir tayin qiymatida, $x \to +\infty$ da $g(x,y)$ funksiya o'z limit funksiyasi $\varphi(y) = 0$ ga tekis yaqinlashsa, u holda

$$\int_a^{+\infty} f(x,y)g(x,y)dx$$

integral E da tekis yaqinlashuvchi bo'ladi.

16.6-misol. Ushbu

$$\int_0^{+\infty} \frac{\sin xy}{x}dx \qquad (y \in E = [1,2])$$

integralning tekis yaqinlashuvchiligi ko'rsatilsin.

◀Agar

$$f(x,y) = \sin xy\,, \quad g(x,y) = \frac{1}{x}$$

deyilsa, unda $\forall t > 0$, $\forall y \in [1,2]$ uchun

$$\left|\int_0^t f(x,y)dx\right| = \left|\int_0^t \sin xy dx\right| = \left|1 - \frac{\cos ty}{y}\right| \le 2$$

bo'ladi. $x \to +\infty$ da $g(x,y) = \frac{1}{x}$ funksiya E to'plamda nolga tekis yaqinlashadi:

$$g(x,y) = \frac{1}{x} \to 0.$$

Demak, berilgan integral Dirixle alomatiga ko'ra $[1,\ 2]$ da tekis yaqinlashuvchidir.▶

Chegaralanmagan funksiya xosmas integralning tekis (notekis) yaqinlashuvchiligi tushunchasi ham yuqoridagidek kiritiladi.

$f(x,y)$ funksiya $M = \{(x,y) \in R^2 :\ x \in [a,в),\ y \in E \subset R\}$ to'plamda berilgan. y o'zgaruvchining E dan olingan har bir tayin qiymatida $f(x,y)$ ni x o'zgaruvchining funksiyasi sifatida qaralganda uning uchun $x = в$ maxsus nuqta bo'lsin va bu funksiya $[a,в)$ da integrallanuvchi bo'lsin. Chegaralanmagan funksiya xosmas integrali ta'rifiga ko'ra ixtiyoriy $[a,t]$ da $(a<t<в)$

$$F_1(t,y)=\int_a^t f(x,y)dx$$

integral mavjud va

$$J_1(y)=\int_a^в f(x,y)dx=\lim_{t\to в-0} F_1(t,y) \qquad (16.22)$$

bo'ladi. Demak, $J_1(y)$ funksiya $F_1(t,y)$ funksiyaning $t\to в-0$ dagi limiti funksiyasi.

8-ta'rif. Agar $t\to в-0$ da $F_1(t,y)$ funksiya o'z limit funksiyasi $J_1(y)$ ga E to'plamda tekis yaqinlashsa,

$$\int_a^в f(x,y)dx$$

integral E to'plamda tekis yaqinlashuvchi deb ataladi.

9-ta'rif. Agar $t\to в-0$ da $F_1(t,y)$ funksiya o'z limit funksiyasi $J_1(y)$ ga E to'plamda notekis yaqinlashsa,

$$\int_a^в f(x,y)dx$$

integral E to'plamda notekis yaqinlashuvchi deb ataladi.

Bu ta'riflarni "$\varepsilon-\delta$" orqali bayon etishni o'quvchiga havola etamiz.

10-ta'rif. Agar $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topilsaki, $в-\delta<t'<в$, $в-\delta<t''<в$ bo'lgan $\forall t',t''$ lar va $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

tengsizlik bajarilsa, (16.22) integral E to'plamda fundamental integral deb ataladi.

11-teorema. $\int_a^в f(x,y)dx$ integralning E to'plamda tekis yaqinlashuvchi bo'lishi uchun uning E to'plamda fundamental bo'lishi zarur va etarli.

4-§. Tekis yaqinlashuvchi parametrga bog'liq xosmas integrallarning xossalari

1^0. Integral belgisi ostida limit o'tish. $f(x,y)$ funksiya $M=\{(x,y)\in R^2:$ $:x\in[a,+\infty),\ y\in E\subset R\}$ to'plamda berilgan. y_0 nuqta E to'plamning limit nuqtasi bo'lsin.

12-teorema. $f(x,y)$ funksiya

1) y o'zgaruvchining E dan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a,+\infty)$ da uzluksiz;

2) $y\to y_0$ da ixtiyoriy $[a,t]$ $(a<t<+\infty)$ oraliqda $\varphi(x)$ limit funksiyaga tekis yaqinlashuvchi bo'lsin.

Agarda

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to'plamda tekis yaqinlashuvchi bo'lsin, u holda $y \to y_0$ da $J(y)$ funksiya limitga ega va

$$\lim_{y\to y_0} J(y)=\lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\varphi(x)dx$$

bo'ladi.

◄Teoremaning 1) va 2) shartlari hamda ushbu bobning 1-§ idagi 2-teoremadan $\varphi(x)$ limit funksiyaning $[a,+\infty)$ da uzluksiz bo'lishi kelib chiqadi. Demak, $\varphi(x)$ funksiya har bir chekli $[a,t]$ $(a<t<+\infty)$ oraliqda integrallanuvchi.

$\varphi(x)$ ni $[a,+\infty)$ da integrallanuvchi ekanligini ko'rsataylik.

Teoremaning shartiga ko'ra

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral E da tekis yaqinlashuvchi. Unda 10-teoremaga asosan, $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $t'>\delta,\ t''>\delta$ bo'lgan $\forall t',t''$ lar va $\forall y\in E$ uchun

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon \qquad (16.23)$$

bo'ladi. $f(x,y)$ funksiyaga qo'yilgan shartlar 2-§ da keltirilgan 3-teorema shartlarining bajarilishini ta'minlaydi. (16.23) tenglikda $y\to y_0$ da limitga o'tib quyidagini topamiz:

$$\left|\int_{t'}^{t''}\varphi(x)dx\right|\le\varepsilon .$$

Bundan esa $\varphi(x)$ ning $[a,+\infty)$ da integrallanuvchi bo'lishi kelib chiqadi (15-bob, 2-§).

Endi

$$\left|\int_a^{+\infty} f(x,y)dx-\int_a^{+\infty}\varphi(x)dx\right|$$

ayirmani quyidagicha yozib,

$$\left|\int_a^{+\infty} f(x,y)dx-\int_a^{+\infty}\varphi(x)dx\right|=\left|\int_a^{t}[f(x,y)-\varphi(x)]dx-\int_t^{+\infty} f(x,y)dx-\right.$$

$$\left.-\int_t^{+\infty}\varphi(x)dx\right|\le\int_a^{t}|f(x,y)-\varphi(x)|dx+\left|\int_t^{+\infty} f(x,y)dx\right|+ \qquad (16.24)$$

$$+\left|\int_t^{+\infty}\varphi(x)dx\right| \qquad (a<t<+\infty)$$

tengsizlikning o’ng tomonidagi har bir qo’shiluvchini baholaymiz.

$\int_a^{+\infty} f(x,y)dx$ integral E da tekis yaqinlashuvchi. Demak, $\forall\varepsilon > 0$ olinganda ham shunday $\delta_1 = \delta_1(\varepsilon) > 0$ topiladiki, barcha $t > \delta_1$ va $\forall y \in E$ uchun

$$\left| \int_t^{+\infty} f(x,y)dx \right| < \frac{\varepsilon}{3} \qquad (16.25)$$

bo’ladi.

$\int_a^{+\infty} \varphi(x)dx$ xosmas integral yaqinlashuvchi. Demak, yuqoridagi $\forall\varepsilon > 0$ olinganda ham shunday $\delta_2 = \delta_2(\varepsilon) > 0$ topiladiki, barcha $t > \delta_2$ uchun

$$\left| \int_t^{+\infty} \varphi(x)dx \right| < \frac{\varepsilon}{3} \qquad (16.26)$$

bo’ladi.

Agar $\delta_0 = max\{\delta_1, \delta_2\}$ deb olinsa, barcha $t > \delta_0$ uchun (16.25) va (16.26) tengsizliklar bir yo’la bajariladi. $y \to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ limit funksiyaga har bir $[a,t]$ (jumladan $t > \delta_0$) da tekis yaqinlashuvchi. Demak, $\forall\varepsilon > 0$ olinganda ham, shunday $\delta' > 0$ topiladiki, $|y_n - y_0| < \delta'$ tengsizlikni qanoatlantiruvchi $y \in E$ va $\forall x \in [a, в]$ uchun

$$|f(x,y) - \varphi(x)| < \frac{\varepsilon}{3(t-a)} \qquad (16.27)$$

bo’ladi. Natijada (16.24), (16.25), (16.26) va (16.27) tengsizliklarga ko’ra

$$\left| \int_a^{+\infty} f(x,y)dx - \int_a^{+\infty} \varphi(x)dx \right| < \varepsilon$$

bo’ladi. Bu esa

$$\lim_{y \to y_0} \int_a^{+\infty} f(x,y)dx = \int_a^{+\infty} \varphi(x)dx \qquad (16.28)$$

bo’lishini bildiradi.▶

(16.28)limit munosabatni quyidagicha ham yozish mumkin:

$$\lim_{y \to y_0} \int_a^{+\infty} f(x,y)dx = \int_a^{+\infty} \left(\lim_{y \to y_0} f(x,y) \right) dx .$$

Bu esa 12-teoremaning shartlari bajarilganda parametrga bog’liq xosmas integrallarda ham integral belgisi ostida limitga o’tish mumkinligini ko’rsatadi.

2^0. Integrallarning parametr bo’yicha uzluksizligi. $f(x,y)$ funksiya $M = \{(x,y) \in R^2 : x \in [a, +\infty), \ y \in [c, \ d]\}$ to’plamda berilgan.

14-teorema. $f(x,y)$ funksiya M to’plamda uzluksiz va

$$J(y) = \int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ da tekis yaqinlashuvchi bo'lsin. U holda $J(y)$ funksiya $[c,d]$ oraliqda uzluksiz bo'ladi.

◀ $f(x,y)$ funksiyaning M to'plamda uzluksizligidan, avvalo bu funksiya y o'zgaruvchining har bir tayin qiymatida x ning uzluksiz funksiyasi bo'lishi kelib chiqadi. Shu bilan birga $f(x,y)$ funksiya $M_t=\{(x,y)\in R^2 : x\in[a,\ t],\ y\in[c,\ d]\}$ $(a<t<+\infty)$ to'plamda ham uzluksiz, demak, shu to'plamda tekis uzluksiz bo'ladi.

$\forall y_0\in[c,d]$ nuqtani olaylik. $y\to y_0$ da $f(x,y)$ funksiya $f(x,y_0)$ limit funksiyaga $[a,t]$ da tekis yaqinlashadi (qaralsin, 16-§ 5-bet). Agar teoremaning ikkinchi shartini e'tiborga olsak, u holda $f(x,y)$ funksiya 12-teoremaning barcha shartlarini bajarishini ko'ramiz. U holda 12-teoremaga asosan

$$\lim_{y\to y_0} J(y)=\lim_{y\to y_0}\int_a^{+\infty} f(x,y)dx=\int_a^{+\infty}\left(\lim_{y\to y_0} f(x,y)\right)dx=\int_a^{+\infty} f(x,y_0)dx=J(y_0)$$

bo'ladi. Bu esa $J(y)$ funksiyaning $[c,d]$ oraliqda uzluksiz ekanini bildiradi.▶

3⁰. Integrallarni parametr bo'yicha differensiallash. $f(x,y)$ funksiya $M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in[c,d]\}$ to'plamda berilgan.

16-teorema. $f(x,y)$ funksiya M to'plamda uzluksiz, $f'_y(x,y)$ xususiy hosilaga ega va u ham uzluksiz hamda y o'zgaruvchining $[c,d]$ dan olingan har bir tayin qiymatida

$$J(y)=\int_a^{+\infty} f(x,y)dx$$

integral yaqinlashuvchi bo'lsin.

Agar $\int_a^{+\infty} f'_y(x,y)dx$ integral $[c,d]$ da tekis yaqinlashuvchi bo'lsa, u holda $J(y)$ funksiya ham $[c,d]$ oraliqda $J'(y)$ hosilaga ega bo'ladi va

$$J'(y)=\int_a^{+\infty} f'_y(x,y)dx$$

munosabat o'rinlidir.

◀ $\forall y_0\in[c,d]$ nuqtani olib, unga shunday Δy $(\Delta y\ge 0)$ orttirma beraylikki, $y_0+\Delta y\in[c,d]$ bo'lsin.

$J(y)$ funksiyaning y_0 nuqtadagi orttirmasini olib, ushbu

$$\frac{J(y_0+\Delta y)-J(y_0)}{\Delta y}=\int_a^{+\infty}\frac{f(x,y_0+\Delta y)-f(x,y)}{\Delta y}dx \qquad (16.29)$$

tenglikni hosil qilamiz. Endi (16.29) tenglikdagi integralda $\Delta y\to 0$ da integral belgisi ostida limitga o'tish mumkinligini ko'rsatamiz.

Lagranj teoremasiga ko'ra

$$\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}=f'_y(x,y_0+\theta\cdot\Delta y) \qquad (16.30)$$

bo'ladi, bunda $0<\theta<1$.

Shunga ko'ra $f'_y(x,y)$ funksiya $M_t = \{(x,y) \in R^2 : \ x \in [a,t], \ y \in [c,d]\}$ to'plamda uzluksiz, demak, tekis uzluksiz. U holda $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $|x'' - x'| < \delta$, $|y'' - y'| < \delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy $(x', y') \in M_t$, $(x'', y'') \in M_t$ nuqtalar uchun

$$\left| f'_y(x'', y'') - f'_y(x', y') \right| < \varepsilon$$

bo'ladi. Agar $x' = x'' = x$, $y' = y_0$, $y' = y_0 + \theta \cdot \Delta y$ deyilsa, unda $|\Delta y| < \delta$ bo'lganda

$$\left| f'_y(x, y_0 + \theta \cdot \Delta y) - f'_y(x, y_0) \right| < \varepsilon \qquad (\forall x \in [a,t])$$

bo'ladi. Yuqoridagi (16.30) tenglikdan foydalanib quyidagini topamiz:

$$\left| \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} - f'_y(x, y_0) \right| < \varepsilon .$$

Bu esa $\Delta y \to 0$ da $\dfrac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y}$ funksiya $f'_y(x, y_0)$ limit funksiyaga tekis yaqinlashishini bildiradi.

Teoremaning shartiga ko'ra

$$\int_a^{+\infty} f'_y(x,y) dx$$

tekis yaqinlashuvchi. Demak, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $t' > \delta$, $t'' > \delta$ bo'lgan ixtiyoriy t', t'' va $\forall y \in [c,d]$ uchun

$$\left| \int_{t'}^{t''} f'_y(x,y) dx \right| < \varepsilon$$

bo'ladi. Jumladan

$$\left| \int_{t'}^{t''} f(x, y_o + \theta \cdot \Delta y) dx \right| < \varepsilon$$

bo'ladi. (16.30) tenglikka asosan

$$\left| \int_{t'}^{t''} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} dx \right| < \varepsilon$$

bo'ladi. Bu esa

$$\int_a^{+\infty} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} dx$$

integralning tekis yaqinlashuvchiligini bildiradi.

Natijada 12-teoremaga ko'ra

$$\lim_{\Delta y \to 0} \int_a^{+\infty} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} dx = \int_a^{+\infty} \left(\lim_{\Delta y \to 0} \frac{f(x, y_0 + \Delta y) - f(x, y_0)}{\Delta y} \right) dx$$

tenglik o'rinli bo'ladi.

Yuqoridagi (16.29) tenglikda $\Delta y \to 0$ da limitga o'tamiz:

$$\lim_{\Delta y\to 0}\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}=\lim_{\Delta y\to 0}\int_a^{+\infty}\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}dx=$$

$$=\int_a^{+\infty}\left(\lim_{\Delta y\to 0}\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}\right)dx=\int_a^{+\infty}f'_y(x,y_0)dx.$$

Demak,

$$J'(y_0)=\int_a^{+\infty}f'_y(x,y_0)dx\,. \blacktriangleright$$

Keyingi munosabatdan quyidagicha ham yozish mumkin:

$$\frac{d}{dy}\int_a^{+\infty}f(x,y)dx=\int_a^{+\infty}\left(\frac{d}{dy}f(x,y)\right)dx\,.$$

Bu esa teorema shartlarida differensiallash amalini integral belgisi ostida o’tkazish mumkinligini ko’rsatadi.

4^0 ***Integrallarni parametr bo’yicha integrallash.*** $f(x,y)$ funksiya $M=\{(x,y)\in R^2:\ x\in[a,+\infty),\ y\in[c,d]\}$ to’plamda berilgan.

18-teorema. Agar $f(x,y)$ funksiya M to’plamda uzluksiz va

$$J(y)=\int_a^{+\infty}f(x,y)dx$$

integral $[c,d]$ oraliqda tekis yaqinlashuvchi bo’lsa, u holda $J(y)$ funksiya $[c,d]$ da integrallanuvchi va

$$\int_c^d J(y)dy=\int_c^d\left(\int_a^{+\infty}f(x,y)dx\right)dy=\int_a^{+\infty}\left(\int_c^d f(x,y)dy\right)dx$$

bo’ladi.

◀Teoremaning shartlaridan $J(y)$ funksiya $[c,d]$ oraliqda uzluksiz bo’lishi kelib chiqadi (qaralsin, 4-teorema). Demak, $J(y)$ funksiya $[c,d]$ da integrallanuvchi.

Endi

$$\int_c^d\left(\int_a^{+\infty}f(x,y)dx\right)dy=\int_a^{+\infty}\left(\int_c^d f(x,y)dy\right)dx$$

tenglikning o’rinli bo’lishini ko’rsatamiz.

Shartga ko’ra

$$J(y)=\int_a^{+\infty}f(x,y)dx$$

integral $[c,d]$ da tekis yaqinlashuvchi. Demak, $\forall\varepsilon>0$ olinganda ham shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $\forall t>\delta$ va $\forall y\in[c,d]$ uchun

$$\left|\int_t^{+\infty}f(x,y)dx\right|<\varepsilon \qquad (16.31)$$

bo’ladi. Mana shunday t bo’yicha

$$\int_c^d\left[\int_a^{+\infty} f(x,y)dx\right]dy$$

integralni quyidagicha yozamiz:

$$\int_c^d\left(\int_a^{+\infty} f(x,y)dx\right)dy=\int_c^d\left(\int_a^{t} f(x,y)dx\right)dy+\int_c^d\left(\int_t^{+\infty} f(x,y)dx\right)dy.$$

6-teoremaga asosan

$$\int_c^d\left(\int_a^{t} f(x,y)dx\right)dy=\int_a^t\left(\int_c^{d} f(x,y)dy\right)dx$$

bo’ladi. Natijada

$$\int_c^d J(y)dy=\int_a^t\left(\int_c^{d} f(x,y)dy\right)dx+\int_c^d\left(\int_t^{+\infty} f(x,y)dx\right)dy$$

bo’ladi. Yuqoridagi (16.31) munosabatni e’tiborga olib topamiz:

$$\left|\int_c^d J(y)dy-\int_a^t\left(\int_c^{d} f(x,y)dy\right)dx\right|\le\int_c^d\left|\int_t^{+\infty} f(x,y)dx\right|dy<\varepsilon(d-c).$$

Bu esa

$$\int_c^d J(y)dy=\lim_{t\to+\infty}\int_a^t\left(\int_c^{d} f(x,y)dy\right)dx=\int_a^{+\infty}\left(\int_c^{d} f(x,y)dy\right)dx$$

ekanini bildiradi. Demak,

$$\int_c^d\left(\int_a^{+\infty} f(x,y)dx\right)dy=\int_a^{+\infty}\left(\int_c^{d} f(x,y)dy\right)dx.\ ▶$$

Endi $f(x,y)$ funksiya $M_2=\{(x,y)\in R^2:\ x\in[a,+\infty),\ y\in[c,+\infty)\}$ to’plamda berilgan bo’lsin.

19-teorema. $f(x,y)$ funksiya M_2 to’plamda uzluksiz va

$$\int_a^{+\infty} f(x,y)dx,\quad \int_c^{+\infty} f(x,y)dy$$

integrallar mos ravishda $[c,+\infty)$ va $[a,+\infty)$ da tekis yaqinlashuvchi bo’lsin.

Agar

$$\int_c^{+\infty}\left(\int_a^{+\infty}|f(x,y)|dx\right)dy\ \ \left(\text{yoki}\ \int_a^{+\infty}\left(\int_c^{+\infty}|f(x,y)|dy\right)dx\right)$$

integral yaqinlashuvchi bo’lsa, u holda

$$\int_a^{+\infty}\left(\int_c^{+\infty} f(x,y)dy\right)dx,\quad \int_c^{+\infty}\left(\int_a^{+\infty} f(x,y)dx\right)dy$$

integrallar yaqinlashuvchi va

$$\int_c^{+\infty}\left(\int_a^{+\infty} f(x,y)dx\right)dy=\int_a^{+\infty}\left(\int_c^{+\infty} f(x,y)dy\right)dx$$

bo’ladi.

Bu teoremaning isbotini o’quvchiga havola qilamiz.

16.7-misol. Ushbu

$$J=\int_0^{+\infty}\frac{\sin x}{x}dx$$

integral hisoblansin.

◀Bu xosmas integralning yaqinlashuchi bo'lishi 15-bobning 2-§ ida ko'rsatilgan edi. Endi berilgan integralni hisoblaymiz. Buning uchun quyidagi

$$J(a)=J=\int_0^{+\infty}e^{-ax}\frac{\sin x}{x}dx$$

parametrga bog'liq xosmas integralni qaraymiz.

Ravshanki,

$$f(x,a)=e^{-ax}\frac{\sin x}{x}\quad (f(0,a)=1)$$

funksiya

$$\{(x,a)\in R^2:\ \ x\in[0,+\infty),\ a\in[0,\ c],\ c>0\}$$

to'plamda uzluksiz,

$$f'_a(x,a)=-e^{-ax}\sin x$$

xususiy hosilaga ega va u ham uzluksiz funksiya. Quyidagi

$$\int_0^{+\infty}f'_a(x,a)dx=-\int_0^{+\infty}e^{-ax}\sin xdx$$

integral esa $a\geq a_0.\ (a_0>0)$ da tekis yaqinlashuvchi. 16-teoremaga ko'ra

$$J'(a)=\int_0^{+\infty}\left(e^{-ax}\frac{\sin x}{x}\right)'_a dx=-\int_0^{+\infty}e^{-ax}\sin xdx=-\frac{1}{1+a^2}$$

bo'ladi (qaralsin, 1-qism, 8-bob, 2-§). Demak,

$$J(a)=-arctga+c.$$

$a=+\infty$ bo'lganda, $J(+\infty)=0$ bo'lib, $-\frac{\pi}{2}+c=0$ ya'ni $c=\frac{\pi}{2}$ bo'ladi.

Demak,

$$J(a)=\frac{\pi}{2}-arctga.$$

Bu tenglikda $a\to 0$ da limitga o'tib quyidagini topamiz:

$$\lim_{a\to 0}J(a)=\frac{\pi}{2}.$$

Shunday qilib, $J(0)=\frac{\pi}{2}$ ya'ni

$$J=\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$$

bo'ladi.▶

5-§. Eyler integrallari

1⁰. Beta funksiya va uning xossalari. Ma'lumki, ushbu

$$\int_0^1 x^{a-1}(1-x)^{в-1}dx \qquad (16.32)$$

chegaralanmagan funksiyaning xosmas integrali $a>0$, $в>0$ ya'ni

$$M=\{(a,в)\in R^2:\ a\in(0,+\infty),\ в\in(0,+\infty)\}$$

to'plamda yaqinlashuvchi (15-bob). Ayni paytda bu integral a va $в$ paraametrlarga ham bog'liq.

11-ta'rif. (16.32) integral beta funksiya yoki I tur Eyler integrali deb ataladi va $B(a,в)$ kabi belgilanadi, demak,

$$B(a,в)=\int_0^1 x^{a-1}(1-x)^{в-1}dx.$$

Shunday qilib, $B(a,в)$ funksiya R^2 fazodagi $M=\{(a,в)\in R^2:$ $a\in(0,+\infty),\ в\in(0,+\infty)\}$ to'plamda berilgandir.

Endi $B(a,в)$ funksiyaning xossalarini o'rganaylik.

1) (16.32) integral

$$B(a,в)=\int_0^1 x^{a-1}(1-x)^{в-1}dx$$

ixtiyoriy $M=\{(a,в)\in R^2:\ a\in(a_0,+\infty),\ в\in(в_0,+\infty)\}$ $(a_0>0,\ в_0>0)$ to'plamda tekis yaqinlashuvchi bo'ladi.

◀Berilgan integralni tekis yaqinlashuvchilikka tekshirish uchun uni quyidagicha

$$\int_0^1 x^{a-1}(1-x)^{в-1}dx=\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{в-1}dx+\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{в-1}dx$$

yozib olamiz.

Ravshanki, $a>0$ bo'lganda $\int_0^{\frac{1}{2}} x^{a-1}dx$ integral yaqinlashuvchi, $в>0$ bo'lganda $\int_{\frac{1}{2}}^1 (1-x)^{в-1}dx$ integral yaqinlashuvchi.

Parametr a ning $a\ge a_0$ $(a_0>0)$ qiymatlari va $\forall в>0$, $\forall x\in\left(0,\frac{1}{2}\right]$ uchun

$$x^{a-1}(1-x)^{в-1}\le x^{a_0-1}(1-x)^{в-1}\le 2x^{a_0-1}$$

bo'ladi. Veyershtrass alomatidan foydalanib,

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{в-1}dx$$

integralning tekis yaqinlashuvchiligini topamiz.

Shuningdek, parametr $в$ ning $в \geq в_0$ $(в_0 > 0)$ qiymatlari va $\forall a > 0$, $\forall x \in \left[\frac{1}{2}, 1\right)$ uchun

$$x^{a-1}(1-x)^{в-1} \leq x^{a-1}(x-1)^{в_0-1} \leq 2(1-x)^{в_0-1}$$

bo'ladi va yana Veyershtrass alomatiga ko'ra $\int_{\frac{1}{2}}^{1} x^{a-1}(1-x)^{в-1}dx$ integralning tekis yaqinlashuvchiligi kelib chiqadi.

Demak, $\int_0^1 x^{a-1}(1-x)^{в-1}dx$ integral $a \geq a_0 > 0$ va $в \geq в_0 > 0$ bo'lganda, ya'ni

$$M_0 = \{(a, в) \in R^2 : \ a \in [a_0, +\infty), \ в \in [в_0, +\infty)\}$$

to'plamda tekis yaqinlashuvchi bo'ladi. ▶

2) $B(a, в)$ funksiya $M = \{(a, в) \in R^2 : \ a \in (a_0, +\infty), \ в \in (в_0, +\infty)\}$ to'plamda uzluksiz funksiyadir.

◀ Haqiqatdan ham,

$$B(a, в) = \int_0^1 x^{a-1}(1-x)^{в-1}dx$$

integralning M_0 to'plamda tekis yaqinlashuvchi bo'lishidan va integral ostidagi funksiyaning $\forall(a, в) \in M$ da uzluksizligidan 15-teoremaga asosan $B(a, в)$ funksiya

$$M = \{(a, в) \in R^2 : \ a \in (0, +\infty), \ в \in (0, +\infty)\}$$

to'plamda uzluksiz bo'ladi. ▶

3) $\forall(a, в) \in M$ uchun $B(a, \ в) = B(в, \ a)$ bo'ladi.

◀ Darhaqiqat $B(a, в) = \int_0^1 x^{a-1}(1-x)^{в-1}dx$ integralda $x = 1 - t$ almashtirish bajarilsa, unda

$$B(a, в) = \int_0^1 x^{a-1}(1-x)^{в-1}dx = \int_0^1 t^{в-1}(1-t)^{a-1}dt = B(в, a)$$

bo'lishini topamiz. ▶

4) $B(a, в)$ funksiya quyidagicha ham ifodalanadi:

$$B(a, в) = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+в}}dt\,.$$

◀ Haqiqatdan ham, (16.32) integralda $x = \frac{t}{1+t}$ almashtirish bajarilsa, u holda

$$B(a, в) = \int_0^1 x^{a-1}(1-x)^{в-1}dx = \int_0^{+\infty} \left(\frac{t}{1+t}\right)^{a-1}\left(1 - \frac{t}{1+t}\right)^{в-1}\frac{dt}{(1+t)^2} = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+в}}dt$$

bo'ladi. ▶

Xususan, $в = 1 - a$ $(0 < a < 1)$ bo'lganda

$$B(a,1-a)=\int_0^{+\infty}\frac{t^{a-1}dt}{1+t}=\frac{\pi}{\sin a\pi}$$

bo'ladi (qaralsin; 17-bob). Keyingi munosabatdan quyidagini topamiz:

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\pi .$$

5) $\forall(a,в)\in M'$ $\left(M'=\{(a,в)\in R^2:\ a\in(0,+\infty),\ в\in(1,+\infty)\}\right)$ uchun

$$B(a,в)=\frac{в-1}{a+в-1}B(a,в-1)$$

bo'ladi.

◀(16.32) integralni bo'laklab integrallaymiz:

$$B(a,в)=\int_0^1 x^{a-1}(1-x)^{в-1}dx=\int_0^1(1-x)^{в-1}d\left(\frac{x^a}{a}\right)=\frac{1}{a}x^a(1-x)^{в-1}\Big|_0^1+$$

$$+\frac{в-1}{a}\int_0^1 x^a(1-x)^{в-2}\,dx=\frac{в-1}{a}\int_0^1 x^a(1-x)^{в-2}dx\qquad(a>0,\quad в>1)$$

Agar $x^a(1-x)^{в-2}=x^{a-1}[1-(1-x)](1-x)^{в-2}=x^{a-1}(1-x)^{в-2}-x^{a-1}(1-x)^{в-1}$ ekanligini e'tiborga olsak, u holda

$$\int_0^1 x^a(1-x)^{в-2}dx=\int_0^1 x^{a-1}(1-x)^{в-2}dx-\int_0^1 x^{a-1}(1-x)^{в-1}dx=B(a,в-1)-B(a,в)$$

bo'lib, natijada

$$B(a,в)=\frac{в-1}{a}(B(a,в-1)-B(a,в))$$

bo'ladi. Bu tenglikdan esa

$$B(a,в)=\frac{в-1}{a+в-1}B(a,в-1)\qquad(a>0,\ \ в>1)$$

bo'lishini topamiz.

Xuddi shunga o'xshash $\forall(a,в)\in M''$ uchun

$$\left(M''=\{(a,в)\in R^2:\ a\in(1,+\infty),\ в\in(0,+\infty)\}\right)$$

$$B(a,в)=\frac{a-1}{a+в-1}B(a-1,в)$$

bo'ladi.▶

Xususan, $в=n$ $(n\in N)$ bo'lganda

$$B(a,n)=\frac{n-1}{a+n-1}B(a,n-1)$$

bo'lib, keyingi formulani takror qo'llab quyidagini topamiz.

$$B(a,n)=\frac{n-1}{a+n-1}\cdot\frac{n-2}{a+n-2}\cdot\ldots\cdot\frac{1}{n+1}B(a,1).$$

Ravshanki, $B(a,1)=\int_0^1 x^{a-1}dx=\frac{1}{a}$. Demak,

$$B(a,n)=\frac{1\cdot 2\cdot\ldots\cdot(n-1)}{a(a+1)(a+2)\ldots(a+n-1)}. \qquad (16.33)$$

Agar (16.33) da $a=m$ $(m\in N)$ bo'lsa, u holda

$$B(m,n)=\frac{1\cdot 2\cdot\ldots\cdot(n-1)}{m(m+1)\ldots(m+n-1)}=\frac{(n-1)!\,(m-1)!}{(m+n-1)!}$$

bo'ladi.

2⁰. Gamma funksiya va uning xossalari. Biz 15-bobning 9-§ ida quyidagi

$$\int_0^{+\infty} x^{a-1}e^{-x}dx \qquad (16.34)$$

xosmas integralni qaradik. Bu chegaralanmagan funksiyaning ($a<1$ da $x=0$ maxsus nuqta) cheksiz oraliq bo'yicha olingan xosmas integrali bo'lishi bilan birga a ga (parametrga) ham bog'liqdir. Usha erda (16.34) xosmas integralning $a>0$ da yaqinlashuvchi ekanligi ko'rsatildi.

12-ta'rif. (16.34) integral gamma funksiya yoki II tur Eyler integrali deb ataladi va $\Gamma(a)$ kabi belgilanadi. Demak,

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx.$$

Shunday qilib, $\Gamma(a)$ funksiya $(0,+\infty)$ da berilgandir. Endi $\Gamma(a)$ funksiyaning xossalarini o'rgandik.

1) (16.34) integral

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx$$

ixtiyoriy $[a_0,в_0]$ $(a<a_0<в_0<\infty)$ oraliqda tekis yaqinlashuvchi bo'ladi.

◀(16.34) integralni quyidagi ikki qismga ajratib,

$$\int_0^{+\infty} x^{a-1}e^{-x}dx=\int_0^1 x^{a-1}e^{-x}dx+\int_1^{+\infty} x^{a-1}e^{-x}dx$$

ularning har birini alohida-alohida tekis yaqinlashuvchilikka tekshiramiz.

Agar a_0 $(a_0>0)$ sonni olib, parametr a ning $a\le a_0$ qiymatlari qaralsa, unda barcha $x\in(0,1]$ uchun $x^{a-1}e^{-x}\le\frac{1}{x^{1-a_0}}$ bo'lib, ushbu bobning 4-§ ida keltirilgan Veyershtrass alomatiga asosan

$$\int_0^1 x^{a-1}e^{-x}dx$$

integral tekis yaqinlashuvchi bo'ladi.

Agar $в_0$ $(в_0>0)$ sonni olib, parametr a ning $a\ge в_0$ qiymatlari qaraladigan bo'lsa, unda barcha $x\ge 1$ uchun

$$x^{a-1}e^{-x}\le x^{в_0-1}e^{-x}\le\left(\frac{в_0+1}{e}\right)^{в_0+1}\frac{1}{x^2}$$

bo'lib,

$$\int_1^{+\infty} \frac{1}{x^2}dx$$

integralning yaqinlashuvchiligidan, yana Veyershtrass alomatiga ko'ra

$$\int_1^{+\infty} x^{a-1}e^{-x}dx$$

integralning tekis yaqinlashuvchi bo'lishini topamiz. Shunday qilib,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx$$

integral $[a_0, в_0]$ $(0 < a_0 < в_0 < +\infty)$ da tekis yaqinlashuvchi bo'ladi. ▶

2) $\Gamma(a)$ funksiya $(0, +\infty)$ da uzluksiz hamda barcha tartibdagi uzluksiz hosilalarga ega va

$$\Gamma^{(n)}(a) = \int_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n dx \quad (n = 1, 2, ...)$$

◀ $\forall a \in (0, +\infty)$ nuqtani olaylik. Unda shunday $[a_0, в_0]$ $(0 < a_0 < в_0 < \ +\infty)$ oraliq topiladiki, $a \in [a_0, в_0]$ bo'ladi.

Ravshanki,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx$$

integral ostidagi $f(x, a) = x^{a-1}e^{-x}$ funksiya $M = \{(x, a) \in R^2.\ x \in (0, +\infty),\ a \in (0, +\infty)\}$ to'plamda uzluksiz funksiyadir. (16.34) integral esa (yuqorida isbot etilgan ko'ra) $[a_0, в_0]$ da tekis yaqinlashuvchi. U holda 4-teoremaga asosan $\Gamma(a)$ funksiya $[a_0, в_0]$ da, binobarin, a nuqtada uzluksiz bo'ladi.

(16.34) integral ostidagi $f(x, a) = x^{a-1}e^{-x}$ funksiya

$$f'_a(x, a) = x^{a-1}e^{-x} \ln x$$

hosilasining M to'plamda uzluksiz funksiya ekanligini payqash qiyin emas.

Endi

$$\int_0^{+\infty} f'_a(x, a)dx = \int_0^{+\infty} x^{a-1}e^{-x} \ln x \, dx$$

integralni $[a_0, в_0]$ da tekis yaqinlashuvchi bo'lishini ko'rsatamiz.

Ushbu $\int_0^1 x^{a-1}e^{-x} \ln x \, dx$ integral ostidagi $x^{a-1}e^{-x} \ln x$ funksiya uchun $0 < x < 1$ da $\left|x^{a-1}e^{-x} \ln x\right| \le x^{a_0 - 1}|\ln x|$ tengsizlik o'rinlidir. $\psi_1(x) = x^2|\ln x|$ funksiya $0 < x \le 1$ da chegaralanganligidan va $\int_0^1 x^{\frac{a_0}{2}-1}dx$ integralning yaqinlashuvchiligidan

$\int_0^1 x^{a-1}|\ln x|dx$ ning ham yaqinlashuvchi bo'lishini va Veyershtrass alomatiga ko'ra qaralayotgan $\int_0^1 x^{a-1}e^{-x}\ln x\, dx$ integralning tekis yaqinlashuvchiligini topamiz.

Shunga o'xshash quyidagi

$$\int_0^{+\infty} x^{a-1}e^{-x}\ln x\, dx$$

integralda, integral ostidagi $x^{a-1}e^{-x}\ln x$ funksiya uchun barcha $x \geq 1$ da

$$x^{a-1}e^{-x}\ln x \leq x^{в_0-1}e^{-x}\ln x < x^{в_0}e^{-x} \leq \left(\frac{в_0+2}{e}\right)^{в_0+2} \cdot \frac{1}{x^2}$$

bo'lib, $\int_1^{+\infty}\frac{dx}{x^2}$ integralning yaqinlashuvchiligidan, yana Veyershtrass alomatiga ko'ra $\int_1^{+\infty} x^{a-1}e^{-x}\ln x\, dx$ ning tekis yaqinlashuvchiligi kelib chiqadi. Demak, $[a_0, в_0]$ da $\int_0^{+\infty} x^{a-1}e^{-x}\ln x\, dx$ integral tekis yaqinlashuvchi. Unda 16-teoremaga asosan

$$\Gamma'(a) = \left(\int_0^{+\infty} x^{a-1}e^{-x}dx\right)' = \int_0^{+\infty}\left(x^{a-1}e^{-x}\right)'_a dx = \int_0^{+\infty} x^{a-1}e^{-x}\ln x\, dx$$

bo'ladi va $\Gamma'(a)$ $[a_0, в_0]$ da binobarin, a nuqtada uzluksizdir.

Xuddi shu yo'l bilan $\Gamma(a)$ funksiyaning ikkinchi, uchinchi va hokazo tartibdagi hosilalarining mavjudligi, uzluksizligi hamda

$$\Gamma^{(n)}(a) = \int_0^{+\infty} x^{a-1}e^{-x}(\ln x)^n\, dx \quad (n = 1, 2, ...)$$

bo'lishi ko'rsatiladi.►

3) $\Gamma(a)$ funksiya uchun ushbu

$$\Gamma(a+1) = a\Gamma(a) \quad (a > 0)$$

formula o'rinli.

◄Haqiqatdan ham,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx = \int_0^{+\infty} e^{-x}d\left(\frac{x^a}{a}\right)$$

integralni bo'laklab integrallasak,

$$\Gamma(a) = e^{-x}\frac{x^a}{a}\Bigg|_0^{+\infty} + \int_0^{+\infty}\frac{x^a}{a}e^{-x}dx = \frac{1}{a}\Gamma(a+1)$$

bo'lib, undan

$$\Gamma(a+1) = a\Gamma(a) \qquad (16.35)$$

bo'lishi kelib chiqadi.►

Bu formula yordamida $\Gamma(a+n)$ ni topish mumkin. Darhaqiqat, (16.35) formulani takror qo'llab

$$\Gamma(a+2)=(a+1)\Gamma(a+1),$$
$$\Gamma(a+3)=(a+2)\Gamma(a+2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\Gamma(a+n)=(a+n-1)\Gamma(a+n-1)$$

bo'lishini, ulardan esa

$$\Gamma(a+n)=(a+n-1)(a+n-2)\ldots(a+2)(a+1)a\Gamma(a)$$

ekanligini topamiz. Xususan, $a=1$ bo'lganda

$$\Gamma(n+1)=n(n-1)\ldots 2\cdot 1\Gamma(1)$$

bo'ladi. Agar $\Gamma(1)=\int\limits_0^{+\infty} e^{-x}dx=1$ bo'lishini e'tiborga olsak, unda $\Gamma(n+1)=n!$ ekanligi kelib chiqadi.

Yana (16.35) formuladan foydalanib $\Gamma(2)=\Gamma(1)=1$ bo'lishini topamiz.

4) $\Gamma(a)$ funksiyaning o'zgarish xarakteri.

$\Gamma(a)$ funksiya $(0,+\infty)$ oraliqda berilgan bo'lib, shu oraliqda istalgan tartibli hosilaga ega. Bu funksiyaning $a=1$ va $a=2$ nuqtalardagi qiymatlari bir-biriga teng:

$$\Gamma(1)=\Gamma(2)=1$$

$\Gamma(a)$ funksiyaga Roll teoremasini (qaralsin, 1-qism, 6-bob, 6-§) tatbiq qila olamiz, chunki yuqorida keltirilgan faktlar Roll teoremasi shartlarining bajarilishini ta'minlaydi. Demak, Roll teoremasiga ko'ra, shunday a^* $(1<a^*<2)$ topiladiki, $\Gamma'(a^*)=0$ bo'ladi.

$\forall a\in(0,+\infty)$ da

$$\Gamma''(a)=\int\limits_0^{+\infty} x^{a-1}e^{-x}\,ln^2 x\,dx>0$$

bo'lishi sababli, $\Gamma'(a)$ funksiya $(0,+\infty)$ oraliqda qat'iy o'suvchi bo'ladi. Demak, $\Gamma'(a)$ funksiya $(0,+\infty)$ da a^* nuqtadan boshqa nuqtalarda nolga aylanmaydi, ya'ni

$$\Gamma'(a)=\int\limits_0^{+\infty} x^{a-1}e^{-x}\,ln\,x\,dx=0$$

tenglama $(0,+\infty)$ oraliqda a^* dan boshqa echimga ega emas. U holda

$$0<a<a^* \text{ da } \Gamma'(a)<0$$
$$a^*<a<+\infty \text{ da } \Gamma'(a)>0$$

bo'ladi. Demak $\Gamma(a)$ funksiya a^* nuqtada minimumga ega. Uning minimum qiymati $\Gamma(a^*)$ ga teng.

Taqribiy hisoblash usuli bilan

$$a^*=1{,}4616\ldots,\ \Gamma(a^*)=min\,\Gamma(a)=0{,}8856\ldots$$

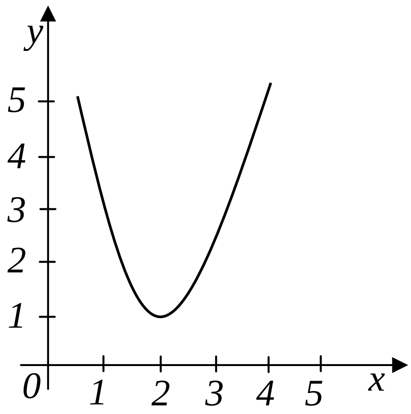
bo'lishi topilgan.

$\Gamma(a)$ funksiya $a > a*$ da o'suvchi bo'lganligi sababli $a > n+1$ $(n \in N)$ bo'lganda $\Gamma(a) > \Gamma(n+1) = n!$ bo'lib, undan

$$\lim_{a \to +\infty} \Gamma(a) = +\infty$$

bo'lishini topamiz.

Ikkinchi tomondan, $a \to +0$ da $\Gamma(a+1) \to \Gamma(1) = 1$ hamda $\Gamma(a) = \dfrac{\Gamma(a+1)}{a}$ ekanligidan $\lim\limits_{a \to +0} \Gamma(a) = +\infty$ kelib chiqadi. $\Gamma(a)$ funksiyaning grafigi 52-chizmada tasvirlangan.



52-chizma

3⁰. Beta va gamma funksiyalar orasidagi bog'lanish. Biz quyida $B(a,в)$ va $\Gamma(a)$ funksiyalar orasidagi bog'lanishni ifodalaydigan formulani keltiramiz.

Ma'lumki, $\Gamma(a)$ funksiya $(0,+\infty)$ da, $B(a,в)$ funksiya esa R^2 fazodagi $M = \{(x,y) \in R^2 : \ a \in (0,+\infty), \ в \in (0,+\infty)\}$ to'plamda berilgan.

21-teorema. $\forall (a, \ в) \in M$ uchun

$$B(a,в) = \frac{\Gamma(a)\Gamma(в)}{\Gamma(a+в)}$$

formula o'rinlidir.

◄Ushbu $\Gamma(a+в) = \int\limits_0^{+\infty} x^{a+в-1} e^{-x} dx \quad (a > 0, \ в > 0)$ gamma funksiyada o'zgaruvchini almashtiramiz:

$$x = (1+t)y \quad (t > 0).$$

Natijada

$$\Gamma(a+в) = \int\limits_0^{+\infty} (1+t)^{a+в-1} y^{a+в-1} e^{-(1+t)y} (1+t) dy = (1+t)^{a+в} \int\limits_0^{+\infty} y^{a+в-1} e^{-(1+t)y} dy.$$

bo'ladi.

Keyingi tenglikdan quyidagini topamiz:

$$\frac{\Gamma(a+в)}{(1+t)^{a+в}} = \int\limits_0^{+\infty} y^{a+в-1} e^{-(1+t)y} dy.$$

Bu tenglikning har ikki tomonini t^{a-1} ga ko'paytirib, natijani $(0,+\infty)$ oraliq bo'yicha integrallaymiz:

$$\Gamma(a+в)\cdot\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+в}}dt=\int_0^{+\infty}\left(\int_0^{+\infty}y^{a+в-1}e^{-(1+t)y}dy\right)t^{a-1}dt\,.$$

Agar

$$\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+в}}dt=B(a,в)$$

ekanini e'tiborga olsak, unda

$$\Gamma(a+в)B(a,в)=\int_0^{+\infty}\left(\int_0^{+\infty}y^{a+в-1}e^{-(1+t)y}dy\right)t^{a-1}dt \qquad (16.36)$$

bo'ladi. Endi (16.36) tenglikning o'ng tomonidagi integral $\Gamma(a)\cdot\Gamma(в)$ ga teng bo'lishini isbotlaysiz. Uning uchun, avvalo bu integrallarda integrallash tartibini almashtirish mumkinligini ko'rsatamiz. Buning uchun 19-teorema shartlari bajarilishini ko'rsatishimiz kerak.

Dastlab $a>1,\ в>1$ bo'lgan holni ko'raylik.

$a>1,\ в>1$ da, ya'ni $\{(a,\ в)\in R^2:\ a\in(1,+\infty),\ в\in(1,+\infty)\}$ to'plamda integral ostidagi

$$f(t,y)=y^{a+в-1}t^{a-1}e^{-(1+t)y}$$

funksiya $\forall(t,y)\in\{(t,\ y)\in R^2:\ t\in[0,+\infty),\ y\in[0,+\infty)\}$ da uzluksiz bo'lib, $f(t,y)=y^{a+в-1}t^{a-1}e^{-(1+t)y}\geq 0$ bo'ladi.

Ushbu $\int_0^{+\infty}f(t,y)dy=\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dy$ integral t o'zgaruvchining $[0,+\infty)$ oraliqda uzluksiz funksiyasi bo'ladi, chunki

$$\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dy=\Gamma(a+в)\cdot\frac{t^{a-1}}{(1+t)^{a+в}}\,.$$

Ushbu

$$\int_0^{+\infty}f(t,y)dt=\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dt$$

integral y o'zgaruvchining $[0,+\infty)$ oraliqda uzluksiz funksiyasi bo'ladi, chunki

$$\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dt=\Gamma(a)y^{в-1}e^{-y}$$

va nihoyat, yuqoridagi (16.36) munosabatga ko'ra

$$\int_0^{+\infty}\left(\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dy\right)dt$$

integral yaqinlashuvchi.

U holda 19-teoremaga asosan

$$\int_0^{+\infty}\left(\int_0^{+\infty}t^{a-1}y^{a+в-1}e^{-(1+t)y}dt\right)dy$$

integral ham yaqinlashuvchi bo'lib,

$$\int_0^{+\infty}\left(\int_0^{+\infty} t^{a-1}y^{a+в-1}e^{-(1+t)y}dy\right)dt=\int_0^{+\infty}\left(\int_0^{+\infty} t^{a-1}y^{a+в-1}e^{-(1+t)y}dt\right)dy$$

bo'ladi. O'ng tomondagi integralni hisoblaymiz:

$$\int_0^{+\infty}\left(\int_0^{+\infty} t^{a-1}y^{a+в-1}e^{-(1+t)y}dy\right)dt=\int_0^{+\infty}\left(\int_0^{+\infty} t^{a-1}y^{a+в-1}e^{-(1+t)y}dt\right)dy=\int_0^{+\infty} y^{a+в-1}e^{-y}\left(\int_0^{+\infty} t^{a-1}e^{-ty}dt\right)dy=$$
$$=\int_0^{+\infty} y^{a+в-1}e^{-y}\frac{1}{y^a}\left(\int_0^{+\infty}(ty)^{a-1}e^{-ty}d(ty)\right)dy=\int_0^{+\infty} y^{в-1}e^{-y}\Gamma(a)dy=\Gamma(a)\Gamma(в) \tag{16.37}$$

Natijada (16.36) va (16.37) munosabatlardan

$$\Gamma(a+в)B(a,в)=\Gamma(a)\,\Gamma(в)$$

ya'ni

$$B(a,в)=\frac{\Gamma(a)\,\Gamma(в)}{\Gamma(a+в)} \tag{16.38}$$

bo'lishi kelib chiqadi. Biz bu formulani $a>1, в>1$ bo'lgan hol uchun isbotladik. Endi umumiy holni ko'raylik.

Aytaylik, $a>0, в>0$ bo'lsin. U holda isbot etilgan (16.38) formulaga ko'ra

$$B(a+1,в+1)=\frac{\Gamma(a+1)\,\Gamma(в+1)}{\Gamma(a+в+2)} \tag{16.39}$$

bo'ladi.

$B(a,в)$ va $\Gamma(a)$ funksiyalarning xossalaridan foydalanib quyidagini topamiz:

$$B(a+1,в+1)=\frac{a}{a+в+1}B(a,в+1)=\frac{a}{a+в+1}\cdot\frac{в}{a+в}B(a,в),$$
$$\Gamma(a+1)=a\Gamma(a),\ \Gamma(в+1)=в\Gamma(в),\ \Gamma(a+в+2)=$$
$$=(a+в+1)\Gamma(a+в+1)=(a+в+1)(a+в)\Gamma(a+в)$$

Natijada (16.39) formula quyidagi

$$\frac{aв}{(a+в)(a+в+1)}B(a,в)=\frac{a\Gamma(a)\,в\Gamma(в)}{(a+в)(a+в+1)\Gamma(a+в)}$$

ko'rinishga keladi. Bu esa (16.38) formula $a>0, в>0$ da ham o'rinli ekanini bildiradi. ▶

1-natija. $\forall a\in(0,1)$ uchun

$$\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin a\pi}$$

bo'ladi.

◀ Haqiqatdan ham, (16.38) formula $в=1-a$ $(0<a<1)$ deyilsa, unda

$$B(a,1-a)=\frac{\Gamma(a)\,\Gamma(1-a)}{\Gamma(1)}$$

bo'lib, $B(a,1-a)=\dfrac{\pi}{\sin a\pi}$ $(0<a<1)$ va $\Gamma(1)=1$ munosabatlarga muvofiq

$$\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin a\pi}\quad(0<a<1).\ ▶ \tag{16.40}$$

Odatda (16.40) formula keltirish formulasi deb ataladi.

Xususan, (16.40) da $a=\frac{1}{2}$ deb olsak, unda

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

bo'lishini topamiz.

2-natija. Ushbu

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a) \quad (a>0)$$

formula o'rinlidir.

◀(16.38) munosabatda $a=в$ deb

$$B(a,a)=\frac{\Gamma(a)\,\Gamma(a)}{\Gamma(2a)}$$

bo'lishini topamiz. So'ngra

$$B(a,a)=\int_0^1[x(1-x)]^{a-1}dx=\int_0^1\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx=2\int_0^{\frac{1}{2}}\left[\frac{1}{4}-\left(\frac{1}{2}-x\right)^2\right]^{a-1}dx$$

integralda $\frac{1}{2}-x=\frac{1}{2}\sqrt{t}$ almashtirishni bajarib,

$$B(a,a)=2\int_0^1\left[\frac{1}{4}(1-t)\right]^{a-1}\frac{1}{4}t^{-\frac{1}{2}}dt=\frac{1}{2^{2a-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{a-1}dt=\frac{1}{2^{2a-1}}B\left(\frac{1}{2},a\right)$$

ga ega bo'lamiz. Natijada

$$\frac{\Gamma^2(a)}{\Gamma(2a)}=\frac{1}{2^{2a-1}}B\left(\frac{1}{2},a\right)$$

bo'ladi.

Yana (16.38) formulaga ko'ra

$$B\left(\frac{1}{2},a\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)}=\sqrt{\pi}\,\frac{\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)}$$

bo'lib, keyingi munosabatlarda

$$\frac{\Gamma(a)}{\Gamma(2a)}=\frac{1}{2^{2a-1}}\sqrt{\pi}\,\frac{1}{\Gamma\left(a+\frac{1}{2}\right)}$$

ekanligi kelib chiqadi. Demak,

$$\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2a-1}}\Gamma(2a).\blacktriangleright \qquad (16.41)$$

Odatda (16.41) formula Lejandr formulasi deb ataladi.

Mashqlar

16.8. Ushbu

$$M=\{(x,y)\in R^2:\ 0\le x\le 1, 0\le y\le 1\}$$

to'plamda berilgan

$$f(x,y)=x^y$$

funksiyaning $y\to 0$ da limit funksiyasi topilsin va unga yaqinlashish notekis bo'lishi ko'rsatilsin.

16.9. Ushbu

$$\int_{-\infty}^{+\infty} f(x,y)dx$$

integral ta'rifi keltirilsin.

16.10. Ushbu

a) $\int_0^{+\infty} e^{-\alpha x^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\quad (\alpha>0)$

b) $\int_0^{+\infty} \frac{cos\,\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}\quad (\alpha>0)$

v) $\int_0^{+\infty} \frac{x\,sin\,\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}\quad (\alpha>0)$

tengliklar isbotlansin.

16.11. Ushbu

$$\int_0^{+\infty} e^{-x}\,cos\,xy dx$$

integralning $(+\infty,-\infty)$ da u parametr bo'yicha tekis yaqinlashishi ko'rsatilsin.

16.12. Ushbu

$$\int_0^1 ln\,\Gamma(x)dx$$

integral hisoblansin.

16.13. Ushbu

$$\lim_{n\to\infty}\int_0^{+\infty} e^{-nx}dx=1$$

tenglik isbotlansin.

17-BOB

Karrali integrallar

Matematika va fanning boshqa tarmoqlarida ko'p o'zgaruvchili funksiyalarning integrallari bilan bog'liq masalalarga duch kelamiz. Binobarin, ularni – karrali integrallarni o'rganish vazifasi yuzaga keladi.

Karrali integrallar nazariyasida ham, aniq integrallar nazariyasidagidek, integralning mavjudligi, uning xossalari, karrali integralni hisoblash, integralning tatbiqlari o'rganiladi. Bunda aniq integral haqidagi ma'lumotlardan muttasil foydalana boriladi.

1-§. Tekis shaklning yuzi hamda fazodagi jismning hajmi haqida ba'zi ma'lumotlar

Aniq integralning tatbiqlari mavzusida tekis shaklning yuzi hamda jismning hajmi haqida ma'lumotlar keltirilgan edi. Bu tushunchalar karrali integrallar nazariyasida muhimligini inobatga olib, ular to'g'risidagi ta'rif va tasdiqlarni talab darajasida bayon etishni lozim topdik.

Aslida, tekis shaklning yuzi, jismning hajmi tushunchalari matematikada muhim bo'lgan to'plamning o'lchovi tushunchasini tekislikdagi shaklga, fazodagi jismga nisbatan aytilishidan iborat.

1^0. Tekis shaklning yuzi va uning mavjudligi. Tekislikda Dekart koordinatalar sistemasi berilgan bo'lsin. Bu tekislikda, sodda yopiq chiziq bilan chegaralangan tekislik qismidan tashkil topgan (Q) shaklni (tekislik nuqtalari to'plamini) qaraylik. (Q) shaklning chegarasini (sodda yopiq chiziq) ∂Q bilan, $(Q)\cup\partial Q$ ni esa $(\overline{Q})$ bilan belgilaymiz:

$$(\overline{Q})=(Q)\cup\partial Q$$

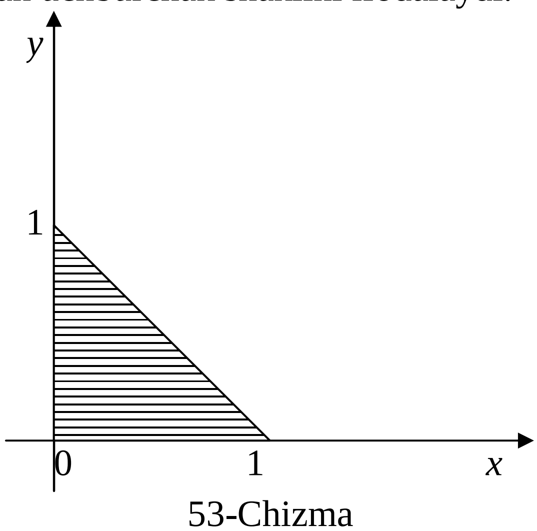
Masalan, koordinatalari ushbu

$$x>0,\ \ y>0,\ \ x+y<1$$

tengsizliklarni qanoatlantiruvchi (x,y) nuqtalardan tashkil topgan to'plam

$$(\Delta)=\{(x,y)\in R^2:\ x>0,\ y>0,\ x+y<1\}$$

53-chizmada tasvirlangan uchburchak shaklini ifodalaydi.



53-Chizma

Ox o'qidagi birlik kesma $(0\le x\le 1)$, Oy o'qidagi birlik kesma $(0\le y\le 1)$ hamda $(1,0)$ va $(0,1)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmalari birgalikda (Δ) uchburchak shaklining chegarasi $\partial\Delta$ ni tashkil etadi.

Tekislikda uchburchaklar, yopiq siniq chiziq bilan chegaralangan tekislik qismidan tashkil topgan ko'pburchaklar yuzaga ega va ularni topish qoidalari o'quvchiga ma'lum deb hisoblaymiz.

Tekislikda (Q) shakl bilan birga (A) va (B) ko'pburchaklarni olaylik.

Agar (A) ko'pburchakning har bir nuqtasi $(\overline{Q})$ ga tegishli bo'lsa, (A) ko'pburchak (Q) shaklning ichiga chizilgan deyiladi (bunda $(A)\subset(\overline{Q})$.

Agar $(\overline{Q})$ ning har bir nuqtasi (B) ko'pburchakka tegishli bo'lsa, (B) ko'pburchak (Q) shaklni o'z ichiga oladi deyiladi (bunda $(\overline{Q})\subset(B)$).

Agar A va B lar mos ravishda (A) va (B) ko'pburchaklarning yuzlari bo'lsa, unda

$$A \le B \qquad (17.1)$$

bo'ladi.

Aytaylik, (Q) shaklning ichiga chizilgan ko'pburchaklar yuzalaridan iborat to'plam $\{A\}$, (Q) shaklni o'z ichiga olgan ko'pburchaklar yuzalaridan iborat to'plam $\{B\}$ bo'lsin. Ravshanki, $\{A\}$ va $\{B\}$ lar sonlar to'plami bo'lib, $\{A\}$ yuqoridan, $\{B\}$ quyidan chegaralangan. Unda to'plamning aniq chegaralari haqidagi teoremaga ko'ra

$$sup\{A\}=Q_*, \quad inf\{B\}=Q^*$$

lar mavjud.

Odatda, Q_* son (Q) shaklning quyi yuzasi, Q^* son esa (Q) shaklning yuqori yuzasi deyiladi.

Tasdiq. Q_* va Q^* miqdorlar uchun

$$Q_* \le Q^* \qquad (17.2)$$

tengsizlik o'rinli bo'ladi.

◀Teskarisini faraz qilaylik, $Q_* > Q^*$ bo'lsin. Bu holda $Q_* - Q^* > 0$ bo'ladi. Aniq chegara ta'riflariga ko'ra $\forall\varepsilon > 0$, jumladan

$$\varepsilon = \frac{1}{2}\left(Q_* - Q^*\right)$$

uchun shunday $(A_0)\subset(\overline{Q})$, $(B_0)\supset(\overline{Q})$ ko'pburchaklar topiladiki,

$$A_0 > Q_* - \varepsilon, \ \ B_0 < Q^* + \varepsilon$$

tengsizliklar bajariladi. Bu tengsizliklardan foydalanib topamiz:

$$B_0 - A_0 < Q^* + \varepsilon - (Q_* - \varepsilon) = Q^* - Q_* + 2\varepsilon = Q^* - Q_* + \left(Q_* - Q^*\right) = 0.$$

Keyingi tengsizlikdan $A_0 > B_0$ bo'lishi kelib chiqadi. Bu esa har doim o'rinli bo'lgan (17.1) munosabatga zid. Demak, (17.2) tengsizlik o'rinli.▶

1-ta'rif. Agar

$$Q_* = Q^*$$

tenglik o'rinli bo'lsa, (Q) shakl yuzaga ega deyiladi.

Ushbu

$$Q_* = Q^*$$

miqdor (Q) shaklning yuzi deyiladi va uni Q orqali belgilanadi:

$$Q = Q_* = Q^*.$$

1-teorema. Tekislikdagi (Q) shakl yuzaga ega bo’lishi uchun $\forall\varepsilon>0$ son olinganda ham (Q) shaklni ichiga chizilgan shunday (A) ko’pburchak, (Q) shaklni o’z ichiga olgan shunday (B) ko’pburchaklar topilib,

$$B-A<\varepsilon \qquad (17.3)$$

tengsizlikning bajarilishi zarur va etarli.

◄***Zarurligi.*** Aytaylik, (Q) shakl yuzaga ega bo’lsin:

$$Q=Q_*=Q^*.$$

Aniq chegara ta’riflariga ko’ra, $\forall\varepsilon>0$ uchun shunday

$$(A)\subset(\overline{Q}),\ \ (B)\supset(\overline{Q})$$

ko’pburchaklar topiladiki,

$$A>Q_*-\frac{\varepsilon}{2},\ \ B<Q^*+\frac{\varepsilon}{2}$$

ya’ni

$$A>Q-\frac{\varepsilon}{2},\ \ B<Q+\frac{\varepsilon}{2}$$

bo’ladi. Bu tengsizliklardan

$$B-A<\varepsilon$$

bo’lishi kelib chiqadi.

Etarliligi. Aytaylik, $(A)\subset(\overline{Q})$, $(B)\supset(\overline{Q})$ ko’pburchaklar uchun

$$B-A<\varepsilon$$

tengsizlik bajarilsin.

Ravshanki, $A\le Q_*$, $B\ge Q^*$. Yuqoridagi (17.2) munosabatdan foydalanib topamiz:

$$A\le Q_*\le Q^*\le B.$$

Bu va (17.3) tengsizlikka ko’ra

$$Q^*-Q_*\le B-A<\varepsilon$$

bo’ladi. Demak, $Q_*=Q^*$.►

Faraz qilaylik, tekislikda l chiziq (u yopiq yoki yopiq bo’lmasligi mumkin) berilgan bo’lsin.

2-ta’rif. Agar shunday (A_0) ko’pburchak topilsaki,

1) $l\subset(A_0)$;

2) $\forall\varepsilon>0$ uchun $A_0<\varepsilon$ bo’lsa, l nol yuzali chiziq deyiladi.

Tasdiq. Agar l chiziq $[a,в]$ segmentda uzluksiz bo’lgan $f(x)$ funksiyaning grafigidan iborat bo’lsa, u nol yuzali chiziq bo’ladi.

◄ $\forall\varepsilon>0$ sonni olib, $[a,в]$ segmentini shunday

$$[x_k,\ x_{k+1}]\quad (k=0,1,2,...,n-1;\ \ x_0=a,\ \ x_n=в)$$

bo’laklarga ajratamizi, har bir $[x_k,\ x_{k+1}]$ da $f(x)$ funksiyaning tebranishi

$$\omega_k<\frac{\varepsilon}{в-a}$$

bo’lsin.U holda l chiziqni o’z ichiga olgan (A_0) ko’pburchakning yuzi

$$A_0=\sum_{k=0}^{n-1}(M_k-m_k)(x_{k+1}-x_k)$$
bo'ladi, bunda
$$\begin{aligned} M_k &= sup\{f(x)\}, \quad x\in[x_{k+1},x_k], \\ m_k &= inf\{f(x)\}, \quad x\in[x_k,x_{k+1}]. \end{aligned} \quad (k=0,1,2,...,n-1)$$
Ravshanki,
$$A_0=\sum_{k=0}^{n-1}\omega_k\Delta x_k<\frac{\varepsilon}{в-a}\sum_{k=0}^{n-1}\Delta x_k=\varepsilon \quad (\Delta x_k=x_{k+1}-x_k).$$
Demak, l nol yuzali chiziq. ▶

Bu tushuncha yordamida yuqoridagi 1-teoremani quyidagicha ifodalasa bo'ladi.

2-teorema. Tekislikdagi (Q) shakl yuzaga ega bo'lishi uchun uning chegarasi ∂Q nol yuzali chiziq bo'lishi zarur va etarli.

Natija. Agar (Q) shaklning chegarasi ∂Q har biri $y=f(x)\in C[a,в]$ yoki $x=g(y)\in C[c,d]$ funksiyalar tasvirlangan bir nechta egri chiziqlardan tashkil topgan bo'lsa, u holda (Q) shakl yuzaga ega bo'ladi.

2⁰. Yuzaning xossalari. Endi yuzaning asosiy xossalarini keltiramiz.

1). Agar tekislikdagi (Q_1), (Q_2) shakllar yuzaga ega bo'lib, $(\overline{Q_1})\subset(\overline{Q_2})$ bo'lsa, u holda
$$Q_1\le Q_2$$
bo'ladi.

2). Agar (Q_1) va (Q_2) shakllar yuzaga ega bo'lsa, u holda $(Q_1)\cup(Q_2)$ ham yuzaga ega bo'lib, $(Q_1)\cup(Q_2)$ shaklning yuzi (Q_1) va (Q_2) shakllar yuzalarining yig'indisidan katta bo'lmaydi.

Agar bu (Q_1) va (Q_2) shakllar chegaralaridan boshqa umumiy nuqtaga ega bo'lmasa, ya'ni
$$(Q_1)\cap(Q_2)=\varnothing$$
bo'lsa, u holda $(Q_1)\cup(Q_2)$ shaklning yuzi (Q_1) va (Q_2) shakllar yuzalarining yig'indisiga teng bo'ladi. Bu yuzaning additivlik xossasi deyiladi.

3⁰. Tekis shaklni bo'laklash. Tekislikda biror yuzaga ega (Q) shakl berilgan bo'lib,
$$(Q_1),(Q_2),...,(Q_n)$$
shakllar uning yuzaga ega bo'lgan qismiy shakllari, ya'ni
$$(Q_k)\subset(\overline{Q}) \quad (k=1,2,...,n)$$
bo'lsin. Agar $(Q_1),(Q_2),...,(Q_n)$ shakllar uchun

1) $(Q_1)\cup(Q_2)\cup...\cup(Q_n)=\overline{Q}$,

2) ixtiyoriy (Q_k) va (Q_i) lar $(k=1,2,...,n,\ i=1,2,...,n)$ umumiy nuqtaga (chegaradagi nuqtalardan boshqa) ega bo'lmasa, $(Q_1),(Q_2),...,(Q_n)$ lar (Q) da bo'laklash bajaradi yoki (Q) shakl $(Q_1),(Q_2),...,(Q_n)$ shakllarga bo'laklangan deyiladi. (Q) shaklni $(Q_1),(Q_2),...,(Q_n)$ larga bo'laklashni P bilan belgilanadi:

$$P=\{(Q_1),(Q_2),\dots,(Q_n)\}.$$

Ushbu

$$d((Q_k))=\sup\rho((x',y'),(x'',y''))$$
$$((x',y')\in(Q_k),\ \ (x'',y'')\in(Q_k)\quad k=1,2,3,\dots,n)$$

miqdorlarning eng kattasi P bo'laklashning diametri deyiladi va λ_P kabi belgilanadi:

$$\lambda_P=\max_{1\le k\le n} d((Q_k))$$

Masalan, ushbu

$$(Q_{ki})=\{(x,y)\in R^2:\ x_k\le x\le x_{k+1},\ y_i\le y\le y_{i+1}\}$$
$$(k=0,1,2,\dots,n-1;\ \ i=0,1,2,\dots,m-1;\ x_0=a,\ x_n=в,y_0=c,y_m=d)$$

to'g'ri to'rtburchaklar

$$(Q)=\{(x,y)\in R^2:\ a\le x\le в,\ c\le y\le d\}$$

shaklni P bo'laklashni hosil qiladi, bunda

$$\lambda_P=\max_{\substack{0\le k\le n-1\\ 0\le i\le m-1}}\left(\sqrt{\Delta x_k^2+\Delta y_i^2}\right)$$

$$\Delta x_k=x_{k+1}-x_k,\ \ \Delta y_i=y_{i+1}-y_i$$

4^0. R^3 ***fazoda jismning hajmi.*** R^3 fazoda Dekart koordinatalar sistemasi berilgan bo'lsin. Bu fazoda, chegaralangan yopiq sirt bilan (yoki bunday sirtlarning bir nechtasi bilan) o'ralgan (V) jismni (R^3 fazo qismini) qaraylik. (V) jismni o'rab to'rgan sirtni - (V) jismning chegarasini ∂V bilan, $(V)\cup\partial V$ ni $(\overline{V})$ bilan belgilaymiz:

$$(\overline{V})=(V)\cup\partial V\ .$$

Masalan, koordinatalari ushbu

$$x^2+y^2+z^2<1$$

tengsizlikni qanoatlantiruvchi (x,y,z) nuqtalardan tashkil topgan

$$(S)=\{(x,y,z)\in R^3:\ \ x^2+y^2+z^2<1\}$$

to'plam, markazi $(0,0,0)$ nuqtada, radiusi 1 ga teng shartni – jismni ifodalaydi. Uning chegarasi

$$\partial S=\{(x,y,z)\in R^3:\ \ x^2+y^2+z^2=1\}$$

sfera bo'ladi. Bunday jism-shar hajmiga ega va $V=\frac{4}{3}\pi$ ga teng. Umuman, fazoda ko'pyoqliklarning hajmga ega bo'lishi va uni topish qoidalari o'quvchiga ma'lum deb hisoblaymiz.

Endi R^3 fazoda (V) jism bilan birga (F) va (G) ko'pyoqlarni qaraymiz.

Agar (F) ko'pyoqlikning har bir nuqtasi $(\overline{V})$ ga tegishli bo'lsa, (F) ko'pyoqlik (V) jismning ichiga joylashgan deyiladi (bunda $(F)\subset(\overline{V})$).

Agar $(\overline{V})$ ning har bir nuqtasi (G) ko'pyoqlikka tegishli bo'lsa, (G) ko'pyoqlik (V) jismni o'z ichiga oladi deyiladi (bunda $(\overline{V})\subset(G)$).

Agar F va G lar mos ravishda (F) va (G) ko'pyoqliklarning hajmlari bo'lsa, unda

$$F \leq G$$

bo'ladi.

Aytaylik, (V) jismning ichiga joylashgan ko'pyoqliklar hajmlaridan iborat to'plam $\{F\}$, jismning o'z ichiga olgan ko'pyoqliklar hajmlaridan iborat to'plam $\{G\}$ bo'lsin. Unda

$$sup\{F\} = V_*, \quad inf\{G\} = V^*$$

lar mavjud.

3-ta'rif. Agar

$$V_* = V^*$$

tenglik o'rinli bo'lsa, (V) jism hajmga ega deyiladi.

Ushbu

$$V_* = V^*$$

miqdor (V) jism hajmga ega deyiladi. Uni V kabi belgilanadi:

$$V = V_* = V^*$$

Tekis shaklning yuzi, fazodagi jismning hajmi tushunchalarida bir-biriga o'xshashlik borligini inobatga olib, jism hajmining mavjudligi haqidagi teoremani keltirish bilan kifoyalanamiz.

3-teorema. Fazodagi (V) jism hajmga ega bo'lishi uchun $\forall \varepsilon > 0$ son olinganda ham (V) jismning ichida joylashgan shunday (F) ko'pyoqlik, (V) jismni o'z ichiga olgan shunday (G) ko'pyoqliklar topilib, ular uchun

$$G - F < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

2-§. Ikki karrali integral ta'riflari

1^0. Integralning ta'rifi. Tekislikda biror chegaralangan (D) soha (shakl) berilgan bo'lsin. Bu sohaning bo'laklashlari to'plamini $\Im$ bilan belgilaymiz.

Aytaylik, (D) sohada $f(x, y)$ aniqlangan. Bu (D) sohaning

$$P = \{(D_1), (D_2), ..., (D_n)\} \subset \Im$$

bo'laklashini va bu bo'laklashning har bir (D_k) $(k = 1, 2, ..., n)$ bo'lagida ixtiyoriy (ξ_k, η_k) $(k = 1, 2, ..., n)$ nuqtani olaylik. Berilgan funksiyaning (ξ_k, η_k) nuqtadagi qiymati $f(\xi_k, \eta_k)$ ni D_k ($D_k - (D_k)$ sohaning yuzi) ga ko'paytirib, quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k$$

yig'indini tuzamiz.

1-ta'rif. Ushbu

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k$$

yig'indi, $f(x,y)$ funksiyaning integral yig'indisi yoki Riman yig'indisi deb ataladi.

Masalan, $f(x,y)=xy$ funksiyaning (D) sohadagi integral yig'indisi

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k=\sum_{k=1}^{n}\xi_k\eta_k D_k$$

bo'ladi, bunda

$$(\xi_k,\eta_k)\in(D_k)\quad(k=1,2,...,n)$$

Yuqorida keltirilgan ta'rifdan ko'rinadiki, $f(x,y)$ funksiyaning integral yig'indisi σ qaralayotgan $f(x,y)$ funksiyaga, (D) sohaning bo'laklash usuliga ham har bir (D_k) dan olingan (ξ_k,η_k) nuqtalarga bog'liq bo'ladi:

$$\sigma_P=\sigma_P(f;\xi_k,\eta_k).$$

Endi (D) sohaning shunday

$$P_1,P_2,...,P_m,... \qquad (17.4)$$

bo'laklashlarni qaraymizki, ularning diametrlaridan tashkil topgan

$$\lambda_{P_1},\lambda_{P_2},...,\lambda_{P_m},...$$

ketma-ketlik nolga intilsin: $\lambda_{P_m}\to 0$. Bunday P_m $(m=1,2,...)$ bo'laklashlarga nisbatan $f(x,y)$ funksiyaning integral yig'indisini tuzamiz:

$$\sigma_m=\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k .$$

Natijada D sohaning (17.4) bo'laklariga mos $f(x,y)$ funksiya integral yig'indilari qiymatlaridan iborat quyidagi

$$\sigma_1,\sigma_2,...,\sigma_m,...$$

ketma-ketlik hosil bo'ladi. Bu ketma-ketlikning har bir hadi (ξ_k,η_k) nuqtalarga bog'liq.

2-ta'rif. Agar (D) sohaning har qanday (17.4) bo'laklashlar ketma-ketligi $\{P_m\}$ olinganda ham, unga mos integral yig'indi qiymatlaridan iborat $\{\sigma_m\}$ ketma-ketlik, (ξ_k,η_k) nuqtalarni tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta J songa intilsa, bu J son σ yig'indining limiti deb ataladi va u

$$\lim_{\lambda_P\to 0}\sigma=\lim_{\lambda_P\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k=J$$

kabi belgilanadi.

Integral yig'indining limitini quyidagicha ham ta'riflash mumkin.

3-ta'rif. Agar $\forall\varepsilon>0$ son olinganda ham, shunday $\delta>0$ topilsaki, (D) sohaning diametri $\lambda_p<\delta$ bo'lgan har qanday P bo'laklashi hamda har bir (D_k) bo'lakdagi ixtiyoriy (ξ_k,η_k) lar uchun

$$|\sigma-J|<\varepsilon$$

tengsizlik bajarilsa, J son σ yig'indining limiti deb ataladi va u

$$\lim_{\lambda_P\to 0}\sigma=J$$

kabi belgilanadi.

4-ta'rif. Agar $\lambda_P \to 0$ da $f(x,y)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, $f(x,y)$ funksiya (D) sohada integrallanuvchi (Riman ma'nosida integrallanuvchi) funksiya deyiladi. Bu σ yig'indining chekli limiti J esa $f(x,y)$ funksiyaning (D) soha bo'yicha ikki karrali integrali (Riman integrali) deyiladi va u

$$\iint\limits_{(D)} f(x,y)dD$$

kabi belgilanadi. Demak,

$$\iint\limits_{(D)} f(x,y)dD = \lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k,\eta_k)D_k .$$

Masalan, $f(x,y)=C$ $(C=const)$ funksiyaning (D) soha bo'yicha integral yig'indisi

$$\sigma = \sum_{k=1}^{n} C \cdot D_k = C \cdot D$$

bo'lib, $\lambda_P \to 0$ da $\lim\limits_{\lambda_P \to 0} \sigma = CD$ bo'ladi. Demak,

$$\iint\limits_{(D)} CdD = CD .$$

Xususan, $f(x,y)=1$ bo'lganda

$$\iint\limits_{(D)} dD = D$$

bo'ladi.

1-eslatma. Agar $f(x,y)$ funksiya (D) sohada chegaralanmagan bo'lsa, u shu sohada integrallanmaydi.

2^0. Darbu yig'indilari. Ikki karrali integralning boshqacha ta'rifi.

1). Darbu yig'indilari. $f(x,y)$ funksiya $(D)\subset R^2$ sohada berilgan bo'lib, u shu sohada chegaralangan bo'lsin. Demak, shunday o'zgarmas m va M sonlar mavjudki, $\forall(x,y)\in(D)$ da

$$m \le f(x,y) \le M$$

bo'ladi.

(D) sohaning biror P bo'laklashni olaylik. Bu bo'laklashning har bir (D_k) $(k=1,2,...,n)$ bo'lagida $f(x,y)$ funksiya chegaralangan bo'lib, uning aniq chegaralari

$$m_k = \inf\{f(x,y):\ (x,y)\in(D_k)\},\ \ M_k = \sup\{f(x,y):\ (x,y)\in(D_k)\}$$

mavjud bo'ladi. Ravshanki, $\forall(x,y)\in(D_k)$ uchun

$$m_k \le f(x,y) \le M_k \tag{17.5}$$

tengsizliklar o'rinli.

5-ta'rif. Ushbu

$$s = \sum_{k=1}^{n} m_k D_k\ ,\ \ S = \sum_{k=1}^{n} M_k D_k$$

yig'indilar mos ravishda Darbuning quyi hamda yuqori yig'indilari deb ataladi.

Bu ta'rifdan, Darbu yig'indilarining $f(x,y)$ funksiyaga hamda (D) sohaning bo'laklashiga bog'liq ekanligi ko'rinadi:

$$s = s_P(f), \quad S = S_P(f).$$

Shuningdek, har doim

$$s \le S$$

bo'ladi.

Yuqoridagi (17.5) tengsizlikdan foydalanib quyidagini topamiz:

$$\sum_{k=1}^{n} m_k D_k \le \sum_{k=1}^{n} f(\xi_k, \eta_k) D_k \le \sum_{k=1}^{n} M_k D_k .$$

Demak,

$$s_P(f) \le \sigma_P(f; \xi_k, \eta_k) \le S_P(f).$$

Shunday qilib, $f(x,y)$ funksiyaning integral yig'indisi har doim uning Darbu yig'indilari orasida bo'lar ekan.

Aniq chegaraning xossasiga ko'ra

$$m \le m_k, \ M_k \le M \quad (k = 1, 2, ..., n)$$

bo'ladi. Natijada ushbu

$$s = \sum_{k=1}^{n} m_k D_k \ge m \sum_{k=1}^{n} D_k = mD,$$

$$S = \sum_{k=1}^{n} M_k D_k \ge m \sum_{k=1}^{n} D_k = MD$$

tengsizliklarga kelamiz. Demak, $\forall P \in \Im$ uchun

$$mD \le s \le S \le MD \tag{17.6}$$

bo'ladi. Bu esa Darbu yig'indilarining chegaralanganligini bildiradi.

2) Ikki karrali integralning boshqacha ta'rifi. $f(x,y)$ funksiya $(D) \subset R^2$ sohada brilgan bo'lib, u shu sohada chegaralangan bo'lsin. (D) sohaning bo'laklashlari to'plami $\Im = \{P\}$ ning har bir $P \in \Im$ bo'laklashiga nisbatan $f(x,y)$ funksiyaning Darbu yig'indilari $s_P(f)$, $S_P(f)$ ni tuzib,

$$\{s_P(f)\}, \ \{S_P(f)\}$$

to'plamlarni qaraymiz. Bu to'plamlar (17.6) ga ko'ra chegaralangan bo'ladi.

6-ta'rif. $\{s_P(f)\}$ to'plamning aniq yuqori chegarasi $f(x,y)$ funksiyaning (D) sohadagi quyi ikki karrali integrali (quyi Riman integrali) deb ataladi va u

$$\underline{J} = \iint_{(\underline{D})} f(x,y)\,dD$$

kabi belgilanadi.

$\{S_P(f)\}$ to'plamning aniq quyi chegarasi $f(x,y)$ funksiyaning (D) sohadagi yuqori ikki karrali integrali (yuqori Riman integrali) deb ataladi va u

$$\overline{J} = \overline{\iint}_{(D)} f(x,y)\,dD$$

kabi belgilanadi. Demak,

$$\underline{J} = \iint\limits_{(\overline{D})} f(x,y)dD = sup\{s\},\ \overline{J} = \overline{\iint\limits_{(D)}} f(x,y)dD = inf\{S\}.$$

7-ta'rif. Agar $f(x,y)$ funksiyaning (D) sohada quyi hamda yuqori ikki karrali integrallar bir-biriga teng bo'lsa, $f(x,y)$ funksiya (D) sohada integrallanuvchi deb ataladi, ularning umumiy qiymati

$$J = \iint\limits_{(\overline{D})} f(x,y)dD = \overline{\iint\limits_{(D)}} f(x,y)dD\ .$$

$f(x,y)$ funksiyaning (D) sohadagi ikki karrali integrali (Riman integrali) deyiladi va u

$$\iint\limits_{(D)} f(x,y)dD$$

kabi belgilanadi. Demak,

$$\iint\limits_{(D)} f(x,y)dD = \iint\limits_{(\overline{D})} f(x,y)dD = \overline{\iint\limits_{(D)}} f(x,y)dD\ .$$

Agar

$$\iint\limits_{(\overline{D})} f(x,y)dD \neq \overline{\iint\limits_{(D)}} f(x,y)dD$$

bo'lsa, $f(x,y)$ funksiya (D) sohada integrallanmaydi deb ataladi.

3-§. Ikki karrali integralning mavjudligi

$f(x,y)$ funksiyaning $(D)\subset R^2$ soha bo'yicha ikki karrali integrali mavjudligi masalasini qaraymiz. Buning uchun avvalo (D) sohaning hamda Darbu yig'indilarining xossalarini keltiramiz.

(D) sohaning bo'laklashlari xossalari 1-qism, 9-bobda o'rganilgan $[a,в]$ segmentning bo'laklashlari xossalari kabidir. Ularni isbotlash deyarli bir xil mulohaza asosida olib borilishini e'tiborga olib, quyidagi u xossalarni isbotsiz keltirishni lozim topdik.

$f(x,y)$ funksiyaning Darbu yig'indilari xossalari haqidagi vaziyat ham xuddi shundaydir.

Faraz qilaylik, $\Im - \{P\} - (D)$ soha bo'laklashlari to'plami bo'lib, $P_1 \in \Im,\ P_2 \in \Im$ bo'lsin:

$$P_1 = \{(D_1),(D_2),...,(D_n)\}$$
$$P_2 = \{(D_1'),(D_2'),...,(D_{n'}')\}.$$

Agar P_1 bo'laklashdagi har bir (D_i) $(i=1,2,...,n)$ P_2 bo'laklashdagi biror (D_i') $(i=1,2,...,n')$ ning qismi bo'lsa, P_1 bo'laklash P_2 ni ergashtiradi deyiladi va $P_1 \subset P_2$ kabi yoziladi. Ravshanki, $P_1 \subset P_2$ bo'lsa,

$$\lambda_{P_1} \leq \lambda_{P_2}$$

bo'ladi.

1⁰. Darbu yig'indilarining xossalari. $f(x,y)$ funksiya (D) sohada berilgan va chegaralangan bo'lsin. (D) sohaning P bo'laklashini olib, bu bo'laklashga nisbatan $f(x,y)$ funksiyaning integral va Darbu yig'indilarini tuzamiz:

$$\sigma=\sigma_P(f,\xi_k,\eta_k)=\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k\,,$$

$$s=s_P(f)=\sum_{k=1}^{n} m_k D_k\,,$$

$$S=S_P(f)=\sum_{k=1}^{n} M_k D_k\,.$$

1) $\forall\varepsilon>0$ olinganda ham $(\xi_k,\eta_k)\in(D_k)$ nuqtalarni $(k=1,2,...,n)$ shunday tanlab olish mumkinki,

$$0\le S_P(f)-\sigma_P(f)<\varepsilon,$$

shuningdek, $(\xi_k,\eta_k)\in(D_k)$ $(k=1,2,...,n)$ nuqtalarini yana shunday tanlab olish mumkinki,

$$0\le\sigma_P(f)-s_P(f)<\varepsilon$$

bo'ladi.

Bu xossa Darbu yig'indilari $s_P(f)$, $S_P(f)$ lar uchun integral yig'indi $\sigma_P(f)$ muayyan bo'laklash uchun mos ravishda aniq quyi hamda aniq yuqori chegara bo'lishini bildiradi.

2) Agar P_1 va P_2 lar (D) sohaning ikki bo'laklashlari bo'lib, $P_1\subset P_2$ bo'lsa, u holda

$$s_{P_1}(f)\le s_{P_2}(f),\ \ S_{P_1}(f)\ge S_{P_2}(f)$$

bo'ladi.

Bu xossa (D) sohaning bo'laklashdagi bo'laklar soni orta borganda ularga mos Darbuning quyi yig'indisining kamaymasligi, yuqori yig'indisining esa oshmasligini bildiradi.

3) Agar P_1 va P_2 lar (D) sohaning ixtiyoriy ikki bo'laklashlari bo'lib, $s_{P_1}(f)$, $S_{P_1}(f)$ va $s_{P_2}(f)$, $S_{P_2}(f)$ lar $f(x,y)$ funksiyaning shu bo'laklashlarga nisbatan Darbu yig'indilari bo'lsa, u holda

$$s_{P_1}(f)\le S_{P_2}(f),\ \ s_{P_2}(f)\le S_{P_1}(f)$$

bo'ladi.

Bu xossa, (D) sohaning bo'laklashlariga nisbatan tuzilgan quyi yig'indilar to'plami $\{s_p(f)\}$ ning har bir elementi (yuqori yig'indilar to'plami $\{S_p(f)\}$ ning har bir elementi) yuqori yig'indilari to'plami $\{S_p(f)\}$ ning istalgan elementidan (quyi yig'indilar to'plami $\{s_p(f)\}$ ning istalgan elementidan) katta (kichik) emasligini bildiradi.

4) Agar $f(x,y)$ funksiya (D) sohada berilgan va chegaralangan bo'lsa, u holda

$$sup\{s_P(f)\}\le inf\{S_P(f)\}$$

bo'ladi.

Bu xossa $f(x,y)$ funksiyaning quyi ikki karrali integrali, uning yuqori ikki karrali integralidan katta emasligini bildiradi:

$$\underline{J} \le \overline{J}$$

5) Agar $f(x,y)$ funksiya (D) sohada berilgan va chegaralangan bo'lsa, u holda $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topiladiki, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan barcha bo'laklashlari uchun

$$\begin{aligned} &S_P(f)<\overline{J}+\varepsilon \quad \left(0\le S_P(f)-\overline{J}<\varepsilon\right) \\ &s_P(f)>\underline{J}-\varepsilon \quad \left(0\le \underline{J}-s_P(f)<\varepsilon\right) \end{aligned} \tag{17.7}$$

bo'ladi.

Bu xossa $f(x,y)$ funksiyaning yuqori hamda quyi integrallari $\lambda_P\to 0$ da mos ravishda Darbuning yuqori hamda quyi yig'indilarining limiti ekanligini bildiradi:

$$\overline{J}=\lim_{\lambda_P\to 0} S_P(f),\ \ \underline{J}=\lim_{\lambda_P\to 0} s_P(f)$$

2⁰. Ikki karrali integralning mavjudligi. Endi ikki karrali integralning mavjud bo'lishining zarur va etarli shartini (kriteriysini) keltiramiz.

1-teorema. $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lishi uchun, $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilib, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashga nisbatan Darbu yig'indilari

$$S_P(f)-s_P(f)<\varepsilon \tag{17.8}$$

tengsizlikni qanoatlantirilishi zarur va etarli.

◄***Zarurligi.*** $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsin. Ta'rifga ko'ra

$$J=\underline{J}=\overline{J}$$

bo'ladi, bunda

$$\underline{J}=sup\{s_P(f)\},\ \ \overline{J}=inf\{S_P(f)\}$$

$\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{2}$ ga ko'ra shunday $\delta>0$ topiladiki, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashiga nisbatan Darbu yig'indilari uchun (17.7) munosabatlarga ko'ra

$$S_p(f)-\overline{J}<\frac{\varepsilon}{2},\ \ \underline{J}-s_P(f)<\frac{\varepsilon}{2}$$

bo'lib, undan

$$S_p(f)-s_P(f)<\varepsilon$$

bo'lishi kelib chiqadi.

Etarliligi. $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilib, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashga nisbatan Darbu yig'indilari uchun

$$S_p(f)-s_P(f)<\varepsilon$$

bo'lsin. Qaralayotgan $f(x,y)$ funksiya (D) sohada chegaralangani uchun, uning quyi hamda yuqori integrallari

$$\underline{J} = sup\{s_P(f)\}, \quad \overline{J} = inf\{S_P(f)\}$$

mavjud va

$$\underline{J} \le \overline{J}$$

bo'ladi. Ravshanki,

$$s_P(f) \le \underline{J} \le \overline{J} \le S_P(f).$$

Bu munosabatdan

$$0 \le \overline{J} - \underline{J} \le S_P(f) - s_P(f)$$

bo'lishini topamiz. Demak, $\forall \varepsilon > 0$ uchun

$$0 \le \overline{J} - \underline{J} < \varepsilon$$

bo'lib, undan $\underline{J} = \overline{J}$ bo'lishi kelib chiqadi. Bu esa $f(x,y)$ funksiyaning (D) sohada integrallanuvchi ekanligini bildiradi.▶

Agar $f(x,y)$ funksiyaning (D_k) $(k = 1, 2, ..., n)$ sohadagi tebranishini ω_k bilan belgilasak, u holda

$$S_P(f) - s_P(f) = \sum_{k=1}^{n}(M_k - m_k)D_k = \sum_{k=1}^{n}\omega_k D_k$$

bo'lib, teoremadagi (17.8) shart ushbu

$$\sum_{k=1}^{n}\omega_k D_k < \varepsilon$$

ya'ni

$$\lim_{\lambda_P \to 0}\sum_{k=1}^{n}\omega_k D_k = 0$$

ko'rinishlarni oladi.

4-§. Integrallanuvchi funksiyalar sinfi

Ushbu paragrafda ikki karrali integralning mavjudligi haqidagi teoremadan foydalanib, ma'lum sinf funksiyalarining integrallanuvchi bo'lishini ko'rsatamiz.

2-teorema. Agar $f(x,y)$ funksiya chegaralangan yopiq $(D) \subset R^2$ sohada berilgan va uzluksiz bo'lsa, u shu sohada integrallanuvchi bo'ladi.

◀ $f(x,y)$ funksiya (D) sohada tekis uzluksiz bo'ladi. U holda Kantor teoremasining natijasiga asosan (12-bob, 6-§), $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topiladiki, (D) sohaning diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'laklashida

$$S_P(f) - s_P(f) = \sum_{k=1}^{n}\omega_k D_k < \varepsilon \sum_{k=1}^{n} D_k = \varepsilon D$$

bo'lib, undan

$$\lim_{\lambda_P \to 0}\sum_{k=1}^{n}\omega_k D_k = 0$$

bo'lishi kelib chiqadi. Demak, $f(x,y)$ funksiya (D) sohada integrallanuvchi.▶

(D) sohada nol yuzli Γ chiziq berilgan bo'lsin.

1-lemma. $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topiladiki, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan P bo'laklashi olinganda bu bo'laklashning Γ chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari yuzlarining yig'indisi ε dan kichik bo'ladi.

◀Shartga ko'ra Γ - nol yuzli chiziq. Demak, uni shunday (Q) ko'pbo'rchak bilan o'rash mumkinki, bu ko'pburchakning yuzi $Q<\varepsilon$ bo'ladi.

Γ chiziq bilan (Q) ko'pbo'rchak chegarasi umumiy nuqtaga ega emas deb, Γ chiziq nuqtalari bilan (Q) ko'pburchak chegarasi nuqtalari orasidagi masofani qaraylik. Bu nuqtalar orasidagi masofa o'zining eng kichik qiymatiga erishadi. Biz uni $\delta>0$ orqali belgilaymiz. Agar (D) sohaning diametri $\lambda_P<\delta$ bo'lgan P bo'laklashi olinsa, ravshanki, bu bo'laklashning Γ chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari butunlay (Q) ko'pburchakda joylashadi. Demak, bunday bo'laklar yuzlarining yig'indisi ε dan kichik bo'ladi.▶

3-teorema. Agar $f(x,y)$ funksiya (D) sohada chegaralangan va bu sohaning chekli sondagi nol yuzli chiziqlarida uzilishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, funksiya (D) sohada integrallanuvchi bo'ladi.

◀ $f(x,y)$ funksiya (D) sohada chegaralangan bo'lib, u shu sohaning faqat bitta nol yuzli Γ chizig'ida $(\Gamma\subset(D))$ uzilishga ega bo'lib qolgan barcha nuqtalarda uzluksiz bo'lsin.

$\forall\varepsilon>0$ sonni olib, Γ chiziqni yuzi ε dan kichik bo'lgan (Q) ko'pburchak bilan o'raymiz. Natijada (D) soha (Q) va $(D)\backslash(Q)$ sohalarga ajraladi.

Shartga ko'ra, $f(x,y)$ funksiya $(D)\backslash(Q)$ da uzluksiz. Demak, $\forall\varepsilon>0$ olinganda ham shunday $\delta_1>0$ topiladiki, diametri $\lambda_{P_1}<\delta_1$ bo'lgan P_1 bo'laklashning har bir bo'lagidagi $f(x,y)$ funksiyaning tebranishi $\omega_k<\varepsilon$ bo'ladi.

Yuqoridagi lemmaning isbot jarayoni ko'rsatadiki, shu $\varepsilon>0$ ga ko'ra, shunday $\delta_2>0$ topiladiki, (D) sohaning diametri $\lambda_P<\delta_2$ bo'lgan bo'laklashi olinsa, bu bo'laklashning (Q) ko'pburchak bilan umumiy nuqtaga ega bo'lgan bo'laklar yuzlarining yig'indisi ε dan kichik bo'ladi.

Endi $min\{\delta_1,\delta_2\}=\delta$ deb, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan P bo'laklashini olamiz. Bu bo'laklashga nisbatan $f(x,y)$ funksiyaning Darbu yig'indilarini tuzib, quyidagi

$$S_P(f)-s_P(f)=\sum_{k=1}^{n}\omega_k D_k \qquad (17.9)$$

ayirmani qaraymiz.

Bu (17.9)yig'indining (Q) ko'pburchakdan tashqari joylashgan (D_k) bo'laklarga mos hadlaridan iborat yig'indi

$$\sum_k{}' \omega_k D_k$$

bo'lsin.

(17.9) yig'indining qolgan barcha hadlaridan tashkil topgan yig'indi

$$\sum_{k}{}'' \omega_k D_k$$

bo’lsin. Natijada (17.9) yig’indi ikki qismga ajraladi:

$$\sum_{k=1}^{n} \omega_k D_k = \sum_{k}{}' \omega_k D_k + \sum_{k}{}'' \omega_k D_k \qquad (17.10)$$

$(D)\setminus(Q)$ sohadagi bo’laklarda $\omega_k < \varepsilon$ bo’lganligidan

$$\sum_{k}{}' \omega_k D_k < \varepsilon \sum_{k}{}' D_k \le \varepsilon D \qquad (17.11)$$

bo’ladi.

Agar $f(x,y)$ funksiyaning (D) sohadagi tebranishini Ω bilan belgilasak, u holda

$$\sum_{k}{}'' \omega_k D_k \le \Omega \sum_{k}{}' D_k$$

bo’ladi. (Q) ko’pburchakda butunlay joylashgan P bo’laklashning bo’laklari yuzlarining yig’indisi ε dan kichik hamda (Q) ko’pburchak chegarasi bilan umumiy nuqtaga ega bo’lgan bo’laklar yuzlarining yig’indisi ham ε dan kichik bo’lishini e’tiborga olsak, unda

$$\sum_{k}{}'' D_k < 2\varepsilon$$

bo’lishini topamiz. Demak,

$$\sum_{k}{}'' \omega_k D_k < 2\Omega\varepsilon . \qquad (17.12)$$

Natijada, (17.10), (17.11) va (17.12) munosabatlardan

$$\sum_{k=1}^{n} \omega_k D_k < \varepsilon D + 2\Omega\varepsilon = \varepsilon(D + 2\Omega)$$

ekanligi kelib chiqadi. Demak,

$$\lim_{\lambda_P \to 0} \sum_{k=1}^{n} \omega_k D_k = 0 .$$

Bu esa $f(x,y)$ funksiyaning (D) sohada integrallanuvchi bo’lishini bildiradi.

$f(x,y)$ funksiya (D) sohaning chekli sondagi nol yuzli chiziqlarida uzilishga ega bo’lib, barcha nuqtalarida uzluksiz bo’lsa, uning (D) da integrallanuvchi bo’lishi yuqoridagidek isbot etiladi.▶

5-§. Ikki karrali integralning xossalari

Quyida $f(x,y)$ funksiya ikki karrali integralining xossalarini o’rganamiz.

Ikki karrali integral ham aniq integralning xossalari singari xossalarga ega. Ularni asosan isbotsiz keltiramiz.

1) $f(x,y)$ funksiya (D) sohada $\left((D)\subset R^2\right)$ integrallanuvchi bo’lsin. Bu funksiyaning (D) sohaga tegishli bo’lgan nol yuzli L chiziqdagi $(L\subset(D))$

qiymatlarinigina (chegaralanganligini saqlagan holda) o'zlashtirishdan hosil bo'lgan $F(x,y)$ funksiya ham (D) sohada integrallanuvchi bo'lib,

$$\iint_{(D)} f(x,y)dD = \iint_{(D)} F(x,y)dD$$

bo'ladi.

◀ Ravshanki, $\forall(x,y)\in(D)\setminus L$ uchun

$$f(x,y)\equiv F(x,y).$$

Shartga ko'ra L nol yuzli chiziq. Unda 1-lemmaga asosan, $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topiladiki, (D) soha diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashi olinganda ham, bu bo'laklashning L chiziq bilan umumiy nuqtaga ega bo'lgan bo'laklari yuzlarining yig'indisi ε dan kichik bo'ladi. Shu P bo'laklashga nisbatan $f(x,y)$ va $F(x,y)$ funksiyalarning ushbu integral yig'indilarini tuzamiz:

$$\sigma_P(f)=\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k\ ,$$

$$\sigma_P(F)=\sum_{k=1}^{n} F(\xi_k,\eta_k)D_k\ .$$

$\sigma_P(f)$ yig'indini quyidagicha ikki qismga ajratamiz:

$$\sigma_P(f)=\sum_k{}' f(\xi_k,\eta_k)D_k+\sum_k{}'' f(\xi_k,\eta_k)D_k$$

bunda $\sum_k{}'$ yig'indi L chiziq bilan umumiy nuqtaga ega bo'lgan (D_k) bo'laklar bo'yicha olingan, $\sum_k{}''$ esa qolgan barcha hadlardan tashkil topgan yig'indi.

Xuddi shunga o'xshash

$$\sigma_P(F)=\sum_k{}' F(\xi_k,\eta_k)D_k+\sum_k{}'' F(\xi_k,\eta_k)\cdot D_k\ .$$

Agar $\forall(x,y)\in(D)\setminus L$ uchun $f(x,y)=F(x,y)$ ekanini e'tiborga olsak, u holda

$$\left|\sigma_P(f)-\sigma_P(F)\right|=\sum_k{}'\left|f(\xi_k,\eta_k)-F(\xi_k,\eta_k)\right|D_k\le M\sum_k{}' D_k<M\varepsilon$$

bo'lishi kelib chiqadi, bunda $M=sup\left|f(x,y)-F(x,y)\right|$ $((x,y)\in(D)\setminus L)$. Demak,

$$\left|\sigma_P(f)-\sigma_P(F)\right|<M\varepsilon\ .$$

Keyingi tengsizlikda $\lambda_P\to0$ da limitga o'tib quyidagini topamiz:

$$\iint_{(D)} f(x,y)dD=\iint_{(D)} F(x,y)dD\ .$$ ▶

2) $f(x,y)$ funksiya (D) sohada berilgan bo'lib, (D) soha nol yuzli L chiziq bilan (D_1) va (D_2) sohalarga ajralgan bo'lsin. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'lsa, funksiya (D_1) va (D_2) sohalarda ham integrallanuvchi bo'ladi. Va aksincha, ya'ni $f(x,y)$ funksiya (D_1) va (D_2) sohalarning har birida integrallanuvchi bo'lsa, (D) sohada ham integrallanuvchi bo'ladi. Bunda

$$\iint_{(D)} f(x,y)dD = \iint_{(D_1)} f(x,y)dD + \iint_{(D_2)} f(x,y)dD$$

3) Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo’lsa, u holda $cf(x,y)$ $(c = const)$ ham shu sohada integrallanuvchi va ushbu

$$\iint_{(D)} cf(x,y)dD = c\iint_{(D)} f(x,y)dD$$

formula o’rinli bo’ladi.

4) Agar $f(x,y)$ va $g(x,y)$ funksiyalar (D) sohada integrallanuvchi bo’lsa, u holda $f(x,y) \pm g(x,y)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iint_{(D)} (f(x,y) \pm g(x,y))dD = \iint_{(D)} f(x,y)dD \pm \iint_{(D)} g(x,y)dD$$

formula o’rinli bo’ladi.

1-natija. Agar $f_1(x,y)$, $f_2(x,y), \ldots,$ $f_n(x,y)$ funksiyalarning har biri (D) sohada integrallanuvchi bo’lsa, u holda ushbu

$$c_1 f_1(x,y) + c_2 f_2(x,y) + \ldots + c_n f_n(x,y) \quad (c_i = const, \quad i = 1, 2, \ldots, n)$$

funksiya ham shu sohada integrallanuvchi va

$$\iint_{(D)} (c_1 f_1(x,y) + c_2 f_2(x,y) + \ldots + c_n f_n(x,y))dD =$$

$$= c_1 \iint_{(D)} f_1(x,y)dD + c_2 \iint_{(D)} f_2(x,y)dD + \ldots + c_n \iint_{(D)} f_n(x,y)dD$$

bo’ladi.

5) Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo’lib, $\forall (x,y) \in (D)$ uchun $f(x,y) \geq 0$ bo’lsa, u holda

$$\iint_{(D)} f(x,y)dD \geq 0$$

bo’ladi.

2-natija. Agar $f(x,y)$ va $g(x,y)$ funksiyalar (D) sohada integrallanuvchi bo’lib, $\forall (x,y) \in (D)$ uchun

$$f(x,y) \leq g(x,y)$$

bo’lsa, u holda

$$\iint_{(D)} f(x,y)dD \leq \iint_{(D)} g(x,y)dD$$

bo’ladi.

6) Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo’lsa, u holda $|f(x,y)|$ funksiya ham shu sohada integrallanuvchi va

$$\left| \iint_{(D)} f(x,y)dD \right| \leq \iint_{(D)} |f(x,y)|dD$$

bo’ladi.

7) ***O’rta qiymat haqidagi teoremalar.*** $f(x,y)$ funksiya (D) sohada berilgan va u shu sohada chegaralangan bo’lsin. Demak, shunday m va M o’zgarmas

sonlar $m = inf\{f(x,y),\ (x,y) \in (D)\}$, $M = sup\{f(x,y),\ (x,y) \in (D)\}$ mavjudki, $\forall (x,y) \in (D)$ uchun

$$m \le f(x,y) \le M$$

bo’ladi.

4-teorema. Agar $f(x,y)$ funksiya (D) sohada integrallanuvchi bo’lsa, u holda shunday o’zgarmas $\mu\ (m \le \mu \le M)$ son mavjudki,

$$\iint_{(D)} f(x,y) dD = \mu \cdot D$$

bo’ladi, bunda $D - (D)$ sohaning yuzi.

3-natija. Agar $f(x,y)$ funksiya yopiq (D) sohada uzluksiz bo’lsa, u holda bu sohada shunday $(a,в) \in (D)$ nuqta topiladiki,

$$\iint_{(D)} f(x,y) dD = f(a,в) dD$$

bo’ladi.

5-teorema. Agar $g(x,y)$ funksiya (D) sohada integrallanuvchi bo’lib, u shu sohada o’z ishorasini o’zgartirmasa va $f(x,y)$ funksiya (D) sohada uzluksiz bo’lsa, u holda shunday $(a,в) \in (D)$ nuqta topiladiki,

$$\iint_{(D)} f(x,y)\, g(x,y)\, dD = f(a,в) \iint_{(D)} g(x,y)\, dD$$

bo’ladi.

8) ***Integrallash sohasi o’zgaruvchi bo’lgan ikki karrali integrallar.*** $f(x,y)$ funksiya (D) sohada berilgan bo’lib, u shu sohada integrallanuvchi bo’lsin. Bu funksiya, (D) sohaning yuzga ega bo’lgan har qanday (d) qismida $((d) \subset (D))$ ham integrallanuvchi bo’ladi. Ravshanki, ushbu

$$\iint_{(d)} f(x,y) dD$$

integral (d) ga bog’liq bo’ladi.

(D) sohaning yuzga ega bo’lgan har bir (d) qismiga yuqoridagi integralni mos qo’yamiz:

$$\Phi : (d) \to \iint_{(d)} f(x,y) dD .$$

Natijaja funksiya hosil bo’ladi. Odatda bu

$$\Phi((d)) = \iint_{(d)} f(x,y) dD$$

funksiya sohaning funksiyasi deb ataladi.

(D) sohada biror (x_0, y_0) nuqtani olaylik. (d) esa shu nuqtani o’z ichiga olgan va $(d) \subset (D)$ bo’lgan soha bo’lsin. Bu sohaning yuzi d diametri esa λ bo’lsin.

Agar $\lambda \to \infty$ da $\dfrac{\Phi((d))}{d}$ nisbatning limiti $\lim\limits_{\lambda \to 0} \dfrac{\Phi((d))}{d}$ mavjud va chekli bo’lsa, bu limit $\Phi((d))$ funksiyaning (x_0, y_0) nuqtadagi soha bo’yicha hosilasi deb ataladi.

Agar $f(x,y)$ funksiya (D) sohada uzluksiz bo'lsa, u holda $\Phi((d))$ funksiyaning (x_0, y_0) nuqtadagi soha bo'yicha hosilasi $f(x_0, y_0)$ ga teng bo'ladi.

6-§. Ikki karrali integrallarni hisoblash

$f(x,y)$ funksiyaning (D) sohadagi $((D)\subset R^2)$ ikki karrali integrali tegishli integral yig'indining ma'lum ma'nodagi limiti sifatida ta'riflanadi. Bu limit tushunchasi murakkab xarakterga ega bo'lib, uni shu ta'rif bo'yicha hisoblash hatto sodda hollarda ham ancha qiyin bo'ladi.

Agar $f(x,y)$ funksiyaning (D) sohada integrallanuvchiligi ma'lum bo'lsa, unda bilamizki, integral yig'indi (D) sohaning bo'laklash usuliga ham, har bir bo'lakda olingan (ξ_n, η_k) nuqtalarga ham bog'liq bo'lmay, $\lambda_P \to 0$ da yagona $\iint\limits_{(D)} f(x,y)dD$ songa intiladi. Natijada funksiyaning ikki karrali integralini topish uchun birorta bo'laklashga nisbatan integral yig'indining limitini hisoblash etarli bo'ladi. Bu hol (D) sohaning bo'laklashini hamda (ξ_n, η_k) nuqtalarni integral yig'indini va uning limitini hisoblashga qulay qilib olish imkonini beradi.

17.1-misol. Ushbu

$$\iint\limits_{(D)} xy\, dD$$

integral hisoblansin, bunda $(D)=\{(x,y)\in R^2 : \ 0\le x\le 1,\ 0\le y\le 1-x\}$.

◀Ravshanki, $f(x,y)=xy$ funksiya (D) da uzluksiz. Demak, bu funksiya (D) sohada integrallanuvchi.

(D) sohani

$$(D_{ik})=\left\{(x,y)\in R^2 : \ \frac{i}{n}\le x\le \frac{i+1}{n},\ \frac{k}{n}\le y\le \frac{k+1}{n};\ \frac{i}{n}+\frac{k}{n}\le 1\right\}$$

$$(i=0,1,2,\dots,n-1,\ k=0,1,2,\dots,n-1)$$

bo'laklarga ajratib, har bir (D_{ik}) da $(\xi_k, \eta_k)=\left(\frac{i}{n},\frac{k}{n}\right)$ deb qaraymiz.

U holda

$$\sigma=\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)D_{ik}=\sum_{i=0}^{n-1}\left[\sum_{k=0}^{n-i-1}\frac{i}{n}\frac{k}{n}\cdot\frac{1}{n^2}+\frac{i}{n}\cdot\frac{n-i}{n}\cdot\frac{1}{2n^2}\right]=$$

$$=\frac{1}{2n^4}\sum_{i=0}^{n-1} i(n-i)^2=\frac{1}{2n^2}\left(\frac{n^2(n-1)n}{2}-2n\frac{n(n-1)(2n-1)}{6}+\frac{n^2(n-1)^2}{4}\right)$$

bo'ladi. Bundan esa

$$\lim_{n\to\infty}\sigma=\frac{1}{24}$$

bo'lishi kelib chiqadi. Demak,

$$\iint\limits_{(D)} xy dD=\frac{1}{24}.$$ ▶

Umuman, ko'p hollarda funksiyalarning karrali integrallarini ta'rifga ko'ra hisoblash qiyin bo'ladi. Shuning uchun karrali integrallarni hisoblashning amaliy jihatdan qulay bo'lgan yo'llarini topish zaruriyati tug'ildi.

Yuqorida aytib o'tganimizdek, $f(x,y)$ funksiyaning karrali integrali va uni hisoblash (D) sohaga bog'liq.

Avvalo sodda holda, (D) soha to'g'ri to'rtburchak sohadan iborat bo'lgan holda funksiyaning karrali integralini hisoblaymiz.

6-teorema. $f(x,y)$ funksiya $(D)=\{(x,y)\in R^2 : a\le x\le в,\ c\le y\le d\}$ sohada berilgan va integrallanuvchi bo'lsin.

Agar x $(x\in[a,в])$ o'zgaruvchining har bir tayin qiymatida

$$J(x)=\int_c^d f(x,y)dy$$

integral mavjud bo'lsa, u holda ushbu

$$\int_a^в\left[\int_c^d f(x,y)dy\right]dx$$

integral ham mavjud va

$$\iint_{(D)} f(x,y)dD=\int_a^в\left[\int_c^d f(x,y)dy\right]dx$$

bo'ladi.

◄ (D) sohani

$$(D_{ik})=\{(x,y)\in R^2 :\ x_i\le x\le x_{i+1},\ y_k\le y\le y_{k+1}\}$$
$$(i=0,1,2,...,n-1,\ k=0,1,2,...,m-1)$$

bo'laklarga ajratamiz. Bu bo'laklashni P_{nm} deb belgilaymiz. Uning diametri

$$\lambda_{P_{nm}}=max\sqrt{\Delta x_i^2+\Delta y_k^2}\qquad (\Delta x_i=x_{i+1}-x_i,\ \Delta y_k=y_{k+1}-y_k).$$

Modomiki, $f(x,y)$ funksiya (D) sohada integrallanuvchi ekan, u shu sohada chegaralangan bo'ladi. Binobarin, $f(x,y)$ funksiya har bir (D_{ik}) da chegaralangan va demak, u shu sohada aniq yuqori hamda aniq quyi chegaralariga ega bo'ladi:

$$m_{ik}=inf\{f(x,y):\quad (x,y)\in(D_{ik})\},$$
$$M_{ik}=sup\{f(x,y):\quad (x,y)\in(D_{ik})\},$$
$$(i=0,1,2,...,n-1,\ k=0,1,2,...,m-1).$$

Ravshanki, $\forall(x,y)\in(D_{ik})$ uchun $m_{ik}\le f(x,y)\le M_{ik}$ xususan, $\xi_i\in[x_i,x_{i+1}]$ uchun ham $m_{ik}\le f(\xi_i,y)\le M_{ik}$ bo'ladi. Teoremaning shartidan foydalanib quyidagini topamiz:

$$\int_{y_k}^{y_{k+1}} m_{ik}dy\le\int_{y_k}^{y_{k+1}} f(\xi_i,y)dy\le\int_{y_k}^{y_{k+1}} M_{ik}dy,$$

ya'ni

$$m_{ik}\Delta y_k\le\int_{y_k}^{y_{k+1}} f(\xi_i,y)dy\le M_{ik}\Delta y_k\quad (\Delta y_k=y_{k+1}-y_k).$$

Agar keyingi tengsizliklarni k ning $(k=0,1,2,...,m-1)$ qiymatlarida yozib, ularni hadlab qo'shsak, u holda

$$\sum_{k=0}^{m-1} m_{ik}\Delta y_k \le \sum_{k=0}^{m-1}\int_{y_k}^{y_{k+1}} f(\xi_i, y)dy \le \sum_{k=0}^{m-1} M_{ik}\Delta y_k ,$$

ya'ni

$$\sum_{k=0}^{m-1} m_{ik}\Delta y_k \le \int_c^d f(\xi_1, y)dy = J(\xi_i) \le \sum_{k=0}^{m-1} M_{ik}\Delta y_k \quad (i=0,1,2,...,n-1)$$

bo'ladi.

Endi keyingi tengsizliklarni Δx_i $(\Delta x_i = x_{i+1} - x_i)$ ga ko'paytirib, so'ng hadlab qo'shamiz. Natijada

$$\sum_{i=0}^{n-1}\left(\sum_{k=0}^{m-1} m_{ik}\Delta y_k\right)\Delta x_i \le \sum_{i=0}^{n-1} J(\xi_i)\Delta x_i \le \sum_{i=0}^{n-1}\left(\sum_{k=0}^{m-1} M_{ik}\Delta y_k\right)\Delta x_i$$

bo'ladi.

Ravshanki,

$$\sum_{i=0}^{n-1}\left(\sum_{k=0}^{m-1} m_{ik}\Delta y_k\right)\Delta x_i = \sum_{i=0}^{n-1}\sum_{k=0}^{m-1} m_{ik}\Delta x_i\Delta y_k = \sum_{i=0}^{n-1}\sum_{k=0}^{m-1} m_{ik} D_{ik} = s$$

$f(x,y)$ funksiya uchun Darbuning quyi yig'indisi,

$$\sum_{i=0}^{n-1}\left(\sum_{k=0}^{m-1} M_{ik}\Delta y_k\right)\Delta x_i = \sum_{i=0}^{n-1}\sum_{k=0}^{m-1} M_{ik} D_{ik} = S$$

esa Darbuning yuqori yig'indisidir. Demak,

$$s \le \sum_{i=0}^{n-1} J(\xi_i)\Delta x_i \le S . \qquad (17.13)$$

Shartga ko'ra $f(x,y)$ funksiya (D) da integrallanuvchi. U holda $\lambda_{P_{mn}} \to 0$ da

$$s \to \iint_{(D)} f(x,y)dD, \quad S \to \iint_{(D)} f(x,y)dD$$

bo'ladi.

(17.13) munosabatda esa,

$$\sum_{i=0}^{n-1} J(\xi_i)\Delta x_i$$

yig'indi limitga ega hamda bu limit

$$\iint_{(D)} f(x,y)dD$$

ga teng bo'lishi kelib chiqadi:

$$\lim_{\lambda_{P_{nm}}\to 0}\sum_{i=0}^{n-1} J(\xi_i)\Delta x_i = \iint_{(D)} f(x,y)dD .$$

Agar

$$\lim_{\lambda_{P_{nm}}\to 0}\sum_{i=0}^{n-1} J(\xi_i)\Delta x_i = \int_a^в J(x)dx$$

va

$$\int_a^в J(x)dx = \int_a^в \left[\int_c^d f(x,y)dy\right]dx$$

ekanligini e’tiborga olsak, unda

$$\iint_{(D)} f(x,y)dD = \int_a^в \left[\int_c^d f(x,y)dy\right]dx$$

bo’lishini topamiz. ▶

7-teorema. $f(x,y)$ funksiya $(D) = \{(x,y) \in R^2 : a \le x \le в;\, c \le y \le d\}$ sohada berilgan va integrallanuvchi bo’lsin. Agar $y\ (y \in [c,d])$ o’zgaruvchining har bir tayin qiymatida

$$J(y) = \int_a^в f(x,y)dx$$

integral mavjud bo’lsa, u holda ushbu

$$\int_c^d \left[\int_a^в f(x,y)dx\right]dy$$

integral ham mavjud va

$$\iint_{(D)} f(x,y)dD = \int_c^d \left[\int_a^в f(x,y)dx\right]dy$$

bo’ladi.

Bu teoremaning isboti yuqoridagi teoremaning isboti kabidir. 6-teorema va 7-teoremalardan quyidagi natijalar kelib chiqadi.

4-natija. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo’lsin. Agar $x\ (x \in [a,в])$ o’zgaruvchining har bir tayin qiymatida $\int_c^d f(x,y)dy$ integral mavjud bo’lsa, $y\ (y \in [c,d])$ o’zgaruvchining har bir tayin qiymatida $\int_a^в f(x,y)dx$ integral mavjud bo’lsa, u holda ushbu

$$\int_a^в \left[\int_c^d f(x,y)dy\right]dx, \quad \int_c^d \left[\int_a^в f(x,y)dx\right]dy \qquad (*)$$

integrallar ham mavjud va

$$\iint_{(D)} f(x,y)dD = \int_a^в \left[\int_c^d f(x,y)dy\right]dx = \int_c^d \left[\int_a^в f(x,y)dx\right]dy$$

bo’ladi.

5-natija. Agar $f(x,y)$ funksiya (D) sohada berilgan va uzluksiz bo’lsa, u holda

$$\iint_{(D)} f(x,y)dD, \quad \int_a^в \left[\int_c^d f(x,y)dy\right]dx, \quad \int_c^d \left[\int_a^в f(x,y)dx\right]dy$$

integrallarning har biri mavjud va ular bir-biriga teng bo’ladi.

(*) integrallar, tuzilishiga ko'ra, ikki argumentli funksiyadan avval bir argumenti bo'yicha (ikkinchi argumentini o'zgarmas hisoblab turib), so'ng ikkinchi argumenti bo'yicha olingan integrallardir. Bunday integrallarni takroriy integrallar deb atash (takroriy limitlar singari) tabiiydir.

Shunday qilib, qaralayotgan holda karrali integrallarni hisoblash takroriy integrallarni hisoblashga keltirilar ekan. Takroriy integralni hisoblash esa ikkita oddiy (bir argumentli funksiyaning integralini) Riman integralini ketma-ket hisoblash demakdir.

2-eslatma. Yuqorida keltirilgan 6-teoremani isbotlash jarayonida ko'rdikki, to'g'ri turtburchak (D) soha, tomonlari mos ravishda Δx_i, Δy_k bo'lgan to'g'ri to'rtburchak sohalar (D_{ik}) larga ajratildi. Ravshanki, bu elementar sohaning yuzi $D_{ik} = \Delta x_i \cdot \Delta y_k$ bo'ladi.

Avval aytganimizdek, Δx ni dx ga, Δy ni dy ga almashtirish mumkinligini hamda $a \le x \le в$, $c \le y \le d$ ekanini e'tiborga olib, bundan buyon integralni ushbu

$$\iint\limits_{(D)} f(x,y)dD$$

ko'rinishda yozish o'rniga

$$\int\limits_a^в \int\limits_c^d f(x,y)dydx \quad \text{(yoki } \int\limits_c^d \int\limits_a^в f(x,y)dxdy\text{)}$$

kabi ham yozib ketaveramiz.

17.2-misol. Ushbu

$$\iint\limits_{(D)} \frac{x}{\left(1+x^2+y^2\right)^{3/2}}dxdy$$

integral hisoblansin, bunda $(D) = \{(x,y) \in R^2 : 0 \le x \le 1; 0 \le y \le 1\}$.

◄Integral ostidagi

$$f(x,y) = \frac{x}{\left(1+x^2+y^2\right)^{3/2}}$$

funksiya (D) sohada uzluksiz. Unda qaralayotgan ikki karrali integral ham,

$$\int\limits_0^1 \frac{x}{\left(1+x^2+y^2\right)^{3/2}}dx$$

integral ham mavjud. 7-teoremaga ko'ra

$$\int\limits_0^1 \left[\int\limits_0^1 \frac{x}{\left(1+x^2+y^2\right)^{3/2}}dx\right]dy$$

integral mavjud bo'ladi va

$$\iint\limits_{(D)} \frac{x}{\left(1+x^2+y^2\right)^{3/2}}dxdy = \int\limits_0^1 \left[\int\limits_0^1 \frac{x}{\left(1+x^2+y^2\right)^{3/2}}dx\right]dy$$

bo'ladi.

Agar

$$\int_0^1 \frac{x\,dx}{\left(1+x^2+y^2\right)^{3/2}} = \frac{1}{2}\int_0^1 \left(1+x^2+y^2\right)^{-3/2} d\left(1+x^2+y^2\right) =$$

$$= -\frac{1}{\sqrt{1+x^2+y^2}}\Bigg|_{x=0}^{x=1} = \frac{1}{\sqrt{y^2+1}} - \frac{1}{\sqrt{y^2+2}}$$

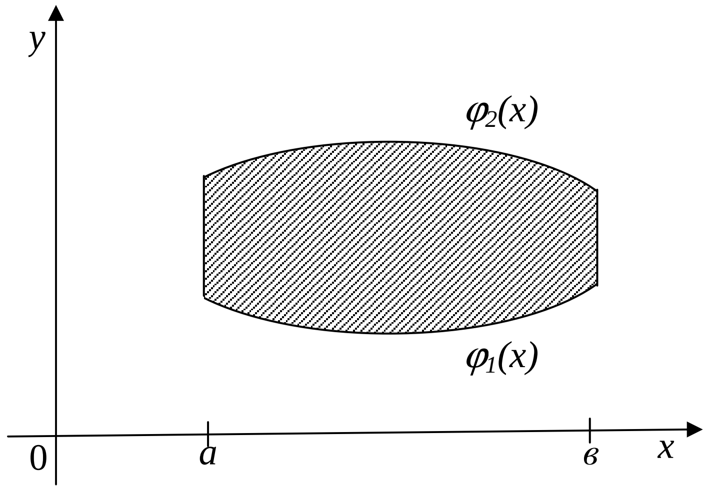
bo'lishini hisobga olsak, unda

$$\iint_{(D)} \frac{x\,dxdy}{\left(1+x^2+y^2\right)^{3/2}} = \int_0^1 \left[\frac{1}{\sqrt{y^2+1}} - \frac{1}{\sqrt{y^2+2}}\right] dy =$$

$$= \left[ln\left(y+\sqrt{y^2+1}\right) - ln\left(y+\sqrt{y^2+2}\right)\right]_0^1 = ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$$

ekanini topamiz. Demak, $\iint_{(D)} \frac{x\,dxdy}{\left(1+x^2+y^2\right)^{2/3}} = ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$. ▶

Endi (D) soha ushbu $(D) = \{(x,y) \in R^2 : a \le x \le в; \varphi_1(x) \le y \le \varphi_2(x)\}$ ko'rinishda bo'lsin. Bunda $\varphi_1(x)$ va $\varphi_2(x)$ $[a,в]$ da berilgan va uzluksiz funksiyalar (54-chizma)



54-chizma

8-teorema. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar x $(x \in [a,в])$ o'zgaruvchining har bir tayin qiymatida

$$J(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

integral mavjud bo'lsa, u holda ushbu

$$\int_a^в \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right] dx$$

integral ham mavjud va

$$\iint_{(D)} f(x,y)dD = \int_a^в \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right] dx$$

bo'ladi.

◀ $\varphi_1(x)$ va $\varphi_2(x)$ funksiyalar $[a,в]$ da uzluksiz. Veyershtrass teoremasiga ko'ra bu funksiyalar $[a,в]$ da o'zining eng katta va eng kichik qiymatlariga erishadi. Ularni

$$\min_{a\le x\le в} \varphi_1(x)=c, \quad \min_{a\le x\le в} \varphi_2(x)=d$$

deb belgilaylik.

Endi

$$(D_1)=\{(x,y)\in R^2 : a\le x\le в; c\le y\le d\}$$

sohada ushbu

$$f^*(x,y)=\begin{cases} f(x,y), & agar\ (x,y)\in(D) \quad bo'lsa, \\ 0, & agar\ (x,y)\in(D_1)\setminus(D)\ bo'lsa \end{cases}$$

funksiyani qaraylik.

Ravshanki, teorema shartlarida bu funksiya (D_1) sohada integrallanuvchi va integral xossasiga ko'ra

$$\iint_{(D_1)} f^*(x,y)dD=\iint_{(D)} f^*(x,y)dD+\iint_{(D_1)\setminus(D)} f^*(x,y)dD=\iint_{(D)} f(x,y)dD \qquad (17.14)$$

bo'ladi. Shuningdek, x $(x\in[a,в])$ o'zgaruvchining har bir tayin qiymatida

$$J_1(x)=\int_c^d f^*(x,y)dy$$

integral mavjud va

$$J_1(x)=\int_c^d f^*(x,y)dy=\int_c^{\varphi_1(x)} f^*(x,y)dy+\int_{\varphi_1(x)}^{\varphi_2(x)} f^*(x,y)dy+$$
$$+\int_{\varphi_2(x)}^{d} f^*(x,y)dy=\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \qquad (17.15)$$

bo'ladi. Unda 6-teoremaga ko'ra

$$\int_a^в\left[\int_c^d f^*(x,y)dy\right]dx$$

integral ham mavjud va

$$\iint_{(D)} f^*(x,y)dD=\int_a^в\left[\int_c^d f^*(x,y)dy\right]dx$$

bo'ladi.

(17.14) va (17.15) munosabatdan

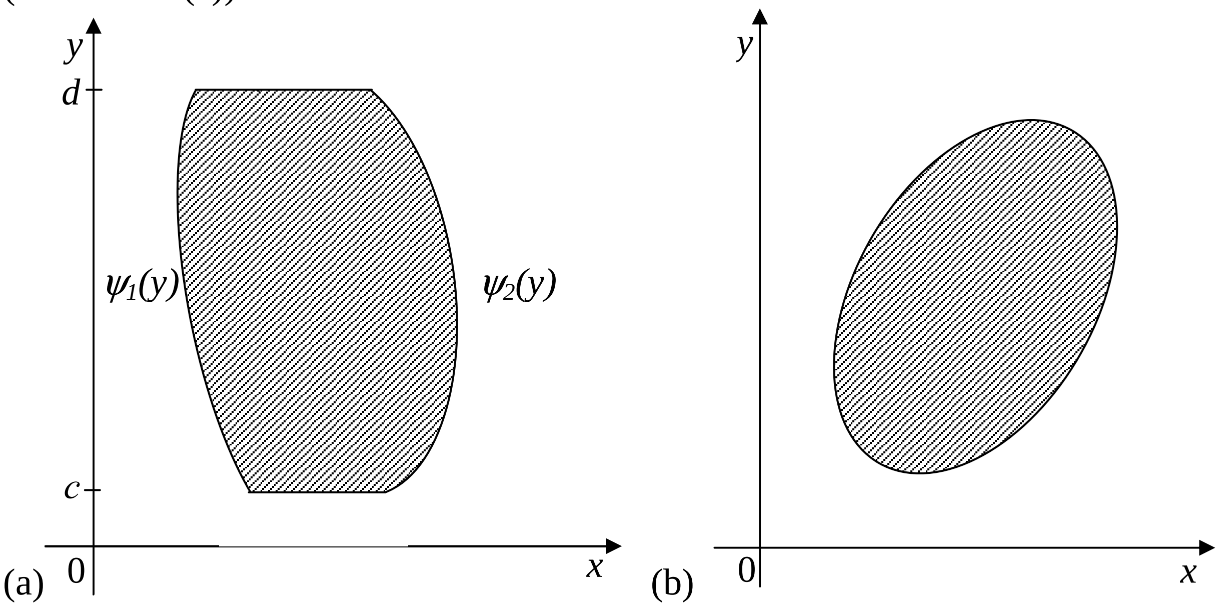
$$\iint_{(D)} f(x,y)dD=\int_a^в\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx$$

bo'lishi kelib chiqadi. ▶

Endi (D) soha ushbu

$$(D)=\{(x,y)\in R^2 : \psi_1(y)\le x\le \psi_2(y); c\le y\le d\}$$

ko'rinishda bo'lsin. Bunda $\psi_1(x)$ va $\psi_2(x)$ $[c,d]$ da berilgan uzluksiz funksiyalar (55-chizma (a)).



55-chizma

9-teorema. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar y $(y\in[c,d])$o'zgaruvchining har bir tayin qiymatida

$$J(y)=\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

integral mavjud bo'lsa, u holda

$$\int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy$$

integral ham mavjud va

$$\iint_{(D)} f(x,y)dD=\int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy$$

bo'ladi.

Bu teoremaning isboti 8-teoremaning isboti kabidir.

Faraz qilaylik, (D) soha $\left((D)\subset R^2\right)$ yuqorida qaralgan sohalarning har birining xususiyatiga ega bo'lsin (55-chizma (b)).

6-natija. $f(x,y)$ funksiya (D) sohada berilgan va integrallanuvchi bo'lsin. Agar x $(x\in[a,в])$ o'zgaruvchining har bir tayin qiymatida

$$\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy$$

integral mavjud bo'lsa, y $(y\in[c,d])$ o'zgaruvchining har bir tayin qiymatida

$$\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

integral mavjud bo'lsa, u holda

$$\int_a^в \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right] dx, \quad \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right] dy$$

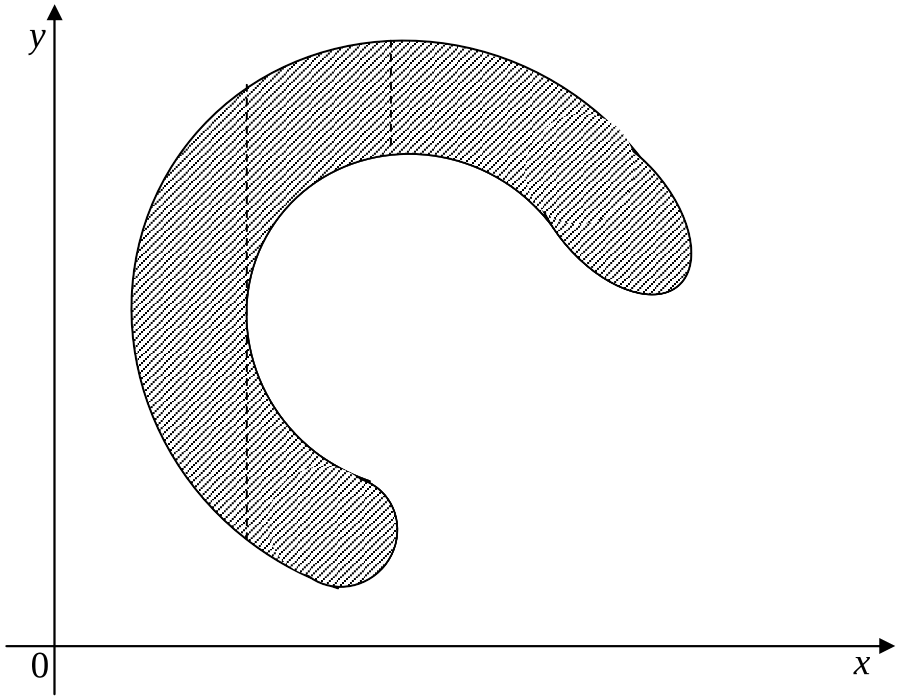
integrallar ham mavjud va

$$\iint_{(D)} f(x,y)dD = \int_a^в \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \right] dx = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right] dy$$

bo'ladi.

Bu natijaning isboti 8-teorema va 9-teoremadan kelib chiqadi.

Agar (D) soha (56-chizma)



56-chizmada

tasvirlangan soha bo'lsa, u holda bu soha yuqorida o'rganilgan sohalar ko'rinishiga keladigan qilib bo'laklarga ajratiladi. Natijada (D) soha bo'yicha ikki karrali integral ajratilgan sohalar bo'yicha ikki karrali integrallar yig'indisiga teng bo'ladi. Shunday qilib, biz integrallash sohasi (D) ning etarli keng sinfi uchun karrali integrallarni takroriy integrallarga keltirib hisoblash mumkinligini ko'ramiz.

17.3-misol. Ushbu

$$\iint_{(D)} e^{-y^2} dxdy$$

integral hisoblansin, bunda $(D) = \{(x,y) \in R^2 : 0 \le x \le y; 0 \le y \le 1\}$.

◀Bu holda 7-teoremaning barcha shartlari bajariladi. Usha teoremaga ko'ra

$$\iint_{(D)} e^{-y^2} dxdy = \int_0^1 \left[\int_0^y e^{-y^2} dx \right] dy$$

bo'ladi. Keyingi tenglikning o'ng tomonidagi integrallarni hisoblab quyidagilarni topamiz:

$$\int_0^y e^{-y^2}dx = ye^{-y^2},$$

$$\int_0^1 ye^{-y^2}dy = \frac{1}{2}\int_0^1 e^{-y^2}d\left(y^2\right) = -\frac{1}{2}e^{-y^2}\Big|_0^1 = \frac{1}{2}\left(1-\frac{1}{e}\right)$$

Demak,

$$\iint\limits_{(D)} e^{-y^2}dxdy = \frac{1}{2}\left(1-\frac{1}{e}\right). ►$$

17.4—misol. Ushbu

$$\iint\limits_{(D)} \sqrt{x+y}\, dxdy$$

integral hisoblansin, bunda $(D) = \{(x,y) \in R^2 : 0 \le x \le 1; 0 \le y \le 1-x\}$.

◄Bu holda 6-teoremaning barcha shartlari bajariladi. Usha teoremaga ko'ra

$$\iint\limits_{(D)} \sqrt{x+y}\, dxdy = \int_0^1 \left[\int_0^{1-x} \sqrt{x+y}\, dy\right]dx$$

bo'ladi. Integrallarni hisoblab topamiz:

$$\int_0^1 \left[\int_0^{1-x} \sqrt{x+y}\, dy\right]dx = \int_0^1 \left(\frac{2}{3}\sqrt{(x+y)^3}\right)\Bigg|_{y=0}^{y=1-x} dx = \frac{2}{3}\int_0^1 \left(1-\sqrt{x^3}\right)dx = \frac{2}{5}.$$

Demak,

$$\iint\limits_{(D)} \sqrt{x+y}\, dxdy = \frac{2}{5}. ►$$

Bu keltirilgan misollarda sodda funksiyalarning sodda soha bo'yicha ikki karrali integrallari qaraldi. Ko'p hollarda sodda funksiyalarni murakkab soha bo'yicha, murakkab funksiyalarni sodda soha bo'yicha va ayniqsa, murakkab funksiyalarni murakkab soha bo'yicha ikki karrali integrallarini hisoblashga to'g'ri keladi. Bunday integrallarni hisoblash esa ancha qiyin bo'ladi.

7-§. Ikki karrali integrallarda o'zgaruvchilarni almashtirish

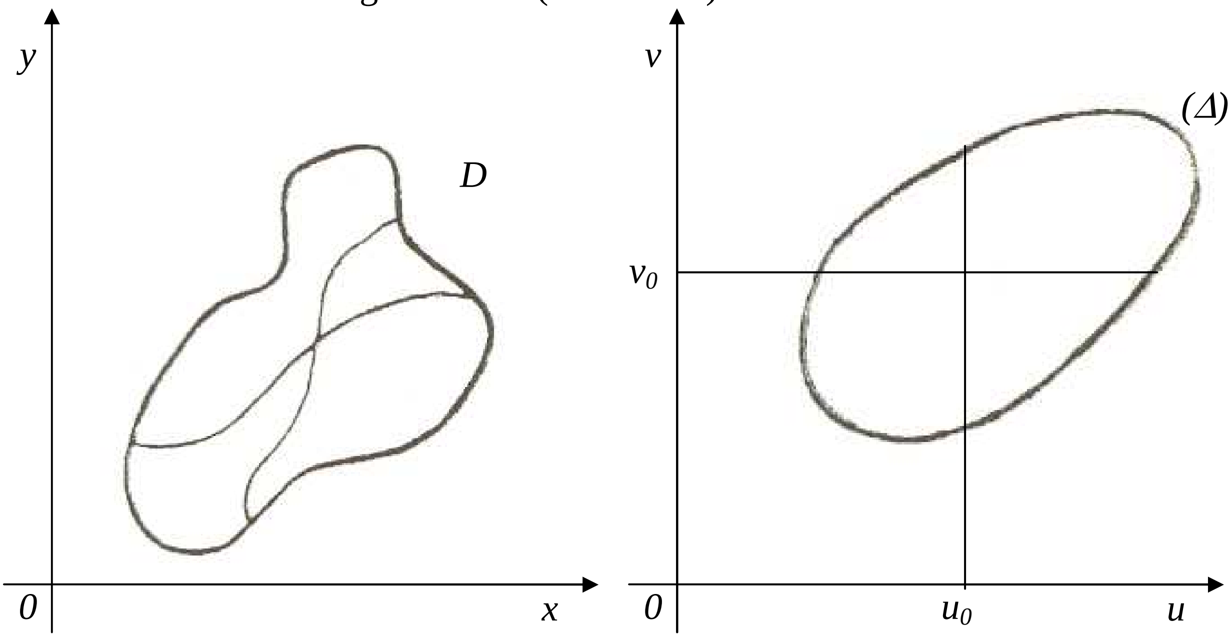
$f(x,y)$ funksiya (D) sohada $\left((D) \subset R^2\right)$ berilgan bo'lsin. Bu funksiyaning ikki karrali

$$\iint\limits_{(D)} f(x,y)dxdy$$

integrali mavjudligi ma'lum bo'lib, uni hisoblash talab etilsin. Ravshanki, $f(x,y)$ funksiya hamda (D) soha murakkab bo'lsa, integralni hisoblash qiyin bo'ladi. Ko'pincha, x va y o'zgaruvchilarni, ma'lum qoidaga ko'ra boshqa o'zgaruvchilarga almashtirish natijasida integral ostidagi funksiya ham, integrallash sohasi ham soddalashib, ikki karrali integralni hisoblash osonlashadi.

Ushbu paragrafda ikki karrali integrallarda o'zgaruvchilarni almashtirish bilan shug'ullanamiz. Avvalo tekislikda sohani sohaga akslantirish, egri chiziqli koordinatalar hamda sohaning yuzini egri chiziqli koordinatalarda ifodalanishini keltiramiz.

Ikkita tekislik berilgan bo’lsin (57-chizma).



57-chizma.

Birinchi tekislikda to’g’ri burchakli Oxy koordinata sistemasini va chegaralangan (D) sohani qaraylik. Bu sohaning chegarasi ∂D sodda, bo’lakli-silliq chiziqdan iborat bo’lsin. Ikkinchi tekislikda esa to’g’ri burchakli Ouv koordinata sistemasini va chegaralangan (Δ) sohani qaraylik. Bu sohaning chegarasi $\partial\Delta$ ham sodda, bo’lakli-silliq chiziqdan iborat bo’lsin.

$\varphi(u,v)$ va $\psi(u,v)$ lar (Δ) sohada berilgan shunday funksiyalar bo’lsinki, ulardan tuzilgan $\{\varphi(u,v),\psi(u,v)\}$ sistema (Δ) sohadagi (u,v) nuqtani (D) sohadagi (x,y) nuqtaga akslantirsin:

$$\varphi: \ (u,v) \to x,$$
$$\psi: \ (u,v) \to y.$$

va bu akslantirishning akslaridan iborat $\{(x,y)\}$ to’plam (D) ga tegishli bo’lsin.

Demak, ushbu

$$\begin{cases} x = \varphi(u,v), \\ y = \psi(u,v) \end{cases} \tag{17.16}$$

sistema (Δ) sohani (D) sohada akslantiradi.

Bu akslantirish quyidagi shartlarni bajarsin:

1⁰. (17.16) akslantirish o’zaro bir qiymatli akslantirish, ya’ni (Δ) sohaning turli nuqtalarini (D) sohaning turli nuqtalariga akslantirib, (D) sohadagi har bir nuqta uchun (Δ) sohada unga mos keladigan nuqta bittagina bo’lsin.

Ravshanki, bu holda (17.16) sistema u va v larga nisbatan bir qiymatli echiladi: $u = \varphi_1(x,y)$, $v = \psi_1(x,y)$ va ushbu

$$\begin{cases} u = \varphi_1(x,y), \\ v = \psi_1(x,y) \end{cases} \tag{17.17}$$

sistema bilan akslantirish yuqoridagi akslantirishga teskari bo'lib, (D) sohani (Δ) sohaga akslantiradi. Demak,

$$\begin{aligned} \varphi(\varphi_1(x,y),\psi_1(x,y)) &= x \\ \psi(\varphi_1(x,y),\psi_1(x,y)) &= y \end{aligned} \qquad (17.18)$$

2⁰. $\varphi(u,v)$, $\psi(u,v)$ funksiyalar (Δ) sohada, $\varphi_1(u,v)$, $\psi_1(u,v)$ funksiyalar (D) sohada uzluksiz va barcha xususiy hosilalarga ega bo'lib, bu xususiy hosilalar ham uzluksiz bo'lsin.

3⁰. (17.16) sistemadagi funksiyalarning xususiy hosilaridan tuzilgan ushbu

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad (17.19)$$

funksional determinantning ham (Δ) sohada noldan farqli (ya'ni (Δ) sohaning har bir nuqtasida noldan farqli) bo'lsin. Odatda (17.19) determinantni sistemaning yakobiani deyiladi va $J(u,v)$ yoki $\frac{D(x,y)}{D(u,v)}$ kabi belgilanadi.

Bu 2^0 va 3^0 – shartlardan, (Δ) bog'lamli soha bo'lganda, (17.19) yakobianning shu sohada o'z ishorasini saqlashi kelib chiqadi.

Haqiqatdan ham, $J(u,v)$ funksiya (Δ) sohaning ikkita turli nuqtalarida turli ishorali qiymatlarga ko'ra, (Δ) da shunday (u_0,v_0) nuqta topiladiki, $J(u_0,v_0)=0$ bo'ladi. Bu esa $J(u,v)\neq 0$ bo'lishiga ziddir.

3-shartdan (17.17) sistemaning yakobiani, ya'ni ushbu

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

funksional determinantning ham (D) sohada noldan farqli bo'lishi kelib chiqadi.

Haqiqatdan ham, (17.18) munosabatdan

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x} = 1,$$

$$\frac{\partial y}{\partial y} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial y} = 1,$$

$$\frac{\partial x}{\partial y} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = 0$$

bo'lishini e'tiborga olsak, u holda

$$\frac{D(x,y)}{D(u,v)}\cdot\frac{D(u,v)}{D(x,y)} = 1$$

bo'lib,

$$J_1(x,y)=\frac{D(u,v)}{D(x,y)}\neq 0$$

bo'lishini topamiz.

Demak, (D) bog'lamli soha bo'lganda $J_1(u,v)$ yakobiani ham (D) sohada o'z ishorasini saqlaydi.

Yuqoridagi shartlardan yana quyidagilar kelib chiqadi.

(17.16) akslantirish (Δ) sohaning ichki nuqtasini (D) sohaning ichki nuqtasiga akslantiradi. Haqiqatdan ham, oshkormas funksiyaning mavjudligi haqida teoremaga ko'ra (17.16) sistema (x_0,y_0) nuqtaning biror atrofida u va v larni x va y o'zgaruvchilarning funksiyasi sifatida aniqlaydi: $u=\varphi_1(x,y)$, $v=\psi_1(x,y)$. Bunda $\varphi_1(x_0,y_0)=u_0$, $\psi_1(x_0,y_0)=v_0$ bo'ladi. Demak, (x_0,y_0) (D) sohaning ichki nuqtasi. Bundan (17.16) akslantirish (Δ) sohaning chegarasi $\partial\Delta$ ni (D) sohaning chegarasi $\partial\Delta$ ga akslantirishi kelib chiqadi.

Shuningdek, (17.16) akslantirish (Δ) sohadagi silliq (bo'lakli -silliq) egri chiziq

$$\begin{array}{l} n=u(t) \\ v=v(t) \end{array} \qquad (\alpha\le t\le\beta)$$

ni (D) sohadagi silliq (bo'lakli-silliq) egri chiziq

$$\begin{array}{l} x=\varphi(u(t),v(t)) \\ y=\psi(u(t),v(t)) \end{array}$$

ga akslantiradi.

(Δ) sohada $u=u_0$ to'g'ri chiziqni olaylik. (17.16) akslantirish bu to'g'ri chiziqni (D) sohadagi

$$\begin{array}{l} x=\varphi(u_0,v) \\ y=\psi(u_0,v) \end{array} \tag{17.20}$$

egri chiziqqa akslantiradi. Xuddi shunday (Δ) sohadagi $v=v_0$ to'g'ri chiziqni (17.16) akslantirish (D) sohadagi

$$\begin{array}{l} x=\varphi(u_1,v_0) \\ y=\psi(u_1,v_0) \end{array} \tag{17.21}$$

egri chiziqqa akslantiradi. Odatda, (17.20) va (17.21) egri chiziqlarni koordinata chiziqlari (17.20) ni v koordinata chizig'i, (17.21) ni esa u koordinata chizig'i) deb ataladi.

Modomiki, (17.16) akslantirish o'zaro bir qiymatli akslantirish ekan, unda (D) sohaning har bir (x,y) nuqtasidan yagona v koordinata chizig'i (u ning tayin o'zgarmas qiymatiga mos bo'lgan chiziq), yagona u koordinata chizig'i (v ning tayin o'zgarmas qiymatiga mos bo'lgan chiziq) o'tadi. Demak, (D) sohaning shu (x,y) nuqtasi yuqorida aytilgan u va v lar bilan, ya'ni (Δ) sohaning (u,v) nuqtasi bilan to'liq aniqlanadi. Shuning uchun u va v larni (D) soha nuqtalarining koordinatalari deb qarash mumkin. (D) soha nuqtalarining bunday koordinatalari egri chiziqli koordinatalari deyiladi.

Masalan, ushbu

$$\begin{matrix} x=\rho\, cos\,\varphi \\ y=\rho\, cos\,\varphi \end{matrix} \quad (\rho\ge 0,\ \ 0\le\varphi<2\pi)$$

sistema $(\Delta)=\{(u,v)\in R^2:\ 0\le\rho<+\infty,\ 0\le\varphi<2\pi\}$ sohani Oxy tekislikka akslantiradi. Bu sistemaning yakobiani

$$J(u,v)=\begin{vmatrix} cos\,\varphi & -\rho\, sin\,\varphi \\ sin\,\varphi & \rho\, cos\,\varphi \end{vmatrix}=\rho$$

bo'ladi.

ρ va φ lar (D) soha nuqtalarining egri chiziqli koordinatalari bo'lib, shu sohaning koordinat chiziqlari esa, markazi $(0,0)$ nuqtada, radiusi ρ ga teng ushbu

$$x^2+y^2=\rho^2$$

aylanalardan (v koordinat chiziqlari) hamda $(0,0)$ nuqtadan chiqqan $\varphi=\rho_0$ $(0\le\rho_0<2\pi)$ nurlardan (v koordinat chiziqlari) iborat.

Faraz qilaylik, ushbu

$$\begin{matrix} x=\varphi(u,v) \\ y=\psi(u,v) \end{matrix}$$

sistema (Δ) sohani (D) sohaga akslantirsin. Bu akslantirish yuqoridagi 1^0-3^0-shartlarni bajarsin. U holda (D) sohaning yuzi

$$D=\iint\limits_{(\Delta)}|J(u,v)|dudv=\iint\limits_{(\Delta)}\left|\frac{D(x,y)}{D(u,v)}\right|dudv \qquad (17.22)$$

bo'ladi.

Bu formulaning isboti keyingi bobda keltiriladi (qarang, 18-bob, 3-§).

$f(x,y)$ funksiya (D) sohada $\left((D)\subset R^2\right)$ berilgan va shu sohada uzluksiz bo'lsin. (D) esa sodda, bo'lakli-silliq chiziq bilan chegaralangan soha bo'lsin. Ravshanki, $f(x,y)$ funksiya (D) sohada integrallanuvchi bo'ladi.

Aytaylik, ushbu

$$\begin{matrix} x=\varphi(u,v) \\ y=\psi(u,v) \end{matrix}$$

sistema (Δ) sohani (D) sohaga akslantirsin va bu akslagtirish yuqoridagi 1^0-3^0-shartlarni bajarsin.

Har bir bo'luvchi chizig'i bo'lakli-silliq bo'lgan (Δ) sohaning P_Δ bo'laklanishini olaylik. (17.16) akslantirish natijasida (D) sohaning P_D bo'laklanishi hosil bo'ladi. Bu bo'laklanishga nisbatan $f(x,y)$ funksiya integral yig'indisi

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k$$

ni tuzamiz. Ravshanki,

$$\lim_{\lambda_{P_D}\to 0}\sigma=\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)D_k=\iint\limits_{(D)} f(x,y)dxdy\,.$$

Yuqorida keltirilgan (17.22) formulaga ko'ra

$$D_k = \iint\limits_{(\Delta_k)} |J(u,v)| dudv$$

bo'ladi. O'rta qiymat haqidagi teoremadan foydalanib quyidagini topamiz:

$$D_k = \left|J\left(u_k^*, v_k^*\right)\right| \cdot \Delta_k \quad \left(\left(u_k^*, v_k^*\right) \in (\Delta_k)\right)$$

bunda $\Delta_k - (\Delta_k)$ ning yuzi. Natijada σ yig'indi ushbu

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k) \left|J\left(u_k^*, v_k^*\right)\right| \Delta_k$$

ko'rinishga keladi.

(ξ_k, η_k) nuqtaning (D_k) sohadagi ixtiyoriy nuqta ekanligidan foydalanib, uni

$$\varphi\left(u_k^*, v_k^*\right) = \xi_k$$
$$\psi\left(u_k^*, v_k^*\right) = \eta_k$$

deb olish mumkin. U holda

$$\sigma = \sum_{k=1}^{n} f\left(\varphi\left(u_k^*, v_k^*\right), \psi\left(u_k^*, v_k^*\right)\right) \left|J\left(u_k^*, v_k^*\right)\right| \Delta_k$$

bo'ladi.

Ravshanki,

$$f(\varphi(u,x), \psi(u,x)) |J(u,v)|$$

funksiya (Δ) sohada uzluksiz. Demak, u shu sohada integrallanuvchi. U holda

$$\lim_{\lambda_{P_\Delta} \to 0} \sigma = \lim_{\lambda_{P_\Delta} \to 0} \sum_{k=1}^{n} f\left(\varphi\left(u_k^*, v_k^*\right), \psi\left(u_k^*, v_k^*\right)\right) \left|J\left(u_k^*, v_k^*\right)\right| \Delta_k =$$
$$= \iint\limits_{(\Delta)} f(\varphi(u,v), \psi(u,v)) |J(u,v)| dudv$$

bo'ladi.

$\lambda_{P_\Delta} \to 0$ da $\lambda_{P_D} \to 0$ bo'lishini e'tiborga olib, topamiz:

$$\iint\limits_{(D)} f(x,y) dxdy = \iint\limits_{(\Delta)} f(\varphi(u,v), \psi(u,v)) |J(u,v)| dudv. \qquad (17.23)$$

Bu ikki karrali integralda o'zgaruvchilarni almashtirish formulasidir.

U berilgan (D) soha bo'yicha integralni hisoblashni (Δ) soha bo'yicha integralni hisoblashga keltiradi. Agarda (17.23) da o'ng tomondagi integralni hisoblash engil bo'lsa, bajarilgan o'zgaruvchilarni almashtirish o'zini oqlaydi.

17.5-misol. Ushbu

$$\iint\limits_{(D)} \sqrt{\frac{1 - x^2 - y^2}{1 + x^2 + y^2}} dxdy$$

integral hisoblansin, bunda

$$(D) = \left\{(x,y) \in R^2 : x^2 + y^2 < 1;\ y > 0\right\}$$

markazi $(0,0)$ nuqtada, radiusi 1 ga teng bo'lgan yuqori tekislikdagi yarim doira.

◀Berilgan integralda o'zgaruvchilarni quyidagi almashtiramiz:

$$x = \rho \cos\varphi,$$
$$y = \rho \sin\varphi$$

Bu almashtirish ushbu

$$(\Delta)=\{(\rho,\varphi)\in R^2:\ 0<\varphi<\pi,\ 0<\rho<1\}$$

to'g'ri turtburchakni (D) sohaga akslantiradi va u 1^0-3^0-shartlarni qanoatlantiradi. Unda (17.23) formulaga ko'ra

$$\iint_{(D)}\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy=\iint_{(\Delta)}\sqrt{\frac{1-\rho^2}{1+\rho^2}}|J(\rho,\varphi)|d\rho d\varphi$$

bo'ladi. Bunda yakobian $J(\rho,\varphi)=\rho$ bo'ladi. Bu tenglikning o'ng tomondagi integralni hisoblab topamiz:

$$\iint_{(\Delta)}\sqrt{\frac{1-\rho^2}{1+\rho^2}}|J(\rho,\varphi)|d\rho d\varphi=\int_0^1\left(\int_0^\pi d\varphi\right)\sqrt{\frac{1-\rho^2}{1+\rho^2}}\rho d\rho=\pi\int_0^1\sqrt{\frac{1-\rho^2}{1+\rho^2}}\rho d\rho=\frac{\pi}{4}(\pi-2).$$

Demak,

$$\iint_{(D)}\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy=\frac{\pi}{4}(\pi-2).\blacktriangleright$$

8-§. Ikki karrali integralni taqribiy hisoblash

$f(x,y)$ funksiya (D) sohada $((D)\in R^2)$ berilgan va shu sohada integrallanuvchi, ya'ni

$$\iint_{(D)}f(x,y)dxdy \qquad (17.23)$$

integral mavjud bo'lsin. Ma'lum ko'rinishga ega bo'lgan (D) sohalar uchun bunday integralni hisoblash 6-§ da keltirildi. Ravshanki, $f(x,y)$ funksiya murakkab bo'lsa, shuningdek, integrallash sohasi murakkab ko'rinishga ega bo'lsa, unda (17.23) integralni hisoblash ancha qiyin bo'ladi va ko'p hollarda bunday integralni taqribiy hisoblashga to'g'ri keladi.

Ushbu paragrafda (17.23) integralni taqribiy hisoblashni amalga oshiradigan sodda formulalardan birini keltiramiz.

Aytaylik, $f(x,y)$ funksiya $(D)=\{(x,y)\in R^2: a\le x\le в; c\le y\le d\}$ to'g'ri turtburchakda berilgan va uzluksiz bo'lsin. Unda 6-§ da keltirilgan formulaga ko'ra

$$\iint_{(D)}f(x,y)dxdy=\int_a^в\left[\int_c^d f(x,y)dy\right]dx \qquad (17.24)$$

bo'ladi.

Endi

$$\int_c^d f(x,y)dy,\quad (x\in[a,в])$$

integralga 1-qism, 9-bob, 2-§ dagi (9.52) formulani -to'g'ri to'rtburchaklar formulasini tatbiq etib, ushbu

$$\int_c^d f(x,y)dy\approx\frac{d-c}{n}\sum_{k=0}^{n-1}f\left(x,y_{k+\frac{1}{2}}\right)\quad (x\in[a,в]) \qquad (17.25)$$

taqribiy formulani hosil qilamiz. So'ng

$$\int_a^{в} f\left(x, y_{k+\frac{1}{2}}\right)dx$$

integralga yana usha (9.53) formulani qo'llab, quyidagi

$$\int_a^{в} f\left(x, y_{k+\frac{1}{2}}\right)dx \approx \frac{в-a}{n}\sum_{i=0}^{m-1} f\left(x_{i+\frac{1}{2}}, y_{k+\frac{1}{2}}\right) \qquad (17.26)$$

taqribiy formulaga kelamiz.

Natijada (17.24), (17.25) va (17.26) munosabatlardan

$$\iint_{(D)} f(x, y)dxdy \approx \frac{(в-a)(d-c)}{nm}\sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f\left(x_{i+\frac{1}{2}}, y_{k+\frac{1}{2}}\right) \qquad (17.27)$$

bo'lishi kelib chiqadi.

Bu ikki karrali integralni tag'ribiy hisoblash formulasi, «to'g'ri turtburchaklar» formulasi deb ataladi.

Shunday qilib, «to'g'ri turtburchaklar» formulasida, ikki karrali integral maxsus tuzilgan yig'indi bilan almashtiriladi. Bu yig'indi esa quyidagicha tuziladi:

$(D)=\{(x,y)\in R^2 : a \le x \le в, c \le y \le d\}$ to'g'ri turtburchak nm ta teng $(D_{ik})=\{(x,y)\in R^2 : x_i \le x \le x_{i+1}, y_k \le y \le y_{k+1}\}$ $(i=0,1,2,...,m-1,\quad k=0,1,2,...,$ $n-1)$ to'g'ri turtburchaklarga ajratiladi. Bunda

$$x_i = a + i\,\frac{в-a}{m}, \quad y_k = c + k\,\frac{d-c}{n}.$$

Har bir (D_{ik}) ning markazi bo'lgan $(x_{i+\frac{1}{2}}, y_{k+\frac{1}{2}})$ $(i=0,1,2,...,m-1,$ $k=0,1,2,...,\quad n-1)$ nuqtada $f(x,y)$ funksiyaning qiymati $f(x_{i+\frac{1}{2}}, y_{k+\frac{1}{2}})$ hisoblanib, uni shu (D_{ik}) ning yuziga ko'paytiriladi. So'ngra ular barcha i va k lar $(i=0,1,2,...,m-1,\ k=0,1,2,...,\ n-1)$ bo'yicha yig'iladi.

Odatda, har bir taqribiy formulaning xatoligi topiladi yoki baholanadi. Keltirilgan taqribiy formulaning xatoligini ham o'rganish mumkin.

17.6-misol. Ushbu

$$\iint_{(D)} \frac{1}{(x+y+1)^2}dxdy$$

integral taqribiy hisoblansin, bunda

$$(D)=\{(x,y)\in R^2 : 0 \le x \le 1, 0 \le y \le 1\}.$$

◄ (D) ni ushbu to'rtta bo'lakka bo'lamiz:

$$(D_{00})=\left\{(x,y)\in R^2 : 0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}\right\},$$

$$(D_{01})=\left\{(x,y)\in R^2 : 0 \le x \le \frac{1}{2}, \frac{1}{2} \le y \le 1\right\},$$

$$(D_{10})=\left\{(x,y)\in R^2 : \frac{1}{2} \le x \le 1, 0 \le y \le \frac{1}{2}\right\},$$

$$(D_{11})=\left\{(x,y)\in R^2: \frac{1}{2}\le x\le 1, \frac{1}{2}\le y\le 1\right\}.$$

Bu bo'laklarning markazlari

$$\left(\frac{1}{4},\frac{1}{4}\right),\ \left(\frac{1}{4},\frac{3}{4}\right),\ \left(\frac{3}{4},\frac{1}{4}\right),\ \left(\frac{3}{4},\frac{3}{4}\right)$$

nuqtalarda

$$f(x,y)=\frac{1}{(1+x+y)^2}$$

funksiyaning qiymatlarini hisoblab, (17.23) formulaga ko'ra

$$\iint_{(D)}\frac{1}{(1+x+y)^2}dxdy\approx 0{,}2761$$

bo'lishini topamiz. Bu integralning aniq qiymati esa

$$\iint_{(D)}\frac{1}{(1+x+y)^2}dxdy=\int_0^1\left[\int_0^1\frac{dx}{(1+x+y)^2}\right]dy=ln\frac{4}{3}=0{,}287682...$$

bo'ladi.▶

9-§. Ikki karrali integrallarning ba'zi bir tatbiqlari

Ushbu paragrafda ikki karrali integrallarning ba'zi bir tatbiqlarini keltiramiz.

1⁰. Jismning hajmini hisoblash. R^3 fazoda (V) jism yuqoridan $z=f(x,y)$ sirt bilan, (bunda $f(x,y)$ funksiya (D) da uzluksiz) yon tomonlaridan yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt hamda pastdan Oxy tekislikdagi (D) soha bilan chegaralangan jism bo'lsin.

(D) yopiq sohaning P bo'laklashni olamiz. $f(x,y)$ funksiya (D) da uzluksiz bo'lganligi sabali, bu funksiya P bo'laklash har bir (D_k) bo'lagida ham uzluksiz bo'lib, unda $inf\{f(x,y): (x,y)\in(D_k)\}=m_k$, $sup\{f(x,y): (x,y)\in(D_k)\}=M_k$ $(k=1,2,3,...,n)$ larga ega bo'ladi.

Quyidagi

$$V_A=\sum_{k=1}^n m_k D_k\ ,$$

$$V_B=\sum_{k=1}^n M_k D_k$$

yig'indilarni tuzamiz. Bu yig'indilarning birinchisi (V) jism ichiga joylashgan ko'pyoqning hajmini, ikkinchisi esa (V) jismni o'z ichiga olgan ko'pyoqning hajmini ifodalaydi.

Ravshanki, bu ko'pyoqlar, demak, ularning hajmlari ham $f(x,y)$ funksiyaga hamda (D) sohaning bo'laklashga bog'liq bo'ladi:

$$V_A=V_A^P(f),\ V_B=V_B^P(f).$$

(D) sohaning turli bo'laklashlari olinsa, ularga nisbatan (V) jismning ichiga joylashgan hamda (V) jismni o'z ichiga olgan turli ko'pyoqlar yasaladi. Natijada bu ko'pyoqlar xajmlaridan iborat quyidagi

$$\left\{V_A^P(f)\right\},\ \left\{V_B^P(f)\right\}$$

to'plamlar hosil bo'ladi. Bunda $\left\{V_A^P(f)\right\}$ to'plam yuqoridan $\left\{V_B^P(f)\right\}$ to'plam esa quyidani chegaralangan bo'ladi. Demak, bu to'plamlarning aniq chegaralari

$$\sup\left\{V_A^P(f)\right\},\ \inf\left\{V_B^P(f)\right\}$$

mavjud. Shartga ko'ra $f(x,y)$ funksiya (D) yopiq sohada uzluksiz. U holda Kantor teoremasining natijasiga asosan, $\forall\varepsilon>0$ son olinganda ham, $\dfrac{\varepsilon}{D}$ songa ko'ra shunday $\delta>0$ son topiladiki, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan har qanday bo'laklashi P uchun har bir (D_k) da funksiyaning tebranishi

$$M_k-m_k<\frac{\varepsilon}{D}$$

bo'ladi. Unda

$$\inf\left\{V_B^P(f)\right\}-\sup\left\{V_A^P(f)\right\}\le V_B^P(f)-V_A^P(f)=\sum_{k=1}^{n}M_kD_k-\sum_{k=1}^{n}m_kD_k=$$

$$=\sum_{k=1}^{n}(M_k-m_k)D_k<\frac{\varepsilon}{D}\sum_{k=1}^{n}D_k=\frac{\varepsilon}{D}D=\varepsilon.$$

Demak, (D) sohaning diametri $\lambda_P<\delta$ bo'lgan har qanday bo'laklanishi olinganda ham bu bo'laklanishga mos (V) jismning ichiga joylashgan hamda bu (V) ni o'z ichiga olgan ko'pyoq hajmlari uchun har doim

$$0\le\inf\left\{V_B^P(f)\right\}-\sup\left\{V_A^P(f)\right\}<\varepsilon$$

tengsizlik o'rinli bo'ladi. Bundan esa

$$\inf\left\{V_B^P(f)\right\}=\sup\left\{V_A^P(f)\right\} \qquad (17.28)$$

tenglik kelib chiqadi. Bu tenglik (V) jism hajmga ega bo'lishini bildiradi.

Endi yuqorida o'rganilgan $V_A^P(f)$, $V_B^P(f)$ yig'indilarni Darbu yig'indilari bilan taqqoslab, $V_A^P(f)$ ham $V_B^P(f)$ yig'indilar $f(x,y)$ funksiyaning (D) sohada mos ravishda Darbu quyi hamda yuqori yig'indilari ekanini topamiz. Shuning uchun ushbu

$$\sup\left\{V_A^P(f)\right\},\ \inf\left\{V_B^P(f)\right\}$$

miqdorlar $f(x,y)$ funksiyaning quyi hamda yuqori ikki karrali integrallari bo'ladi, ya'ni

$$\sup\left\{V_A^P(f)\right\}=\underline{\iint_{(D)}}f(x,y)dD,\ \ \inf\left\{V_B^P(f)\right\}=\overline{\iint_{(D)}}f(x,y)dD$$

Yuqoridagi (17.28) munosabatga ko'ra

$$\underline{\iint_{(D)}}f(x,y)dD=\overline{\iint_{(D)}}f(x,y)dD$$

tenglik o'rinli ekani ko'rinadi. Demak,

$$\iint\limits_{(D)} f(x,y)dD = \iint\limits_{(\overline{D})} f(x,y)dD = \overline{\iint\limits_{(D)}} f(x,y)dD .$$

Shunday qilib, bir tomondan, qaralayotgan (V) jism hajmga ega ekani ikkinchi tomondan, uning hajmi $f(x,y)$ funksiyaning (D) soha bo'yicha ikki karrali integraliga teng ekani isbot etildi. Demak, (V) jismning hajmi uchun ushbu

$$V = \iint\limits_{(D)} f(x,y)dD \qquad (17.29)$$

formula o'rinli.

17.7-misol. Ushbu

$$\frac{x^2}{a^2}+\frac{y^2}{в^2}+\frac{z^2}{c^2}\le 1$$

ellipsoidning hajmi topilsin.

◄Bu ellipsoid $z=0$ tekislikka nisbatan simmetrikdir. Yuqori qismini $(z\ge 0)$ o'rab turgan sirt

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}$$

bo'ladi.

Yuqoridagi (17.29) formulaga ko'ra ellipsoidning hajmi (V):

$$V=2c\iint\limits_{(D)}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}dxdy$$

bo'ladi, bunda

$$(D)=\left\{(x,y)\in R^2:\ \frac{x^2}{a^2}+\frac{y^2}{в^2}\le 1\right\}.$$

Integralda

$$\begin{aligned} x &= a\rho\cos\varphi \\ y &= в\rho\sin\varphi \end{aligned} \qquad (17.30)$$

almashtirishni bajaramiz. Bu sistemaning yakobiani

$$J(\rho,\varphi)=\begin{vmatrix} a\cos\varphi & в\sin\varphi \\ -a\rho\sin\varphi & в\rho\cos\varphi \end{vmatrix} = a\cdot в\cdot\rho$$

bo'ladi. (17.30) sistema $(\Delta)=\{(\rho,\varphi)\in R^2:\ \ 0\le\rho\le 1\ \ 0\le\varphi\le 2\pi\}$ sohani (D) sohaga akslantiradi. (17.23) formulaga ko'ra

$$\iint\limits_{(D)}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}dxdy=\iint\limits_{(\Delta)}\sqrt{1-\rho^2}\,aв\rho d\rho d\varphi$$

bo'ladi. Demak,

$$V = 2a\,в\,c \iint\limits_{(\Delta)} \sqrt{1-\rho^2}\,\rho d\rho d\varphi = 2a\,в\,c \int_0^1 \left[\int_0^{2\pi} d\varphi \right] \sqrt{1-\rho^2}\,\rho d\rho =$$

$$= 4\pi a\,в\,c \int_0^1 \sqrt{1-\rho^2}\,\rho d\rho = \frac{4\pi}{3} a\,в\,c.$$

Shunday qilib, ellipsoidning hajmi

$$V = \frac{4}{3}\pi a\,в\,c$$

bo’ladi. ►

2⁰. Yassi shaklning yuzi. Ushbu bobning 1-§ ida (D) sohaning yuzi quyidagi

$$D = \iint\limits_{(D)} dD = \iint\limits_{(D)} dxdy$$

integralga teng bo’lishini ko’rdik. Demak, ikki karrali integral yordamida yassi shaklning yuzini hisoblash mumkin ekan.

Xususan, soha

$$(D) = \{(x, y) \in R^2 :\ a \le x \le в\ \ 0 \le y \le f(x)\}$$

egri chiziqli trapesiyadan iborat bo’lsa ($f(x)$ funksiya $[a,\ в]$ da uzluksiz) u holda

$$D = \iint\limits_{(D)} dxdy = \int_a^в \left[\int_0^{f(x)} dy \right] dx = \int_a^в f(x) dx$$

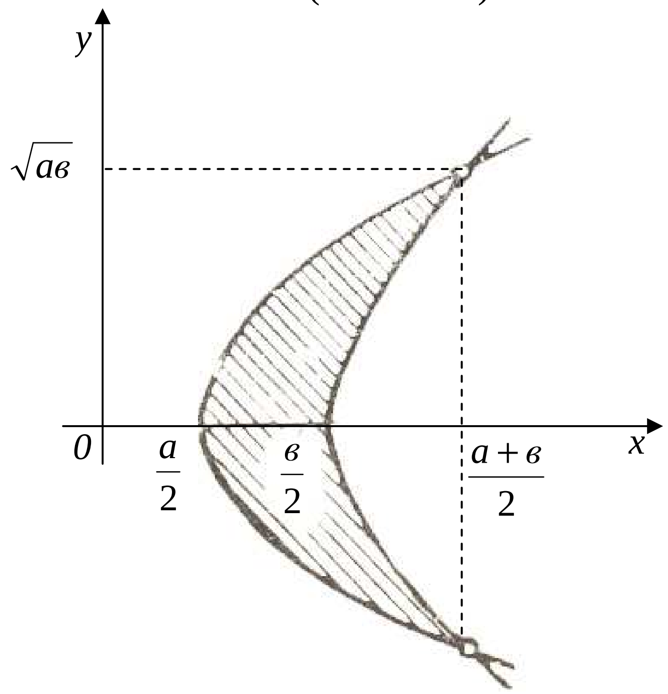
bo’lib, 1-qism, 10-bob, 2-§ da topilgan formulaga kelamiz.

17.8-misol. Ushbu

$$x = \frac{y^2 + a^2}{2a}, \quad x = \frac{y^2 + в^2}{2в}, \quad (0 < a < в)$$

chiziqlar bilan chegaralangan shaklning yuzi topilsin.

◄ Bu chiziqlar parabolalardan iborat (58-chizma).



58-chizma

Quyidagi

$$\begin{cases} x-\dfrac{y^2+a^2}{2a}=0 \\ x-\dfrac{y^2+в^2}{2в}=0 \end{cases}$$

sistemani echib, parabolalarning kesishgan nuqtalari

$$\left(\frac{a+в}{2},\ \sqrt{aв}\right), \quad \left(\frac{a+в}{2},\ -\sqrt{aв}\right)$$

ekanini topamiz. Qaralayotgan shakl Ox o'qiga simmetrik bo'lishini e'tiborga olsak, u holda (D) ning yuzi

$$D=2\iint\limits_{(D_1)} dxdy$$

bo'ladi, bunda

$$(D_1)=\left\{(x,y)\in R^2:\ \frac{y^2+a^2}{2a}\le x\le\frac{y^2+в^2}{2в}\ \ 0\le y\le\sqrt{aв}\right\}.$$

Integralni hisoblab, quyidagini topamiz:

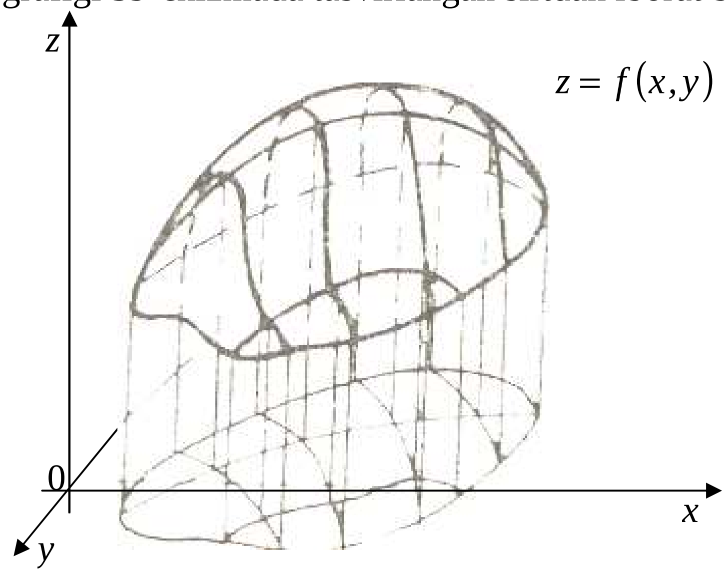
$$\iint\limits_{(D_1)} dxdy=\int\limits_0^{\sqrt{aв}}\left[\int\limits_{\frac{y^2+a^2}{2a}}^{\frac{y^2+в^2}{2в}} dx\right]dy=\int\limits_0^{\sqrt{aв}}\left(\frac{y^2+в^2}{2в}-\frac{y^2+a^2}{2a}\right)dy=\frac{1}{3}(в-a)\sqrt{aв}\ .$$

Demak,

$$D=\iint\limits_{(D)} dxdy=\frac{2}{3}(в-a)\cdot\sqrt{aв}\ .\blacktriangleright$$

3^0. Sirtning yuzi va uning karrali integral orqali ifodalanishi. Ikki karrali integral yordamida sirt yuzini hisoblash mumkin. Avvalo sirtning yuzi tushunchasini keltiramiz.

Faraz qilaylik, $z=f(x,y)$ funksiya (D) sohada berilgan va uzluksiz bo'lsin. Bu funksiyaning grafigi 59-chizmada tasvirlangan sirtdan iborat bo'lsin.



59-chizma

(D) sohaning P bo'laklashni olaylik. Uning bo'laklari (D_1), (D_2),..., (D_n) bo'lsin. Bu bo'laklashning buluvchi chiziqlarini yo'naltiruvchilar sifatida qarab, ular orqali yasovchilari Oz o'qiga parallel bo'lgan silindrik sirtlar o'tkazamiz. Ravshanki, bu silindrik sirtlar (S) sirtni $(S_1),(S_2),...,(S_n)$ bo'laklarga ajratadi. Har bir (D_k) $(k=1,2,...,n)$ da ixtiyoriy (ξ_n,η_n) nuqta olib, (S) sirtda unga mos nuqta (ξ_n,η_n,z_k) $(z_k = f(\xi_n,\eta_n))$ ni topamiz. So'ng (S) sirtga shu (ξ_n,η_n,z_k) nuqtada urinma tekislik o'tkazamiz. Bu urinma tekislik bilan yuqorida aytilgan silindrik sirtning kesishishidan hosil bo'lgan urinma tekislik qismini (T_k) bilan, uning yuzini esa T_k bilan belgilaymiz.

Geometriyadan ma'lumki, (D_k) soha (T_k) ning ortogonal proeksiyasi bo'lib,

$$D_k = T_k|cos\,\gamma_k| \qquad (17.31)$$

bo'ladi, bunda $\gamma_k - (S)$ sirtga (ξ_n,η_n,z_k) $(z_k = f(\xi_n,\eta_n))$ nuqtada o'tkazilgan urinma tekislik normalining Oz o'qi bilan tashkil etgan burchak.

Ravshanki, $\lambda_P \to 0$ da (S_k) $(k=1,2,...,n)$ ning diametri ham nolga intiladi.

Agar $\lambda_P \to 0$ da

$$\sum_{k=1}^{n} T_k$$

yig'indi chekli limitga ega bo'lsa, bu limit (S) sirtning yuzi deb ataladi. Demak, (S) sirtning yuzi

$$S = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} T_k \qquad (17.32)$$

bo'ladi.

Yuqorida qaralayotgan $z = f(x,y)$ funksiya (D) sohada $f'_x(x,y)$, $f'_y(x,y)$ xususiy hosilalarga ega bo'lib, bu xususiy hosilalar (D) sohada uzluksiz bo'lsin. U holda

$$cos\,\gamma_k = \frac{1}{\sqrt{1+f_x'^2(\xi_k,\eta_k)+f_y'^2(\xi_k,\eta_k)}}$$

bo'ladi.

(17.31) munosabatdan

$$T_k = \frac{1}{cos\,\gamma_k} D_k$$

bo'lishini topamiz. Demak,

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{cos\,\gamma_k} D_k = \sum_{k=1}^{n} \sqrt{1+f_x'^2(\xi_k,\eta_k)+f_y'^2(\xi_k,\eta_k)}\,D_k \qquad (17.33)$$

Tenglikning o'ng tomonidagi yig'indi

$$\sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}$$

funksiyaning integral yig'indisidir (qarang 1-§). Bu funksiya (D) sohada uzluksiz, demak, integrallanuvchi. Shuning uchun

$$\lim_{\lambda_P \to 0} \sum_{k=1}^{n} \sqrt{1+f_x'^2(\xi_k,\eta_k)+f_y'^2(\xi_k,\eta_k)}\, D_k = \iint_{(D)} \sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}\, dD$$

bo'ladi.

Shunday qilib, (17.32) va (17.33) munosabatlardan

$$S = \iint_{(D)} \sqrt{1+f_x'^2(x,y)+f_y'^2(x,y)}\, dD \qquad (17.34)$$

bo'lishi kelib chiqadi.

17.9-misol. Asosning radiusi r balandligi h bo'lgan doiraviy konusning yon sirti topilsin.

◄Bunday konus sirtning tenglamasi

$$z = \frac{h}{r}\sqrt{x^2+y^2}$$

bo'ladi. Yuqoridagi (17.34) formulaga ko'ra

$$S = \iint_{(D)} \sqrt{1+z_x'^2+z_y'^2}\, dxdy$$

bo'ladi, bunda

$$(D) = \{(x,y) \in R^2 : x^2+y^2 \le r^2\}.$$

Endi

$$z_x' = \frac{h}{r}\frac{x}{\sqrt{x^2+y^2}}, \qquad z_y' = \frac{h}{r}\frac{y}{\sqrt{x^2+y^2}}$$

va

$$\sqrt{1+z_x'^2+z_y'^2} = \sqrt{1+\frac{h^2}{r^2}\frac{x^2}{x^2+y^2}+\frac{h^2}{r^2}\frac{y^2}{x^2+y^2}} = \sqrt{1+\frac{h^2}{r^2}}$$

ekanini e'tiborga olib, quyidagini topamiz:

$$S = \iint_{(D)} \sqrt{1+\frac{h^2}{r^2}}\, dxdy = \sqrt{1+\frac{h^2}{r^2}} \iint_{(D)} dxdy = \sqrt{1+\frac{h^2}{r^2}}\pi r^2 = \pi r\sqrt{r^2+h^2}\ . ►$$

10-§. Uch karrali integral

Yuqorida Riman integrali tushunchasining ikki o'zgaruvchili funksiya uchun qanday kiritilishini ko'rdik va uni batafsil o'rgandik. Xuddi shunga o'xshash bu tushuncha uch o'zgaruvchili funksiya uchun ham kiritildi. Uni o'rganishda Riman integrali hamda ikki karrali integralda yuritilgan barcha mulohazalar (integrallash sohasining bo'laklanishini olish, bo'laklarda ixtiyoriy nuqta tanlab olib, integral yig'indi tuzish, tegishlicha limitga o'tish va hokazo) qaytariladi. Shuni e'tiborga olib, quyida uch karrali integral haqida faktlarni keltirish bilan chegaralanamiz.

1^0. Uch karrali integral ta'rifi. $f(x,y,z)$ funksiya R^3 fazodagi chegaralangan (V) sohada berilgan bo'lsin. (Bu erda va kelgusida hamma vaqt funksiyaning berilish sohasi (V) ni hajmga ega bo'lgan deb qaraymiz). (V) sohaning P bo'laklashini va bu bo'laklashning har bir (V_k) $(k=1,2,...,n)$ bo'lagida ixtiyoriy (ξ_k,η_k,ζ_k) nuqtani olaylik. So'ngra quyidagi

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot V_k$$

yig'indini tuzamiz, bunda $V_k - (V_k)$ ning hajmi.

Bu yig'indi $f(x,y,z)$ funksiyaning integral yig'indisi yoki Riman yig'indisi deb ataladi.

Endi (V) sohaning shunday

$$P_1, P_2, ..., P_m, ...$$

bo'laklanishlarini qaraymizki, ularning diametridan tashkil topgan

$$\lambda_{P_1}, \lambda_{P_2}, ..., \lambda_{P_m}, ...$$

ketma-ketlik nolga intilsin: $\lambda_{P_m} \to 0$. Bunday P_m $(m = 1, 2, ...)$ bo'laklanishlarga nisbatan $f(x,y,z)$ funksiyaning integral yig'indisini tuzamiz:

$$\sigma_m = \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot V_k$$

Natijada quyidagi

$$\sigma_1, \sigma_2, ..., \sigma_m, ...$$

ketma-ketlik hosil bo'ladi. Bu ketma-ketlikning har bir hadi (ξ_k, η_k, ζ_k) nuqtalarga bog'liq.

9-ta'rif. Agar (V) ning har qanday bo'laklanishlar ketma-ketligi $\{P_m\}$ olinganda ham, unga mos integral yig'indi qiymatlaridan iborat $\{\sigma_m\}$ ketma-ketlik (ξ_k, η_k, ζ_k) nuqtalarni tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta J songa intilsa, bu J son σ yig'indining limiti deb ataladi va u

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) V_k = J$$

kabi belgilanadi.

10-ta'rif. Agar $\lambda_P \to 0$ da $f(x,y,z)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, $f(x,y,z)$ funksiya (V) da integrallanuvchi (Riman ma'nosida integrallanuvchi) funksiya deyiladi. Bu σ yig'indining chekli limiti J esa $f(x,y,z)$ funksiyaning (V) bo'yicha uch karrali integrali (Riman integrali) deyiladi va u

$$\iiint_{(V)} f(x,y,z) dV$$

kabi belgilanadi. Demak,

$$\iiint_{(V)} f(x,y,z) dV = \lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) V_k .$$

$f(x,y,z)$ funksiya (V) da $((V) \subset R^3)$ berilgan bo'lib, u shu sohada chegaralangan bo'lsin:

$$m \le f(x,y,z) \le M \qquad (\forall (x,y,z) \in (V)).$$

(V) sohaning bo'laklanishlar to'plami $\{P\}$ ning har bir bo'laklanishiga nisbatan $f(x,y,z)$ funksiyaning Darbu yig'indilari

$$s_p(f)=\sum_{k=1}^{n} m_k V_k\ , \quad S_p(f)=\sum_{k=1}^{n} M_k V_k$$

ni tuzib, ushbu

$$\{s_P(f)\}, \quad \{S_P(f)\}$$

to’plamlarni qaraylik. Ravshanki, bu to’plamlar chegaralangan bo’ladi.

11-ta’rif. $\{s_P(f)\}$ to’plamning aniq yuqori chegarasi $f(x,y,z)$ funksiyaning quyi uch karrali integrali deb ataladi va u

$$\underline{J}=\underline{\iiint_{(V)}} f(x,y,z)dV$$

kabi belgilanadi.

$\{S_P(f)\}$ to’plamning aniq quyi chegarasi $f(x,y,z)$ funksiyaning yuqori uch karrali integrali deb ataladi va u

$$\overline{J}=\overline{\iiint_{(V)}} f(x,y,z)dV$$

kabi belgilanadi.

12-ta’rif. Agar $f(x,y,z)$ funksiyaning quyi hamda yuqori uch karrali integrallari bir-biriga teng bo’lsa, $f(x,y,z)$ funksiya (V) da integrallanuvchi deb ataladi va ularning umumiy qiymati

$$J=\underline{\iiint_{(V)}} f(x,y,z)dV=\overline{\iiint_{(V)}} f(x,y,z)dV$$

$f(x,y,z)$ funksiyaning uch karrali integrali (Riman integrali) deyiladi.

$$\iiint_{(V)} f(x,y,z)dV=\underline{\iiint_{(V)}} f(x,y,z)dV=\overline{\iiint_{(V)}} f(x,y,z)dV\ .$$

2^0. Uch karrali integralning mavjudligi. $f(x,y,z)$ funksiya $(V)\subset R^3$ sohada berilgan bo’lsin.

10-teorema. $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo’lishi uchun $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topilib, (V) sohaning diametri $\lambda_P<\delta$ bo’lgan har qanday P bo’laklashiga nisbatan Darbu yig’indilari

$$S_p(f)-s_p(f)<\varepsilon$$

tengsizlikni qanoatlantirishi zarur va etarli.

3^0. Integrallanuvchi funksiyalar sinfi. Uch karrali integralning mavjudligi haqidagi teoremadan foydalanib, ma’lum sinf funksiyalarining integrallanuvchi bo’lishi ko’rsatildi.

11-teorema. Agar $f(x,y,z)$ funksiya chegarlangan yopiq $(V)\subset R^3$ sohada berilgan va uzluksiz bo’lsa, u shu sohada integrallanuvchi bo’ladi.

12-teorema. Agar $f(x,y,z)$ funksiya (V) sohada chegaralangan va bu sohaning chekli sondagi nol hajmi sirtlarida uzilishga ega bo’lib, qolgan barcha nuqtalarda uzluksiz bo’lsa, funksiya (V) da integrallanuvchi bo’ladi.

4⁰. Uch karrali integralning xossalari. Uch karrali integrallar ham ushbu bobning 5-§ ida keltirilgan ikki karrali integralning xossalari kabi xossalarga ega.

1). $f(x,y,z)$ funksiya (V) sohada berilgan bo'lib, (V) soha nol hajmi (S) sirt bilan (V_1) va (V_2) sohalarga ajratilgan bo'lsin. Agar $f(x,y,z)$ funksiya (V) sohada integrallanuvchi bo'lsa, funksiya (V_1) va (V_2) sohalarda ham integrallanuvchi bo'ladi, va aksincha, ya'ni $f(x,y,z)$ funksiya (V_1) va (V_2) sohalarning har birida integrallanuvchi bo'lsa, funksiya (V) sohada ham integrallanuvchi bo'ladi. Bunda

$$\iiint\limits_{(V)} f(x,y,z)dV = \iiint\limits_{(V_1)} f(x,y,z)dV + \iiint\limits_{(V_2)} f(x,y,z)dV$$

bo'ladi.

2). Agar $f(x,y,z)$ funksiya (V) da integrallanuvchi bo'lsa, u holda $c\cdot f(x,y,z)$ $(c = const)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iiint\limits_{(V)} cf(x,y,z)dV = c\iiint\limits_{(V)} f(x,y,z)dV$$

formula o'rinli bo'ladi.

3). Agar $f(x,y,z)$ va $g(x,y,z)$ funksiyalar (V) da integrallanuvchi bo'lsa, u holda $f(x,y,z)\pm g(x,y,z)$ funksiya ham shu sohada integrallanuvchi va ushbu

$$\iiint\limits_{(V)} [f(x,y,z)\pm g(x,y,z)]dV = \iiint\limits_{(V)} f(x,y,z)dV \pm \iiint\limits_{(V)} g(x,y,z)dV$$

formula o'rinli bo'ladi.

4). Agar $f(x,y,z)$ funksiya (V) da integrallanuvchi bo'lib, $\forall(x,y,z)\in(V)$ uchun $f(x,y,z)\geq 0$ bo'lsa, u holda

$$\iiint\limits_{(V)} f(x,y,z)dV \geq 0$$

bo'ladi.

5). Agar $f(x,y,z)$ funksiya (V) da integrallanuvchi bo'lsa, u holda $|f(x,y,z)|$ funksiya ham shu sohada integrallanuvchi va

$$\left|\iiint\limits_{(V)} f(x,y,z)dV\right| \leq \iiint\limits_{(V)} |f(x,y,z)|dV$$

bo'ladi.

6). Agar $f(x,y,z)$ funksiya (V) da integrallanuvchi bo'lsa, u holda shunday o'zgarmas μ $(m\leq\mu\leq M)$ son mavjudki,

$$\iiint\limits_{(V)} f(x,y,z)dV = \mu\cdot V$$

bo'ladi, bunda $V - (V)$ sohaning hajmi.

7). Agar $f(x,y,z)$ funksiya yopiq (V) sohada uzluksiz bo'lsa, u holda bu sohada shunday $(a,в,c)\in(V)$ nuqta topiladiki,

$$\iiint\limits_{(V)} f(x,y,z)dV = f(a,в,c)\cdot V$$

bo'ladi.

5⁰. Uch karrali integrallarni hisoblash. $f(x,y,z)$ funksiya

$(V)=\{(x,y,z)\in R^3 :\ a\le x\le в,\ c\le y\le d,\ e\le z\le l\}$ sohada (parallelepipedda) berilgan va uzluksiz bo'lsin. U holda

$$\iiint\limits_{(V)} f(x,y,z)dV=\int\limits_a^в\left[\int\limits_c^d\left(\int\limits_e^l f(x,y,z)dz\right)dy\right]dx$$

bo'ladi.

Endi $(V)\subset R^3$ soha – pastdan $z=\psi_1(x,y)$ yuqoridan $z=\psi_2(x,y)$ sirtlar bilan, yon tomondan esa Oz o'qiga parallel silindrik sirt bilan chegaralangan soha bo'lsin. Bu sohaning Oxy tekislikdagi proeksiyasi esa (D) bo'lsin.

Agar $f(x,y,z)$ funksiya shunday (V) sohada uzluksiz bo'lib, $z=\psi_1(x,y)$, $z=\psi_2(x,y)$ funksiyalar (D) da uzluksiz bo'lsa, u holda

$$\iiint\limits_{(V)} f(x,y,z)dV=\iint\limits_{(D)}\left(\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right)dxdy$$

bo'ladi. Agar yuqoridagi holda $(D)=\{(x,y)\in R^2 :\ a\le x\le в,\ \varphi_1(x)\le y\le \varphi_2(x)$ bo'lib, $\varphi_1(x)$ va $\varphi_2(x)$ funksiyalar $[a,\ в]$ da uzluksiz bo'lsa, u holda

$$\iiint\limits_{(V)} f(x,y,z)dV=\int\limits_a^в\left[\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left(\int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right)dy\right]dx$$

bo'ladi.

6⁰. Uch karrali integrallarda o'zgaruvchilarni almashtirish. Uch karrali integrallarda o'zgaruvchilarni almashtirish ushbu bobning 7-§ da keltirilgan ikki karrali integrallarda o'zgaruvchilarni almashtirish kabidir. Shuni hisobga olib, quyida uch karrali integrallarda o'zgaruvchilarni almashtirish formulasini keltirish bilan kifoyalanamiz.

$f(x,y,z)$ funksiya $(V)\subset R^3$ sohada berilgan va uzluksiz bo'lsin, (V) soha esa silliq yoki bo'lakli-silliq sirtlar bilan chegaralangan bo'lsin.

Ushbu

$$x=\varphi(u,v,w),$$
$$y=\psi(u,v,w),$$
$$z=\zeta(u,v,w)$$

sistema (Δ) $\left((\Delta)\in R^3\right)$ sohani (V) sohaga akslantirsin va bu akslantirish 7-§ da keltirilgan 1^0-3^0-shartlarni bajarsin. U holda

$$\iiint\limits_{(V)} f(x,y,z)dV=\iiint\limits_{(\Delta)} f\left(\varphi(u,v,w),\ \psi(u,v,w),\ \zeta(u,v,w)\right)\cdot|J|dudvdw$$

bo'ladi. Bunda

$$J=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w}\\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{vmatrix}$$

7^0. ***Uch karrali integralning ba'zi bir tatbiqlari.*** Uch karrali integral yordamida R^3 fazodagi jismning hajmini, jismning massasini, inersiya momentlarini topish mumkin.

Mashqlar

17.10. Ikki karrali integral ta'riflarining ekvivalentligi isbotlansin.

17.11. Ushbu

$$\text{a) } \int_{-1}^{1} dx \int_{x^2}^{1} f(x,y)dy$$

$$\text{b) } \int_{0}^{2\pi} dx \int_{0}^{\sin x} f(x,y)dy$$

integrallarda integral tartibi o'zgartirilsin.

17.12. Quyidagi

$$-\frac{\pi}{2} < \iint_{(D)} \left(x^2 - y^2\right) dxdy < 4\pi$$

tengsizliklar isbotlansin, bunda $(D) - x^2 + y^2 - 2x = 0$ aylana bilan chegaralangan soha.

17.13. Agar $f(x)$ funksiya $[a,\ в]$ da uzluksiz bo'lsa.

$$\left[\int_a^в f(x)dx\right]^2 \le (в - a)\int_a^в f^2(x)dx$$

bo'lishi isbotlansin.

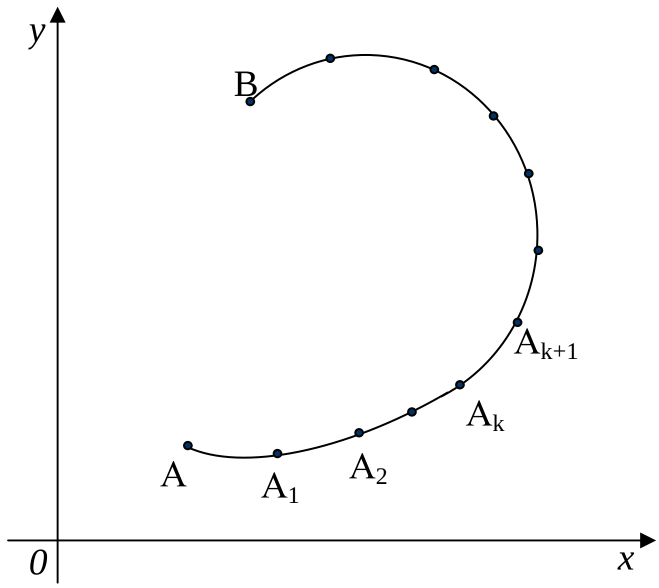
18-BOB

Egri chiziqli integrallar

Yuqoridagi bobda Riman integrali tushunchasini ikki o'zgaruvchili funksiya uchun qanday kiritilishini ko'rdik va uni o'rgandik. Shuni ham aytish kerakki, ko'p o'zgaruvchili funksiyalar uchun integral tushunchasi turlicha kiritilishi mumkin. Biz quyida keltirgan egri chiziqli integrallar ham konkret amaliy masalalardan paydo bo'lgandir.

1-§. Birinchi tur egri chiziqli integrallar

1^0. ***Birinchi tur egri chiziqli integral ta'rifi.*** Tekislikda biror sodda $\breve{AB}$ $\left(A = (a_1, a_2) \in R^2, B = (в_1, в_2) \in R^2\right)$ egri chiziqni (yoyni) olaylik (60-chizma). Bu egri chiziqda ikki yo'nalishdan birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik



60-chizma

$\breve{AB}$ egri chiziqni A dan B ga qarab $A_0 = A,\ A_1, \ldots, A_n = B$, $\left(A_k = (x_k, y_k) \in \breve{AB},\ k = 0,1,2,\ldots,n,\ A_0 = (x_0, y_0) = (a_1, a_2),\ A_n = (x_n, y_n) = (в_1, в_2)\right)$ nuqtalar yordamida n ta bo'lakka bo'lamiz. Bu $A_0, A_1, \ldots, A_n$ nuqtalar sistemasi $\breve{AB}$ yoyining bo'laklash deb ataladi va u

$$P = \{A_0, A_1, \ldots, A_n\}$$

kabi belgilanadi. $\breve{A_k A_{k+1}}$ yoy (bo'laklash yoylari) uzunliklari Δs_k $(k = 0,1,2,\ldots,n)$ ning eng kattasi P bo'laklash diametri deyiladi va u λ_P bilan belgilanadi:

$$\lambda_P = \max_k \{\Delta s_k\}.$$

Ravshanki, $\breve{AB}$ egri chiziqni turli usullar bilan istalgan sonda bo'laklashlarini tuzish mumkin.

$\breve{AB}$ egri chiziqda $f(x, y)$ funksiya berilgan bo'lsin. Bu egri chiziqning

$$P = \{A_0, A_1, \ldots, A_n\}$$

bo'laklanishi va uning har bir $\breve{A_k A_{k+1}}$ yoyida ixtiyoriy $Q_k = (\xi_k, \eta_k)$ $\left(Q_k = (\xi_k, \eta_k) \in \breve{A_k A_{k+1}},\ k = 0,1,\ldots,n-1\right)$ nuqtani olamiz. Berilgan funksiyaning $Q_k = (\xi_k, \eta_k)$ nuqtadagi $f(\xi_k, \eta_k)$ qiymatini $\breve{A_k A_{k+1}}$ ning Δs_k uzunligiga ko'pay-tirib quyidagi yig'indini tuzamiz:

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \Delta s_k. \qquad (18.1)$$

Endi $\breve{AB}$ egri chiziqning shunday

$$P_1,\ P_2,\ \ldots,\ P_m,\ \ldots \qquad (18.2)$$

bo'laklashlari ketma-ketligini qaraymizki, ularning mos diametrlaridan tashkil topgan $\lambda_{P_1}, \lambda_{P_2}, \ldots \lambda_{P_m}, \ldots$ ketma-ketlik nolga intilsin: $\lambda_{P_m} \to 0$. Bunday bo'laklash-larga nisbatan (18.1) kabi yig'indilarni tuzib, ushbu

$$\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_m,\ \ldots$$

ketma-ketlikni hosil qilamiz. Ravshanki, bu ketma-ketlikning har bir hadi $Q_k = (\xi_k, \eta_k)$ nuqtalarga bog'liq.

1-ta'rif. Agar $\breve{A}B$ egri chiziqning har qanday (18.2) ko'rinishdagi bo'laklash-lari ketma-ketligi $\{P_m\}$ olinganda ham, unga mos yig'indilardan iborat $\{\sigma_m\}$ ketma-ketlik (ξ_k, η_k) nuqtalarning tanlab olinishiga bog'liq bo'lmagan holda ham-ma vaqt bitta J songa intilsa, bu son σ yig'indining limiti deb ataladi va

$$\lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta s_k = J \qquad (18.3)$$

kabi belgilanadi.

2-ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ topilsaki, $\breve{A}B$ egri chiziqning diametri $\lambda_P < \delta$ bo'lgan har qanday P bo'laklash uchun tuzilgan σ yig'indi ixtiyoriy $(\xi_k, \eta_k) \in \breve{A}_k A_{k+1}$ nuqtalarda

$$|\sigma - J| < \varepsilon$$

tengsizlikni bajarsa, J son σ yig'indining $\lambda_P \to 0$ dagi limiti deb ataladi va (18.3) kabi belgilanadi.

(18.1) yig'indi limitining bu ta'riflari ekvivalent ta'riflardir.

3-ta'rif. Agar $\lambda_P \to 0$ da σ yig'indi chekli limitga ega bo'lsa, $f(x,y)$ funksiya $\breve{A}B$ egri chiziq bo'yicha integrallanuvchi deyiladi. Bu limit $f(x,y)$ funksiyaning egri chiziq bo'yicha birinchi tur egri chiziqli integrali deb ataladi va u

$$\int_{\breve{A}B} f(x,y)dS$$

kabi belgilanadi.

Shunday qilib, kiritilgan egri chiziqli integral tushunchasining o'ziga xosligi qaralayotgan ikki argumentli funksiyaning berilish sohasi tekislikda biror $\breve{A}B$ egri chiziq ekanligidir. Qolgan boshqa mulohazalar (bo'laklashlarining olinishi, bo'laklardan ixtiyoriy nuqta tanlab integral yig'indi tuzish, tegishlicha limitga o'tish) yuqorida kiritilgan integral tushunchalari singaridir.

2⁰. Uzluksiz funksiya birinchi tur egri chiziqli integrali. Yuqorida keltirilgan 3-ta'rifdan ko'rinadiki, birinchi tur egri chiziqli integral $\breve{A}B$ egri chiziqqa hamda unda berilgan $f(x,y)$ funksiyaga bog'liq bo'ladi.

Faraz qilaylik, $\breve{A}B$ egri chiziq ushbu

$$\begin{cases} x = x(s), \\ y = y(s) \end{cases} \qquad (0 \le s \le S) \qquad (18.4)$$

sistema bilan berilgan bo'lsin. bunda $s - \breve{A}Q$ yoyining uzunligi $\left(Q = (x,y) \in \breve{A}B\right)$ S esa $\breve{A}B$ ning uzunligi. $f(x,y)$ funksiya shu $\breve{A}B$ egri chiziqda berilgan bo'lsin. Modomiki, $x = x(s)$, $y = y(s)$ $(0 \le s \le S)$ ekan, unda $f(x,y) = f(x(s), y(s))$ bo'lib, natijada ushbu

$$f(x(s), y(s)) = F(s) \qquad (0 \le s \le S)$$

murakkab funksiyaga ega bo'lamiz.

$\breve{AB}$ egri chiziqning $P=\{A_0, A_1, ..., A_n\}$ bo'laklashini va har bir $\breve{A_k A_{k+1}}$ da ixtiyoriy $Q_k=(\xi_k, \eta_k)$ nuqtani olaylik. Har bir A_k nuqtaga mos keladigan $\breve{AA_k}$ ning uzunligi s_k, har bir Q_k nuqtaga mos keladigan $\breve{AQ_k}$ ning uzunligi s_k^* deylik. Ravshanki, $\breve{A_k A_{k+1}}$ ning uzunligi $s_{k+1}-s_k=\Delta s_k$ bo'ladi.

Natijada P bo'laklashga nisbatan tuzilgan

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta s_k$$

yig'indi ushbu

$$\sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta s_k=\sum_{k=0}^{n-1} f\left(x\left(s_k^*\right), y\left(s_k^*\right)\right)\Delta s_k=\sum_{k=0}^{n-1} F\left(s_k^*\right)\Delta s_k$$

ko'rinishga keladi. Demak,

$$\sigma=\sum_{k=0}^{n-1} F\left(s_k^*\right)\Delta s_k. \qquad (18.5)$$

Bu yig'indini $[0, S]$ oraliqdagi $F(s)$ funksiyaning integral yig'indisi (Riman yig'indisi) ekanligini payqash qiyin emas (qaralsin, 1-qism, 9-bob, 1-§).

Agar $f(x,y)$ funksiya $\breve{AB}$ egri chiziqda uzluksiz bo'lsa, u holda $F(s)$ funksiya $[0, S]$ da uzluksiz bo'ladi. Demak, bu holda $F(s)$ funksiya $[0, S]$ da integrallanuvchi:

$$\lim_{\lambda_P\to 0}\sum_{k=0}^{n-1} F\left(s_k^*\right)\Delta s_k=\int_0^S F(s)ds. \qquad (18.6)$$

Shunday qilib, (18.5), (18.6) munosabatlardan $\lambda_P\to 0$ da σ yig'indining limiti mavjud va

$$\lim_{\lambda_P\to 0}\sigma=\int_0^S F(s)ds$$

ekanligini topamiz. Natijada quyidagi teoremaga kelamiz.

1-teorema. Agar $f(x,y)$ funksiya $\breve{AB}$ egri chiziqda uzluksiz bo'lsa, u holda bu funksiyaning $\breve{AB}$ egri chiziq bo'yicha birinchi tur egri chiziqli integrali mavjud va

$$\int_{\breve{AB}} f(x,y)ds=\int_0^S f(x(s), y(s))ds$$

bo'ladi.

Bu teorema, bir tomondan uzluksiz funksiya birinchi tur egri chiziqli integralining mavjudligini aniqlab bersa, ikkinchi tomondan bu integralning aniq integralga (Riman integraliga) kelishini ko'rsatadi.

1-eslatma. Ushbu

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta s_k$$

yig'indidagi Δs_k har doim musbat bo'lib, $\breve{AB}$ egri chiziqning yo'nalishiga bog'liq emas. Demak,

$$\int_{\breve{AB}} f(x,y)ds = \int_{\breve{BA}} f(x,y)ds$$

3⁰. Birinchi tur egri chiziqli integrallarning xossalari. Yuqorida ko'rdikki, uzluksiz funksiyalarning birinchi tur egri chiziqli integrallari Riman integrallariga keladi. Binobarin, egri chiziqli integrallar ham Riman integrallari xossalari kabi xossalarga ega bo'ladi. Shuni e'tiborga olib, egri chiziqli integrallarning asosiy xossalarini sanab o'tish bilan kifoyalanamiz.

(18.4) sistema bilan aniqlangan $\breve{AB}$ egri chiziqda $f(x,y)$ funksiya uzluksiz bo'lsin

1). Agar $\breve{AB} = \breve{AC} + \breve{CB}$ bo'lsa, u holda

$$\int_{\breve{AB}} f(x,y)ds = \int_{\breve{AC}} f(x,y)ds + \int_{\breve{CB}} f(x,y)ds$$

bo'ladi.

2). Ushbu

$$\int_{\breve{AB}} cf(x,y)ds = c\int_{\breve{AB}} f(x,y)ds \quad (c = const)$$

tenglik o'rinli.

$\breve{AB}$ egri chiziqda $f(x,y)$ funksiya va $g(x,y)$ funksiyalar uzluksiz bo'lsin.

3). Quyidagi

$$\int_{\breve{AB}} [f(x,y) \pm g(x,y)]ds = \int_{\breve{AB}} f(x,y)ds \pm \int_{\breve{AB}} g(x,y)ds$$

formula o'rinli bo'ladi.

4). Agar $\forall (x,y) \in \breve{AB}$ da $f(x,y) \geq 0$ bo'lsa, u holda

$$\int_{\breve{AB}} f(x,y)ds \geq 0$$

bo'ladi.

5). $f(x,y)$ funksiya shu $\breve{AB}$ da integrallanuvchi va

$$\left|\int_{\breve{AB}} f(x,y)ds\right| \leq \int_{\breve{AB}} |f(x,y)|ds$$

bo'ladi.

6). Shunday $(c_1, c_2) \in \breve{AB}$ nuqta topiladiki,

$$\int_{\breve{AB}} f(x,y)ds = f(c_1, c_2) \cdot S$$

bo'ladi, bunda $S - \breve{AB}$ ning uzunligi.

Bu xossa o'rta qiymat haqidagi teorema deb ataladi.

4⁰. Birinchi tur egri chiziqli integrallarni hisoblash. Birinchi tur egri chiziqli integrallar, asosan Riman integrallariga keltirilib hisoblanadi.

$\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t), \\ y = \psi(t) \end{cases} \qquad (\alpha \leq t \leq \beta) \qquad \textit{(18.7)}$$

sistema bilan (parametrik formada) berilgan bo'lsin. Bunda $\varphi(t)$, $\psi(t)$ funksiyalar $[\alpha,\ \beta]$ da $\varphi'(t)$, $\psi'(t)$ hosilalarga ega va bu hosilalar shu oraliqda uzluksiz hamda $(\varphi(\alpha),\psi(\alpha))=A$ va $(\varphi(\beta),\psi(\beta))=B$ bo'lsin.

Ravshanki, (18.7) sistema $[\alpha,\ \beta]$ oraliqni $\breve{A}B$ egri chiziqqa akslantiradi. Bunda $[\gamma,\ \delta]\subset[\alpha,\ \beta]$ ning $\breve{A}B$ chiziqdagi $\breve{A}_\gamma B_\delta$ aksning uzunligi

$$\int_\gamma^\delta \sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

bo'ladi. (qaralsin, 1-qism, 10-bob, 1-§).

2-teorema. Agar $f(x,y)$ funksiya $\breve{A}B$ da uzluksiz bo'lsa, u holda

$$\int_{\breve{A}B} f(x,y)dS = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

bo'ladi.

◀ $[\alpha,\ \beta]$ oraliqning

$$P=\{t_0,\ t_1,\ ...,\ t_n\},\ (\alpha=t_0<t_1<...<t_n=\beta)$$

bo'laklashini olaylik. Bu bo'laklashning bo'luvchi nuqtalari t_k $(k=0,1,2,...,n)$ ning $\breve{A}B$ dagi mos akslarini A_k $(k=0,1,2,...,n)$ deylik. Ravshanki, bu A_k $(k=0,1,2,...,n)$ nuqtalar $\breve{A}B$ egri chiziqning

$$\{A_0,A_1,...,A_n\}$$

bo'laklashini hosil qiladi. Bunda $A_k=(\varphi(t_k),\ \psi(t_k))$ $(k=0,1,2,...,n)$ va $\breve{A}_k A_{k+1}$ ning uzunligi

$$\Delta s_k=\int_{t_k}^{t_{k+1}} \sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

bo'ladi. O'rta qiymat haqidagi teoremadan foydalanib quyidagini topamiz:

$$\Delta s_k=\sqrt{\varphi'^2(r_k)+\psi'^2(r_k)}\cdot(t_{k+1}-t_k)=\sqrt{\varphi'^2(r_k)+\psi'^2(r_k)}\cdot\Delta t_k$$

bunda $t_k<r_k<t_{k+1}$.

Endi $\varphi(r_k)=\xi_k$, $\psi(r_k)=\eta_k$ deb olamiz. Ravshanki, $(\xi_k,\eta_k)\in\breve{A}_k A_{k+1}$ $(k=0,\ 1,\ 2,\ ...,\ n-1)$ bo'ladi. $\breve{A}B$ egri chiziqning yuqorida aytilgan

$$\{A_1,A_2,...,A_n\}$$

bo'laklashni va har bir $\breve{A}_k A_{k+1}$ bo'lakchasida (ξ_k,η_k) nuqtani olib,

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta s_k$$

yig'indini tuzamiz. Uni quyidagicha ham yozish mumkin:

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta s_k=\sum_{k=0}^{n-1} f(\varphi(r_k),\psi(r_k))\sqrt{\varphi'^2(r_k)+\psi'^2(r_k)}\Delta t_k\,. \qquad (18.8)$$

Bu tenglikning o'ng tomonidagi yig'indi $f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}$ funksiyaning $[\alpha,\ \beta]$ oraliqdagi Riman yig'indisidir.

Shartga ko'ra $f(x,y)$ va $\varphi'(t)$, $\psi'(t)$ funksiyalar uzluksiz. Demak, murakkab funksiyaning uzluksizligi haqidagi teoremaga ko'ra $f(\varphi(t),\psi(t))$ va, demak, $f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}$ funksiya $[\alpha,\ \beta]$ oraliqda uzluksiz. Demak, bu funksiya $[\alpha,\ \beta]$ da integrallanuvchi bo'ladi. Ya'ni

$$\lim_{max\{\Delta t_k\}\to 0}\sum_{k=0}^{n-1} f(\varphi(r_k),\psi(r_k))\sqrt{\varphi'^2(r_k)+\psi'^2(r_k)}\Delta t_k = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt\,.$$

Modomiki, $x=\varphi(t)$, $y=\psi(t)$ funksiyalar $[\alpha,\ \beta]$ da uzluksiz ekan, unda $max\{\Delta t_k\}\to 0$ da $\Delta x_k\to 0$, $\Delta y_k\to 0$ va, demak, $\Delta s_k\to 0$. Bundan esa $\lambda_P\to 0$ bo'lishi kelib chiqadi. (18.8) munosabatdan foydalanib

$$\lim_{\lambda_P\to 0}\sigma = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

bo'lishini topamiz. Bu esa

$$\int_{\breve{A}B} f(x,y)dS = \int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

ekanini bildiradi.►

Bu teoremadan quyidagi natijalar kelib chiqadi.

1-natija. $\breve{A}B$ egri chiziq ushbu

$$y=y(x)\quad (a\le x\le в,\ \ y(a)=A,\quad y(в)=B)$$

tenglama bilan aniqlangan bo'lib, $y(x)$ funksiya $[a,\ в]$ da hosilaga ega va u uzluksiz bo'lsin. Agar $f(x,y)$ funksiya shu $\breve{A}B$ da uzluksiz bo'lsa, u holda

$$\int_{\breve{A}B} f(x,y)dS = \int_a^в f(x,y(x))\sqrt{1+y'^2(x)}dx$$

bo'ladi.

2-natija. $\breve{A}B$ egri chiziq ushbu

$$\rho=\rho(\theta)\quad (\theta_0\le\theta\le\theta_1)$$

tenglama bilan (qutb koordinata sistemasida) berilgan bo'lib, $\rho(\theta)$ funksiya $[\theta_0,\ \theta_1]$ da hosilaga ega va u uzluksiz bo'lsin. Agar $f(x,y)$ funksiya shu $\breve{A}B$ da uzluksiz bo'lsa, u holda

$$\int_{\breve{A}B} f(x,y)dS = \int_{\theta_0}^{\theta_1} f(\rho\cos\theta,\ \rho\sin\theta)\sqrt{\rho^2(\theta)+\rho'^2(\theta)}d\theta$$

bo'ladi.

Bu natijalarni isbotlashni o'quvchiga havola etamiz.

18.1-misol. Ushbu

$$\int_{\breve{A}B}\sqrt{x^2+y^2}ds$$

egri chiziqli integral hisoblansin, bunda $\breve{A}B$ -markazi koordinata boshida, radiusi $r>0$ ga teng bo'lgan aylananing yuqori yarim tekislikdagi qismi.

◄Ravshanki, bu egri chiziq quyidagi

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases} \qquad (0 \le t \le \pi)$$

sistema bilan aniqlanadi. $\breve{A}B$ da $f(x,y) = \sqrt{x^2+y^2} = \sqrt{(r\cos t)^2 + (r\sin t)^2}$ funksiya uzluksiz. Demak,

$$\int_{\breve{A}B} \sqrt{x^2+y^2}\,ds = \int_0^{\pi} \sqrt{(r\cos t)^2 + (\sin t)^2}\sqrt{(r\cos t)'^2 + (r\sin t)'^2}\,dt = r^2\int_0^{\pi} dt = \pi r^2$$

bo'ladi.▶

5⁰. Birinchi tur egri chiziqli integrallarning ba'zi bir tatbiqlari. Birinchi tur egri chiziqli integrallar yordamida yoy uzunligini, jismning massasini, og'irlik markazlarini topish mumkin.

Tekislikda sodda $\breve{A}B$ egri chiziq berilgan bo'lsin. Bu chiziqda $f(x,y) = 1$ funksiyani qaraylik. Ravshanki, bu funksiya $\breve{A}B$ da uzluksiz. $f(x,y)$ funksiyaning birinchi tur egri chiziqli integrali ta'rifidan quyidagini topamiz:

$$\int_{\breve{A}B} 1ds = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} 1\Delta s_k = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \Delta s_k = S\,.$$

Demak,

$$S = \int_{\breve{A}B} ds \qquad (18.9)$$

18.2-misol. Ushbu

$$x = x(t) = a\cos^3 t\,,$$
$$y = y(t) = a\sin^3 t$$

sistema bilan berilgan $\breve{A}B$ chiziqning uzunligi topilsin. (Bu chiziq astroidani ifodalaydi).

◀Yuqoridagi formulaga ko'ra astroidaning uzunligi

$$S = \int_{\breve{A}B} ds$$

bo'ladi. Astroida koordinata o'qlariga nisbatan simmetrik bo'lishini e'tiborga olib, yuqorida keltirilgan (18.9) formuladan foydalanib quyidagini topamiz:

$$\int_{\breve{A}B} ds = 4\int_0^{\pi/2} \sqrt{x'^2(t) + y'^2(t)}\,dt = 4\int_0^{\pi/2} \sqrt{\left(-3a\cos^2 t\sin t\right)^2 + \left(3a\sin^2 t\cos t\right)^2}\,dt =$$

$$= 4\int_0^{\pi/2} \sqrt{\frac{9a^2}{4}\sin^2 2t}\,dt = 6a\int_0^{\pi/2} \sin 2t\,dt = 6a\left(-\frac{\cos 2t}{2}\right)\Bigg|_0^{\pi/2} = 6a\,.$$ ▶

2-§. Ikkinchi tur egri chiziqli integrallar

1⁰. Ikkinchi tur egri chiziqli integrallar ta'rifi. Tekislikda biror sodda $\breve{A}B$ egri chiziqni qaraylik. Bu egri chiziqda $f(x,y)$ funksiya berilgan bo'lsin. $\breve{A}B$ egri chiziqning

$$P = \{A_0, A_1, \ldots, A_n\}$$

bo'laklanishini va uning har bir $\breve{A}_k A_{k+1}$ $(k=0,1,2,...,n-1)$ yoyida ixtiyoriy $Q_k=(\xi_k,\eta_k)$ nuqtani $\left(Q_k=(\xi_k,\eta_k)\in \breve{A}_k A_{k+1},\ k=0,1,2,...,n-1\right)$ olaylik. Berilgan funksiyaning $Q_k=(\xi_k,\eta_k)$ nuqtadagi $f(\xi_k,\eta_k)$ qiymatini $\breve{A}_k A_{k+1}$ ning Ox (Oy) o'qidagi Δx_k (Δy_k) proeksiyasiga ko'paytirib quyidagi yig'indini tuzamiz:

$$\sigma'=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta x_k \quad \left(\sigma''=\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta y_k\right). \tag{18.10}$$

Endi $\breve{A}B$ egri chiziqning shunday

$$P_1,\ P_2,\ ...,\ P_m,\ ... \tag{18.11}$$

bo'laklari ketma-ketligini qaraymizki, ularning diametrlaridan tashkil topgan mos

$$\lambda_{P_1},\ \lambda_{P_2},\ ...,\ \lambda_{P_m},\ ...$$

ketma-ketlik nolga intilsin:

$$\lambda_{P_m}\to 0.$$

Bunday bo'laklashlarga nisbatan (18.10) kabi yig'indilarni tuzib ushbu

$$\sigma_1',\ \sigma_2',\ ...,\ \sigma_m',\ ...\ (\sigma_1'',\ \sigma_2'',\ ...,\ \sigma_m'',\ ...)$$

ketma-ketlik hosil qilamiz.

4-ta'rif. Agar $\breve{A}B$ egri chiziqning har qanday (18.11) ko'rinishdagi bo'laklashlari ketma-ketligi $\{P_m\}$ olinganda ham, unga mos yig'indilardan iborat $\{\sigma_m'\}$ $(\{\sigma_m''\})$ ketma-ketlik (ξ_k,η_k) nuqtalarning $\left((\xi_k,\eta_k)\in \breve{A}_k A_{k+1}\right)$ tanlab olinishiga bog'liq bo'lmagan ravishda hamma vaqt bitta J' songa (J'' songa) intilsa, bu son σ' (σ'') yig'indining limiti deb ataladi va

$$\lim_{\lambda_p\to 0}\sigma'=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta x_k=J'$$
$$\left(\lim_{\lambda_p\to 0}\sigma''=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1} f(\xi_k,\eta_k)\Delta y_k=J''\right) \tag{18.12}$$

kabi belgilanadi. σ' (σ'') yig'indining bu limitini quyidagicha ham ta'riflash mumkin.

5-ta'rif. Agar $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topiladiki, $\breve{A}B$ egri chiziqning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklash uchun tuzilgan σ' (σ'') yig'indi ixtiyoriy (ξ_k,η_k) nuqtalarda $\left((\xi_k,\eta_k)\in \breve{A}_k A_{k+1},\ k=0,1,2,...,n-1\right)$

$$|\sigma'-J'|<\varepsilon \qquad \left(|\sigma''-J''|<\varepsilon\right)$$

tengsizlikni bajarsa, J' son (J'' son) σ' yig'indining ((σ'') yig'indining) $\lambda_P\to 0$ dagi limiti deb ataladi va (18.12) kabi belgilanadi.

Yig'indi limitining bu ta'riflari ekvivalent ta'riflardir.

6-ta'rif. Agar $\lambda_P\to 0$ da σ' yig'indi ((σ'') yig'indi) chekli limitga ega bo'lsa, $f(x,y)$ funksiya $\breve{A}B$ egri chiziq bo'yicha integrallanuvchi deyiladi. Bu limit $f(x,y)$ funksiyaning $\breve{A}B$ egri chiziq bo'yicha ikkinchi tur egri chiziqli integrali deb ataladi va u

$$\int\limits_{\breve{A}B} f(x,y)dx \qquad \left(\int\limits_{\breve{A}B} f(x,y)dy \right)$$

kabi belgilanadi. Demak,

$$\int\limits_{\breve{A}B} f(x,y)dx = \lim_{\lambda_p \to 0} \sigma' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k ,$$

$$\left(\int\limits_{\breve{A}B} f(x,y)dy = \lim_{\lambda_p \to 0} \sigma'' = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta y_k \right).$$

Shunday qilib, $\breve{A}B$ egri chiziqda berilgan $f(x,y)$ funksiyadan ikkita Ox o'qidagi proeksiyalar vositasida va Oy o'qidagi proeksiyalar vositasida olingan ikkinchi tur egri chiziqli integral tushunchalar kiritildi.

Faraz qilaylik, $\breve{A}B$ egri chiziqda ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib, $\int\limits_{\breve{A}B} P(x,y)dx$, $\int\limits_{\breve{A}B} Q(x,y)dy$ lar esa ularning ikkinchi tur egri chiziqli integrallari bo'lsin. Ushbu

$$\int\limits_{\breve{A}B} P(x,y)dx + \int\limits_{\breve{A}B} Q(x,y)dy$$

yig'indi ikkinchi tur egri chiziqli integralning umumiy ko'rinishi deb ataladi va

$$\int\limits_{\breve{A}B} P(x,y)dx + Q(x,y)dy$$

kabi yoziladi. Demak,

$$\int\limits_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int\limits_{\breve{A}B} P(x,y)dx + \int\limits_{\breve{A}B} Q(x,y)dy .$$

Ikkinchi tur egri chiziqli integral ta'rifidan quyidagi natijalar kelib chiqadi.

3-natija. Ikkinchi tur egri chiziqli integral egri chiziqning yo'nalishiga bog'liq bo'ladi. Shuni isbotlaylik

Ma'lumki, $\breve{A}B$ egri chiziqda ikkita yo'nalish (A nuqtadan B nuqtaga va B nuqtadan A nuqtaga) olish mumkin ($\breve{A}B, B\breve{A}$, $A \neq B$).

$\breve{A}B$ egri chiziqning yuqoridagi P bo'laklashni olib, bu bo'laklashga nisbatan (18.10) yig'indini tuzamiz:

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k \qquad (\Delta x_k = x_{k+1} - x_k).$$

Aytaylik, $\lambda_P \to 0$ da bu yig'indi chekli limitga ega bo'lsin:

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k = \int\limits_{\breve{A}B} f(x,y)dx .$$

Endi $\breve{A}B$ ning usha P bo'laklashini hamda har bir $\breve{A}_k A_{k+1}$ dagi usha (ξ_k, η_k) nuqtalarni olib, $\breve{A}B$ egri chiziqning yo'nalishini esa B dan A ga qarab deb ushbu yig'indini tuzamiz:

$$\overline{\sigma'} = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)(x_k - x_{k+1})$$

$\lambda_P \to 0$ da bu yig'indi chekli limitga ega bo'lsa, u ta'rifga binoan ushbu

$$\int\limits_{\breve{B}A} f(x,y)dx$$

integral bo'ladi:

$$\lim_{\lambda_p \to 0} \overline{\sigma'} = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot (x_k - x_{k+1}) = \int\limits_{\breve{B}A} f(x,y)dx .$$

Agar

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot \Delta x_k = -\sum_{k=0}^{n-1} f(\xi_k, \eta_k) \cdot (x_k - x_{k+1}) = -\overline{\sigma'}$$

ekanligini e'tiborga olsak, u holda $\lambda_P \to 0$ da σ_1 yig'indining chekli limitga ega bo'lishidan $\overline{\sigma}_1$ yig'indining ham chekli limitga ega bo'lishi va $\lim\limits_{\lambda_P \to 0} \overline{\sigma}_1 = -\lim\limits_{\lambda_P \to 0} \sigma_1$ tenglikning bajarilishini topamiz. Demak,

$$\int\limits_{\breve{B}A} f(x,y)dx = -\int\limits_{A\breve{B}} f(x,y)dx .$$

Xuddi shunga o'xshash

$$\int\limits_{\breve{B}A} f(x,y)dy = -\int\limits_{A\breve{B}} f(x,y)dy$$

bo'ladi.

4-natija. $\breve{A}B$ egri chiziq Ox o'qiga (Oy o'qiga) perpendikulyar bo'lgan to'g'ri chiziq kesmasidan iborat bo'lib, $f(x,y)$ funksiya shu chiziqda berilgan bo'lsin.

U holda

$$\int\limits_{\breve{A}B} f(x,y)dx \quad \left(\int\limits_{\breve{A}B} f(x,y)dy\right)$$

mavjud va

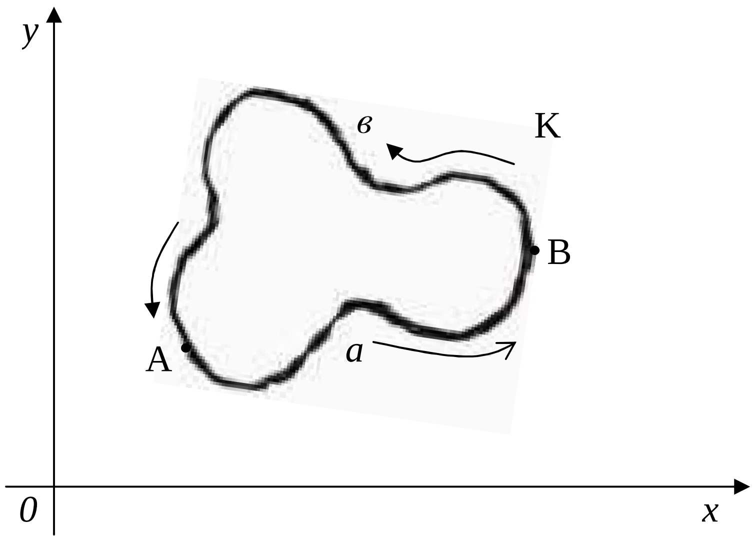
$$\int\limits_{\breve{A}B} f(x,y)dx = 0 \quad \left(\int\limits_{\breve{A}B} f(x,y)dy = 0\right)$$

bo'ladi.

Bu tenglik bevosita ikkinchi tur egri chiziqli integral ta'rifidan kelib chiqadi.

Endi $\breve{A}B$ -sodda yopiq egri chiziq bo'lsin, ya'ni A va B nuqtalar ustma-ust tushsin. Bu yopiq chiziqni K deb belgilaylik. Bu sodda yopiq chiziqda ham ikki yo'nalish bo'ladi. Ularning birini musbat yo'nalish, ikkinchisini manfiy yo'nalish deb qabul qilaylik. Shunday yo'nalishni musbat deb qabul qilamizki, kuzatuvchi yopiq chiziq bo'ylab harakat qilganda, yopiq chiziq bilan chegaralangan soha unga nisbatan har doim chap tomonda yotsin.

Faraz qilaylik, sodda yopiq chiziqda $f(x,y)$ funksiya berilgan bo'lsin. Bu K chiziqda ixtiyoriy ikkita turli nuqtalarni olib, ularni A va B bilan belgilaylik. Natijada, K yopiq chiziq ikkita $A\breve{a}B$ va $B\breve{в}A$ chiziqlarga ajraladi (61-chizma).



61-chizma

Ushbu

$$\int_{A\breve{a}B} f(x,y)dx + \int_{B\breve{в}A} f(x,y)dx$$

integral (agar u mavjud bo'lsa) $f(x,y)$ funksiyaning K yopiq chiziq bo'yicha ikkinchi tur egri chiziqli integrali deb ataladi va

$$\int_K f(x,y)dx \quad \text{yoki} \quad \oint_K f(x,y)dx$$

kabi belgilanadi. Bunda K yopiq chiziqning musbat yo'nalishi olingan. (Bundan buyon yopiq chiziq bo'yicha olingan integrallarda, yopiq chiziq musbat yo'nalishda deb qaraymiz). Demak,

$$\oint_K f(x,y)dx = \int_{A\breve{a}B} f(x,y)dx + \int_{B\breve{в}A} f(x,y)dx .$$

Xuddi shunga o'xshash

$$\oint_K f(x,y)dy$$

hamda, umumiy holda

$$\oint_K P(x,y)dx + Q(x,y)dy$$

integrallar ta'riflanadi.

$\breve{AB}$ fazoviy egri chiziq bo'lib, bu chiziqda $f(x,y,z)$ funksiya berilgan bo'lsin. Yuqoridagidek, $f(x,y,z)$ funksiyaning $\breve{AB}$ egri chiziq bo'yicha ikkinchi tur egri chiziqli integrallari ta'riflanadi va ular

$$\int_{\breve{AB}} f(x,y,z)dx, \quad \int_{\breve{AB}} f(x,y,z)dy, \quad \int_{\breve{AB}} f(x,y,z)dz$$

kabi belgilanadi. Umumiy holda, $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar berilgan bo'lib, ushbu

$$\int_{\breve{AB}} P(x,y,z)dx, \quad \int_{\breve{AB}} Q(x,y,z)dy, \quad \int_{\breve{AB}} R(x,y,z)dz$$

integrallar mavjud bo'lsa,

$$\int_{\breve{AB}} P(x,y,z)dx + \int_{\breve{AB}} Q(x,y,z)dy + \int_{\breve{AB}} R(x,y,z)dz$$

yig'indi ikkinchi tur egri chiziqli integralning umumiy ko'rinishi deb ataladi va u

$$\int\limits_{\breve{A}B} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

kabi belgilanadi. Demak,

$$\int\limits_{\breve{A}B} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int\limits_{\breve{A}B} P(x,y,z)dx + \int\limits_{\breve{A}B} Q(x,y,z)dy + \int\limits_{\breve{A}B} R(x,y,z)dz .$$

2⁰. Uzluksiz funksiya ikkinchi tur egri chiziqli integrali. Faraz qilaylik, $\breve{A}B$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \qquad (\alpha \le t \le \beta) \qquad \textit{(18.13)}$$

sistema bilan (parametrik ko'rinishda) berilgan bo'lsin. Bunda $\varphi(t)$ funksiya $[\alpha,\ \beta]$ da $\varphi'(t)$ hosilaga ega va bu hosila shu oraliqda uzluksiz, $\psi(t)$ funksiya ham $[\alpha,\ \beta]$ da uzluksiz hamda $(\varphi(\alpha),\ \psi(\alpha)) = A$ va $(\varphi(\beta),\ \psi(\beta)) = B$ bo'lsin.

t parametr α dan β ga qarab o'zgarganda $(x,y) = (\varphi(t),\ \psi(t))$ nuqta A dan B ga qarab $\breve{A}B$ ni chiza borsin.

3-teorema. Agar $f(x,y)$ funksiya $\breve{A}B$ da uzluksiz bo'lsa, u holda bu funksiyaning $\breve{A}B$ egri chiziq bo'yicha ikkinchi tur egri chiziqli integrali

$$\int\limits_{\breve{A}B} f(x,y)dx$$

mavjud va

$$\int\limits_{\breve{A}B} f(x,y)dx = \int\limits_{\alpha}^{\beta} f(\varphi(t),\psi(t)) \cdot \varphi'(t)dt$$

bo'ladi.

◄ $[\alpha,\ \beta]$ oraliqning

$$P = \{t_0, t_1, ..., t_n\}, \ \ (\alpha = t_0 < t_1 < ... < t_n = \beta)$$

bo'laklashini olaylik. Bu bo'laklashning bo'luvchi nuqtalari t_k $(k = 0,1,2,...,n)$ ning $\breve{A}B$ dagi mos akslarini A_k deylik $(k = 0,1,2,...,n)$. Ravshanki, bu A_k nuqtalar AB egri chiziqning

$$\{A_0,\ A_1,\ ...,\ A_n\}$$

bo'laklashini hosil qiladi. Bundan $A_k = (\varphi(t_k),\ \psi(t_k))$ $(k = 0,1,2,...,n)$ bo'ladi. Bu bo'laklashga nisbatan (18.10) yig'indini

$$\sigma' = \sum_{k=0}^{n-1} f(\xi_k, \eta_k)\Delta x_k$$

tuzamiz. Keyingi tenglikda $\Delta x_k - \breve{A}_k A_{k+1}$ ning Ox o'qdagi proeksiyasi

$$\Delta x_k = x_{k+1} - x_k = \varphi(t_{k+1}) - \varphi(t_k)$$

ga tengdir.

Lagranj teoremasidan foydalanib topamiz:

$$\varphi(t_{k+1}) - \varphi(t_k) = \varphi'(\theta_k)(t_{k+1} - t_k) = \varphi'(\theta_k)\Delta t_k \qquad (\theta_k \in [t_k, t_{k+1}]).$$

Ma’lumki, $(\xi_k, \eta_k) \in \breve{A}_k A_{k+1}$, $(k = 0, 1, 2, ..., n-1)$. Agar bu (ξ_k, η_k) nuqtaga akslanuvchi nuqtani r_k $(r_k \in [t_k, t_{k+1}])$ deyilsa, unda

$$\xi_k = \varphi(r_k), \quad \eta_k = \psi(r_k)$$

bo’ladi. Natijada σ' yig’indi quyidagi ko’rinishga keladi:

$$\sigma' = \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot \varphi'(\theta_k) \cdot \Delta t_k .$$

Endi $\lambda'_P = \max_k \{\Delta t_k\} \to 0$ da (bu holda λ_P ham nolga intiladi) σ' yig’indining limitini topish maqsadida uning ifodasini o’zgartirib quyidagicha yozamiz:

$$\sigma' = \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot \varphi'(r_k) \cdot \Delta t_k + \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot [\varphi'(\theta_k) - \varphi'(r_k)] \cdot \Delta t_k . \quad (18.14)$$

Bu tenglikning o’ng tomonidagi ikkinchi qo’shiluvchini baholaymiz:

$$\left| \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot [\varphi'(\theta_k) - \varphi'(r_k)] \cdot \Delta t_k \right| \le$$

$$\le \sum_{k=0}^{n-1} |f(\varphi(r_k), \psi(r_k))| \cdot |\varphi'(\theta_k) - \varphi'(r_k)| \cdot \Delta t_k \le$$

$$\le M \sum_{k=0}^{n-1} |\varphi'(\theta_k) - \varphi'(r_k)| \cdot \Delta t_k$$

bunda

$$M = \max_{\alpha \le t \le \beta} |f(\varphi(t), \psi(t))| .$$

$\varphi'(t)$ funksiya $[\alpha, \beta]$ da uzluksiz. U holda Kantor teoremasining natijasiga ko’ra, $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ topiladiki, $[\alpha, \beta]$ oraliqning diametri $\lambda'_P < \delta$ bo’lgan har qanday P bo’linish uchun

$$|\varphi'(\theta_k) - \varphi'(r_k)| < \frac{\varepsilon}{M \cdot (\beta - \alpha)} \qquad (\theta_k, r_k \in [t_k, t_{k+1}])$$

bo’ladi. Unda

$$\left| \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot [\varphi'(\theta_k) - \varphi'(r_k)] \cdot \Delta t_k \right| <$$

$$< M \sum_{k=0}^{n-1} \frac{\varepsilon}{M(\beta - \alpha)} \Delta t_k = \frac{\varepsilon}{\beta - \alpha} \sum_{k=0}^{n-1} \Delta t_k = \varepsilon.$$

Demak,

$$\lim_{\lambda'_P \to 0} \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \cdot [\varphi'(\theta_k) - \varphi'(r_k)] \cdot \Delta t_k = 0$$

bo’ladi. Bu munosabatni e’tiborga olib, (18.14) tenglikda $\lambda_P \to 0$ da limitga o’tib quyidagini topamiz:

$$\lim_{\lambda_P \to 0} \sigma' = \lim_{\lambda'_P \to 0} \sum_{k=0}^{n-1} f(\varphi(r_k), \psi(r_k)) \varphi'(r_k) \Delta t_k = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \varphi'(t) dt .$$

Demak,

$$\int_{\breve{A}B} f(x,y)dx = \int_{\alpha}^{\beta} f(\varphi(t),\ \psi(t))\varphi'(t)dt\,. \blacktriangleright$$

Endi (18.13) sistemada $\varphi(t)$ funksiya $[\alpha,\ \beta]$ da $\psi'(t)$ hosilaga ega va bu hosila shu oraliqda uzluksiz bo'lsin.

4-teorema. Agar $f(x,y)$ funksiya $\breve{A}B$ da uzluksiz bo'lsa, u holda bu funksiyaning $\breve{A}B$ egri chiziq bo'yicha olingan ikkinchi tur egri chiziqli integrali

$$\int_{\breve{A}B} f(x,y)dy$$

mavjud va

$$\int_{\breve{A}B} f(x,y)dy = \int_{\alpha}^{\beta} f(\varphi(t),\ \psi(t))\psi'(t)dt$$

bo'ladi.

Bu teorema yuqoridagi 3-teorema kabi isbotlanadi.

Yuqoridagi teoremalar, bir tomondan, uzluksiz funksiya ikkinchi tur egri chiziqli integralining mavjudligini aniqlab bersa, ikkinchi tomondan, bu integral aniq integral (Riman integrali) orqali ifodalanishini ko'rsatadi.

$\breve{A}B$ egri chiziq (18.13) sistema bilan berilgan bo'lib, $\varphi(t),\ \psi(t)$ funksiyalar $[\alpha,\ \beta]$ da $\varphi'(t),\ \psi'(t)$ hosilalarga ega va u bu hosilalar uzluksiz bo'lsin.

Agar $\breve{A}B$ egri chiziqda ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo'lib, ular shu chiziqda uzluksiz bo'lsa, u holda

$$\int_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int_{\alpha}^{\beta} [P(\varphi(t),\ \psi(t))\varphi'(t) + Q(\varphi(t),\ \psi(t))\psi'(t)]dt$$

bo'ladi.

3⁰. Ikkinchi tur egri chiziqli integralning xossalari. Yuqorida keltirilgan teoremalar uzluksiz funksiyalarning ikkinchi tur egri chiziqli integrallarini, bizga ma'lum bo'lgan aniq integral-Riman integrallariga kelishini ko'rsatadi. Binobarin, bu egri chiziqli integrallar Riman integrallari xossalari kabi xossalarga ega bo'ladi. O'tgan paragrafda esa xuddi shunday mulohaza birinchi tur egri chiziqli integrallarga nisbatan bo'lgan edi. Shularni e'tiborga olib, ikkinchi tur egri chiziqli integrallarning xossalarini keltirishni va tegishli xulosalar chiqarishni o'quvchiga havola etamiz.

4⁰. Ikkinchi tur egri chiziqli integrallarni hisoblash. Yuqorida keltirilgan teoremalar funksiyaning ikkinchi tur egri chiziqli integrallarining mavjudligini tasdiqlabgina qolmasdan ularni hisoblash yo'lini ko'rsatadi. Demak, ikkinchi tur egri chiziqli integrallar ham, asosan Riman integrallariga keltirilib hisoblanadi:

$$\int_{\breve{A}B} f(x,y)dx = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\varphi'(t)dt\,, \qquad (18.15)$$

$$\int_{\breve{A}B} f(x,y)dy = \int_{\alpha}^{\beta} f(\varphi(t),\psi(t))\cdot\psi'(t)dt\,, \qquad (18.16)$$

$$\int\limits_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int\limits_{\alpha}^{\beta} [P(\varphi(t),\psi(t))\cdot\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]dt \quad (18.17)$$

Xususan, $\breve{A}B$ egri chiziq

$$y = y(x) \quad (a \le x \le в)$$

tenglama bilan aniqlangan bo'lib, $y(x)$ funksiya $[a,\ в]$ da hosilaga ega va u uzluksiz bo'lsa, (18.15), (18.17) formulalar quyidagi

$$\int\limits_{\breve{A}B} f(x,y)dx = \int\limits_{a}^{в} f(x, y(x))dx, \quad (18.18)$$

$$\int\limits_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int\limits_{a}^{в} [P(x,y(x)) + Q(x,y(x))y'(x)]dx$$

ko'rinishga keladi.

Shuningdek, $\breve{A}B$ egri chiziq

$$x = x(y) \quad (c \le y \le d)$$

tenglama bilan aniqlangan bo'lib, $x(y)$ funksiya $[c,\ d]$ oraliqda hosilaga ega va u uzluksiz bo'lsa, (18.16) va (18.17) formulalar quyidagi

$$\int\limits_{\breve{A}B} f(x,y)dy = \int\limits_{c}^{d} f(x(y), y)dy, \quad (18.19)$$

$$\int\limits_{\breve{A}B} P(x,y)dx + Q(x,y)dy = \int\limits_{c}^{d} [P(x(y),y)\cdot x'(y) + Q(x(y),y)]dy \quad (18.20)$$

ko'rinishga keladi.

18.3-misol. Ushbu

$$\int\limits_{\breve{A}B} y^2dx + x^2dy$$

integral hisoblansin, bunda $\breve{A}B - \frac{x^2}{a^2} + \frac{y^2}{в^2} = 1$ ellipsning yuqori yarim tekislikdagi qismidan iborat.

◀ Ma'lumki, ellipsning parametrik tenglamasi quyidagicha bo'ladi:

$$x = a\cos t,$$
$$y = в\sin t.$$

$A = (a,\ o)$ nuqtaga parametr t ning $t = 0$ qiymati, $B = (-a,\ o)$ nuqtaga esa $t = \pi$ qiymati mos kelib, t parametr 0 dan π gacha o'zgarganda (x,y) nuqta A dan B ga qarab ellipsning yuqori yarim tekislikdagi qismini chizib chiqadi.

$P(x,y) = y^2$, $Q(x,y) = x^2$ funksiyalar esa $\breve{A}B$ da uzluksiz. (18.17) fomuladan foydalanib quyidagini topamiz:

$$\int\limits_{\breve{A}B} y^2dx + x^2dy = \int\limits_{0}^{\pi} \left[в^2 \sin^2 t(-a\sin t) + a^2\cos^2 t\cdot в\cos t\right]dt =$$

$$= aв\int\limits_{0}^{\pi} \left(a\cos^3 t - в\sin^3 t\right)dt = -\frac{4}{3}aв^2. ▶$$

18.4-misol. Ushbu

$$\int\limits_{\breve{A}B} 3x^2 y dx + (x^3+1)dy$$

integral hisoblansin, bunda $\breve{A}B$ egri chiziq:

a) $(0,0)$ nuqtadan chiqqan $(0,0)$ va $(1,1)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi;

b) $(0,0)$ nuqtadan chiqqan $(0,0)$ va $(1,1)$ nuqtalarni birlashtiruvchi $y = x^2$ parabolaning yoyi;

v) $(0,0)$ nuqtadan chiqqan $(0,0)$, $(1,0)$ va $(1,1)$ nuqtalarni birlashtiruvchi siniq chiziqdan iborat.

Yuqoridagi (18.18), (18.19) va (18.20) formulalardan foydalanib quyidagilarni topamiz:

a) holda

$$\int\limits_{\breve{A}B} 3x^2 y dx + (x^3+1)dy = \int\limits_0^1 \left[3x^2 x + (x^3+1)\right]dx = \int\limits_0^1 (4x^3+1)dx = 2,$$

b) holda

$$\int\limits_{\breve{A}B} 3x^2 y dx + (x^3+1)dy = \int\limits_0^1 \left[3x^2 x^2 + (x^3+1)2x\right]dx = \int\limits_0^1 (5x^4+2x)dx = 2,$$

v) holda

$$\int\limits_{\breve{A}B} 3x^2 y dx + (x^3+1)dy = \int\limits_{\breve{A}C} 3x^2 y dx + (x^3+1)dy + \int\limits_{\breve{C}B} 3x^2 y dx + (x^3+1)dy$$

bunda $\breve{A}C$ -$(0,0)$ va $(1,0)$ nuqtalarni $\breve{C}B$ - $(1,0)$ va $(1,1)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmalaridan iborat

Ravshanki,

$$\int\limits_{\breve{A}C} 3x^2 y dx + (x^3+1)dy = 0 \quad \int\limits_{\breve{C}B} 3x^2 y dx + (x^3+1)dy = \int\limits_0^1 2dy = 2.$$

Demak,

$$\int\limits_{\breve{A}B} 3x^2 y dx + (x^3+1)dy = 2. \blacktriangleright$$

3-§. Grin formulasi va uning tatbiqlari

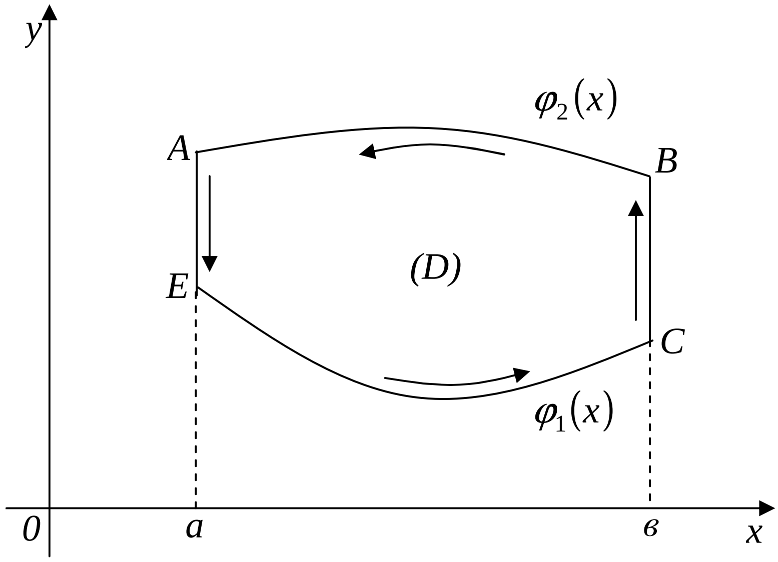
Ma'lumki, Nyuton-Leybnits formulasi $f(x)$ funksiyaning $[a, в]$ oraliq bo'yicha olingan aniq integralini shu funksiya boshlang'ich funksiyasining oraliq chekkalari (chegaralari) dagi qiymatlari orqali ifodalar edi.

Biror (D) sohada $((D) \subset R^2)$ berilgan $f(x,y)$ uzluksiz funksiyaning ikki karrali

$$\iint\limits_{(D)} f(x,y)dxdy$$

integralini tegishli funksiyaning shu soha chegarasidagi qiymatlari orqali (aniqrog'i, soha chegarasi bo'yicha olingan egri chiziqli integrali orqali) ifodalaydigan formula ham mavjud. Quyida bu formulani keltiramiz.

1^0. Grin formulasi. Yuqoridan $y=\varphi_2(x)$ $\quad(a\le x\le в)$ funksiya grafigi, yon tomonlardan $x=a$, $x=в$ vertikal chiziqlar hamda pastdan $y=\varphi_1(x)$ $\quad(a\le x\le в)$ funksiya grafigi bilan chegaralangan soha egri chiziqli trapesiyani qaraylik. Bu sohani (D) bilan, uning chegarasi – yopiq chiziqni ∂D bilan belgilaylik (62-chizma).



62-chizma

Ravshanki, $\breve{A}B-\varphi_2(x)$ funksiya grafigi, $\breve{E}C-\varphi_1(x)$ funksiya grafigi hamda

$$\partial D=\breve{E}C+CB+\breve{B}A+AC\,.$$

$P(x,y)$ funksiya shu (D) sohada uzluksiz bo’lib, $\dfrac{\partial P(x,y)}{\partial y}$ xususiy hosilaga ega va u ham (D) da uzluksiz bo’lsin. U holda ushbu

$$\iint\limits_{(D)}\frac{\partial P(x,y)}{\partial y}dxdy$$

integral mavjud bo’ladi va 18-bobning 6-§ idagi formulaga ko’ra

$$\iint\limits_{(D)}\frac{\partial P(x,y)}{\partial y}dxdy=\int\limits_a^в\left(\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial P(x,y)}{\partial y}dy\right)dx$$

bo’ladi. Endi

$$\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial P(x,y)}{\partial y}dy=P(x,y)\Big|_{y=\varphi_1(x)}^{y=\varphi_2(x)}=P(x,\varphi_2(x))-P(x,\varphi_1(x))$$

bo’lishini e’tiborga olib, quyidagini topamiz:

$$\iint\limits_{(D)}\frac{\partial P(x,y)}{\partial y}dxdy=\int\limits_a^в P(x,\varphi_2(x))dx-\int\limits_a^в P(x,\varphi_1(x))dx$$

Ushbu bobning 2-§ idagi (18.18) formulaga binoan

$$\int\limits_a^в P(x,\varphi_2(x))dx=\int\limits_{\breve{A}B}P(x,y)dx\,,\quad\int\limits_a^в P(x,\varphi_1(x))dx=\int\limits_{\breve{B}C}P(x,y)dx$$

bo’ladi. Demak,

$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = \int\limits_{\breve{A}B} P(x,y)dx - \int\limits_{\breve{E}C} P(x,y)dx = -\int\limits_{\breve{B}A} P(x,y)dx - \int\limits_{\breve{E}C} P(x,y)dx\,.$$

Ravshanki,

$$\int\limits_{CB} P(x,y)dx = 0\,,\ \int\limits_{EA} P(x,y)dx = 0\,.$$

Bu tengliklarni hisobga olib quyidagini topamiz:

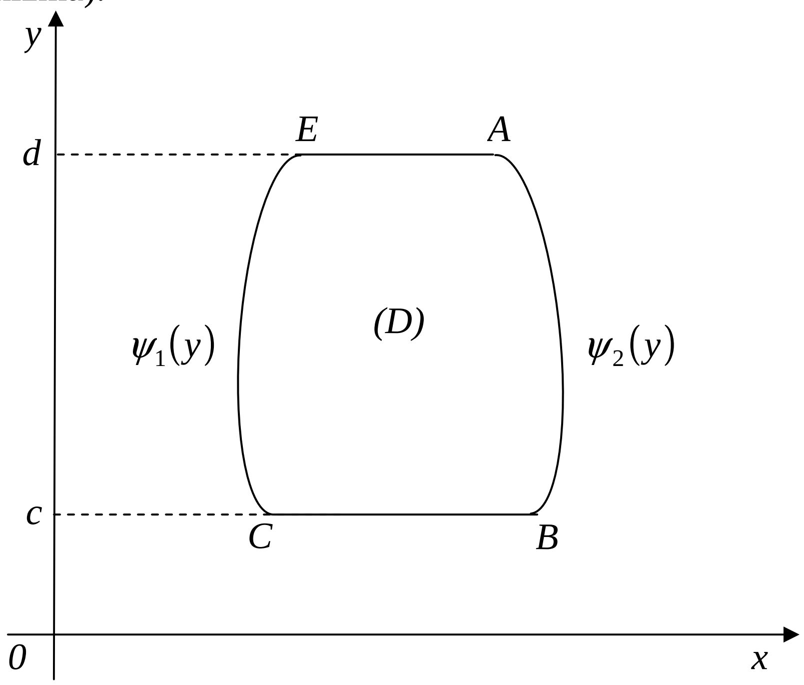
$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = -\int\limits_{\breve{E}C} P(x,y)dx - \int\limits_{\breve{C}B} P(x,y)dx - \int\limits_{\breve{B}A} P(x,y)dx - \int\limits_{\breve{A}E} P(x,y)dx =$$

$$= -\left(\int\limits_{\breve{E}C} P(x,y)dx + \int\limits_{\breve{C}B} P(x,y)dx + \int\limits_{\breve{B}A} P(x,y)dx + \int\limits_{\breve{A}E} P(x,y)dx \right) = -\int\limits_{\partial D} P(x,y)dx.$$

Demak,

$$\iint\limits_{(D)} \frac{\partial P(x,y)}{\partial y} dxdy = -\int\limits_{\partial D} P(x,y)dx. \qquad (18.21)$$

Endi, yuqoridan $y = c$, pastdan $y = d$ chiziqlar, yon tomondan esa $x = \psi_1(y)$, $x = \psi_2(y)$ funksiyalar grafiklari bilan chegaralangan soha egri chiziqli trapesiyani qaraylik. Bu sohani (D) bilan, uning chegarasi – yopiq chiziqli ∂D bilan belgilaylik (63-chizma).



63-chizma

$Q(x,y)$ funksiya shu (D) sohada uzluksiz bo'lib, $\frac{\partial Q(x,y)}{\partial x}$ xususiy hosilaga ega va bu hosila (D) da uzluksiz bo'lsin. U holda

$$\iint\limits_{(D)} \frac{\partial Q(x,y)}{\partial x} dxdy = \int\limits_{\partial D} Q(x,y)dy \qquad (18.22)$$

bo'ladi.

Bu formulaning to'g'riligi yuqoridagidek mulohaza yuritish bilan isbotlanadi.

Endi R^2 fazoda qaraladigan (D) soha yuqoridagi ikki holda qaralgan sohaning har birining xarakteriga ega bo'lgan soha bo'lsin, ∂D esa uning chegarasi bo'lsin. Bu (D) sohada ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar uzluksiz bo'lib, ular $\frac{\partial P(x,y)}{\partial y}$, $\frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega hamda bu hosilalar ham (D) da uzluksiz bo'lsin. Ravshanki, bu holda (18.21) va (18.22) formulalar o'rinli bo'ladi. Ularni hadlab qo'shib ushbuni topamiz:

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_{(D)} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy . \qquad (18.23)$$

Bu Grin formulasi deb ataladi.

Demak, Grin formulasi sohasi bo'yicha olingan ikki karrali integralni shu soha chegarasi bo'yicha olingan egri chiziqli integral bilan bog'laydigan formula ekan.

Biz yuqorida Grin formulasi maxsus ko'rinishdagi (D) sohalar (egri chiziqli trapesiyalar) uchun keltirdik. Aslida bu formula ancha keng sinfdagi sohalar uchun ham to'g'ri bo'lib, bu fakt u sohalarni chekli sondagi egri chiziqli trapesiyalar yig'indisi sifatida tasvirlash bilan isbot qilinadi.

2⁰. Grin formulasining ba'zi bir tatbiqlari.

1). ***Shaklning yuzini topish.*** Grin formulasidan foydalanib, yassi shaklning yuzini sodda funksiyalarning egri chiziqli integrallari yordamida hisoblanishini ko'rsatish qiyin emas. Haqiqatdan ham, (18.23) formulada $P(x,y) = -y$, $Q(x,y) = 0$ deyilsa, u holda

$$\int_{\partial D} (-y)dx = \iint_{(D)} dxdy = D$$

bo'ladi. Demak,

$$D = -\int_{\partial D} ydx .$$

Agar (18.23) formulada $P(x,y) = 0$, $Q(x,y) = x$ deyilsa, u holda

$$D = \int_{\partial D} xdy \qquad (18.24)$$

bo'ladi.

(18.23) formulada $P(x,y) = -\frac{1}{2}y$, $Q(x,y) = \frac{1}{2}x$ deb olinsa, (D) sohaning yuzi

$$D = \frac{1}{2}\int_{\partial D} xdy - ydx \qquad (18.25)$$

bo'ladi.

18.5-misol. Ushbu

$$\begin{cases} x = a\cos t, \\ y = в\sin t \end{cases} \qquad (0 \le t \le 2\pi)$$

ellips bilan chegaralangan shaklning yuzi topilsin.

◄(18.25) formulaga ko'ra

$$D=\frac{1}{2}\int_{\partial D} xdy-ydx=\frac{1}{2}\int_0^{2\pi}(a\cos t\cdot в\cos t+в\sin t\cdot a\sin t)dt=$$

$$=\frac{1}{2}aв\int_0^{2\pi}(\cos^2 t+\sin^2 t)dt=\pi aв.\blacktriangleright$$

2^0. Ikki karrali integrallarni o'zgaruvchilarni almashtirib hisoblash. Mazkur kursning 18-bob, 7-§ ida (Δ) sohani (D) sohaga akslantiruvchi

$$x=\varphi(u,v),$$
$$y=\psi(u,v)$$

sistema usha paragrafda keltirilgan 1-3-shartlarni bajarganda (D) sohaning yuzi

$$D=\iint_{(\Delta)}|J(u,v)|dudv=\iint_{(D)}\left|\frac{D(x,y)}{D(u,v)}\right|dudv$$

bo'lishi aytilgan edi. Grin formulasidan foydalanib, shu formulaning to'g'riligini isbotlaymiz.

Avvalo (18.24) formuladan foydalanib, (D) sohaning yuzi

$$D=\int_{\partial D} xdy \qquad (18.26)$$

bo'lishini topamiz. Faraz qilaylik, $\partial\Delta$ chiziq parametrik formada ushbu

$$\begin{matrix} u=u(t) \\ v=v(t) \end{matrix} \qquad (\alpha\le t\le\beta) \text{ yoki } (\alpha\ge t\ge\beta)$$

sistema bilan ifodalansin. U holda quyidagi

$$x=\varphi(u,v)=\varphi(u(t),v(t)),$$
$$y=\psi(u,v)=\psi(u(t),v(t))$$

sistema (D) sohaning ∂D chegarasini ifodalaydi. Bunda parametrning o'zgarish chegarasini shunday tanlab olamizki, t parametr α dan β ga qarab o'zgarganda ∂D egri chiziq musbat yo'nalishda bo'lsin. U holda (18.26) tenglik ushbu

$$D=\int_{\partial D} xdy=\int_{\partial D}\varphi(u,v)d\psi(u,v)=\int_\alpha^\beta \varphi(u(t),v(t))\cdot\left[\frac{\partial\psi}{\partial u}u'(t)+\frac{\partial\psi}{\partial v}v'(t)\right]dt \qquad (18.27)$$

ko'rinishga keladi.

Agar

$$\int_{\partial(\Delta)}\varphi(u,v)\left[\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv\right]=\int_\alpha^\beta\varphi(u(t),v(t))\left[\frac{\partial\psi}{\partial u}u'(t)+\frac{\partial\psi}{\partial v}v'(t)\right]dt$$

bo'lishini e'tiborga olsak, u holda

$$D=\pm\int_{\partial(\Delta)} x\frac{\partial y}{\partial u}du+x\frac{\partial y}{\partial v}dv \qquad (18.28)$$

bo'lishini topamiz. Bu tenglikdagi integral belgisi oldiga quyilgan ishorani tushuntiramiz. Yuqorida, t parametr α dan β ga qarab o'zgarganda ∂D egri chiziq musbat yo'nalishda bo'lishini aytdik. Bu holda $\partial\Delta$ egri chiziqning yo'nalishi musbat ham bo'lishi mumkin, manfiy ham bo'lishi mumkin. Shuning uchun (18.27) va (18.28) munosabatlar bir-biridan ishora bilan farq qiladi. Agar

∂D egri chiziq musbat yo’nalishga $\partial\Delta$ egri chiziqning ham musbat yo’nalishi mos kelsa, unda “Q” ishora olinadi, aks holda esa “-” ishora olinadi.

Endi ushbu

$$\int_{\partial\Delta} P(u,v)du + Q(u,v)dv = \iint_{(\Delta)} \left(\frac{\partial Q(u,v)}{\partial u} - \frac{\partial P(u,v)}{\partial v} \right) dudv \qquad (18.29)$$

Grin formulasida

$$P(u,v) = x\frac{\partial y}{\partial u}, \;\; Q(x,y) = x\frac{\partial y}{\partial v}$$

deb olsak, u holda bu formula quyidagi ko’rinishga keladi:

$$\int_{\partial\Delta} x\frac{\partial y}{\partial u}du + x\frac{\partial y}{\partial v}dv = \iint_{(\Delta)} \left[\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right) \right] dudv\,. \qquad (18.30)$$

Agar

$$\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial v\partial u}, \;\; \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right) = \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} + x\frac{\partial^2 y}{\partial u\partial v}$$

va

$$\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right) = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = \frac{D(x,y)}{D(u,v)}$$

ekanini e’tiborga olsak, unda (18.28), (18.29) va (18.30) munosabatlardan

$$D = \iint_{(\Delta)} \frac{D(x,y)}{D(u,v)} dudv$$

bo’lishi kelib chiqadi.

Ma’lumki,

$$J(u,v) = \frac{D(x,y)}{D(u,v)}$$

yakobian aniq ishorali, (D) esa ma’nosiga ko’ra musbat bo’lishi kerak. Demak, integral belgisi oldidagi ishora yakobianning ishorasi bilan bir xil bo’lish kerak. Shuning uchun

$$D = \iint_{(\Delta)} \left| \frac{D(x,y)}{D(u,v)} \right| dudv$$

bo’ladi. Shuni isbotlash lozim edi.

3⁰. Egri chiziqli integral qiymatining integrallash yo’liga bog’liq bo’lmasligi. Chegaralangan yopiq bog’lamli (D) $\left((D)\subset R^2\right)$ sohada ikkita $P(x,y)$ va $Q(x,y)$ funksiyalar berilgan bo’lsin. Bu funksiyalar (D) sohada uzluksiz va $\frac{\partial P(x,y)}{\partial y}, \frac{\partial Q(x,y)}{\partial x}$ xususiy hosilalarga ega va bu hosilalar ham shu sohada uzluksiz bo’lsin.

1) Agar (D) sohada

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} \qquad (18.31)$$

bo'lsa, u holda (D) sohaga tegishli bo'lgan har qanday K yopiq chiziq bo'yicha olingan ushbu

$$\int_K P(x,y)dx + Q(x,y)dy$$

integral nolga teng bo'ladi:

$$\int_K P(x,y)dx + Q(x,y)dy = 0.$$

◀ K yopiq chiziq chegaralagan sohani (G) deylik. Ravshanki, $(G) \subset (D)$. Grin formulasiga ko'ra

$$\int_K P(x,y)dx + Q(x,y)dy = \iint_{(G)} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$$

bo'ladi. Shartga ko'ra (D) da, demak, (G) da

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}.$$

U holda (18.31) munosabatdan

$$\iint_{(G)} \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy = 0$$

bo'ladi. Demak,

$$\int_K P(x,y)dx + Q(x,y)dy = 0. \blacktriangleright$$

2) Agar (D) sohaga tegishli bo'lgan har qanday K yopiq chiziq bo'yicha olingan ushbu integral

$$\int_K P(x,y)dx + Q(x,y)dy = 0$$

bo'lsa, u holda quyidagi

$$\int_{\breve{A}B} P(x,y)dx + Q(x,y)dy \qquad \left(\breve{A}B \subset (D)\right) \qquad (18.32)$$

integral A va B nuqtalarni birlashtiruvchi egri chiziqqa bog'liq bo'lmaydi, ya'ni (18.32) integral qiymati integrallash yo'liga bog'liq bo'lmaydi.

◀ (D) sohaning A va B nuqtalarni birlashtiruvchi va shu sohaga tegishli bo'lgan ixtiyoriy ikkita $A\breve{a}B$ hamda $A\breve{в}B$ egri chiziqni olaylik. Bu holda $A\breve{a}B$ va $A\breve{в}B$ egri chiziqlar birgalikda (D) sohaga tegishli bo'lgan yopiq chiziqni tashkil etadi. Uni K bilan belgilaylik:

$$K = AaBвa.$$

Shartga ko'ra

$$\int_K P(x,y)dx + Q(x,y)dy = \iint_{AaBвA} P(x,y)dx + Q(x,y)dy = 0$$

bo'ladi. Integralning xossasidan foydalanib ushbuni topamiz:

$$\int_{AaBвA} P(x,y)dx + Q(x,y)dy = \iint_{A\breve{a}B} P(x,y)dx + Q(x,y)dy + \int_{B\breve{в}A} P(x,y)dx + Q(x,y)dy =$$

$$= \int_{A\breve{a}B} P(x,y)dx + Q(x,y)dy - \int_{A\breve{в}B} P(x,y)dx + Q(x,y)dy.$$

Demak,

$$\int\limits_{Aa B} P(x,y)dx+Q(x,y)dy-\iint\limits_{A\breve{e}B} P(x,y)dx+Q(x,y)dy=0$$

Bundan esa,

$$\int\limits_{Aa B} P(x,y)dx+Q(x,y)dy=\iint\limits_{A\breve{e}B} P(x,y)dx+Q(x,y)dy$$

ekanligi kelib chiqadi.▶

19.2-eslatma. Yuqoridagi tasdiq, isbot jarayonidan ko'rinadiki, $\breve{A}B$ egri chiziq sodda egri chiziqlar to'plamidan ixtiyoriy olinganda o'rinlidir.

3) Agar ushbu

$$\int\limits_{\breve{A}B} P(x,y)dx+Q(x,y)dy \quad \left(\breve{A}B\subset(D)\right) \qquad (18.33)$$

integral A va B nuqtalarni birlashtiruvchi egri chiziqqa bog'liq bo'lmasa, ya'ni integral integrallash yo'liga bog'liq bo'lmasa, u holda

$$P(x,y)dx+Q(x,y)dy$$

ifoda (D) sohada berilgan biror funksiyaning to'liq differensiali bo'ladi.

◀Modomiki, (19.33) integrallash yo'liga bog'liq emas ekan, u holda integral $A=(x_0,y_0)$ va $B=(x_1,y_1)$ nuqtalar bilan bir qiymatli aniqlanadi. Shuning uchun bu holda (19.33) integralni quyidagicha ham yozish mumkin:

$$\int\limits_{(x_0,y_0)}^{(x_1,y_1)} P(x,y)dx+Q(x,y)dy\,.$$

Endi A nuqtani tayinlab, B nuqta sifatida (D) sohaning ixtiyoriy (x,y) nuqtasini olib, ushbu

$$\int\limits_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy$$

integralni qaraymiz. Ravshanki, bu integral (x,y) ga bog'liq bo'ladi:

$$F(x,y)=\int\limits_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy\,.$$

Bu funksiyaning xususiy hosilalarini hisoblaymiz. (x,y) nuqtaning x koordinatasiga shunday Δx orttirma beraylikki, $(x+\Delta x,y)$ nuqta va (x,y), $(x+\Delta x,y)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi ham (D) sohaga tegishli bo'lsin. Natijada $f(x,y)$ funksiya ham xususiy orttirmaga ega bo'ladi:

$$F(x+\Delta x,y)-F(x,y)=\int\limits_{(x_0,y_0)}^{(x+\Delta x,y)} P(x,y)dx+Q(x,y)dy-\int\limits_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy=$$

$$=\int\limits_{(x,y)}^{(x+\Delta x,y)} P(x,y)dx+Q(x,y)dy=\int\limits_{(x,y)}^{(x+\Delta x,y)} P(x,y)dx.$$

O'rta qiymat haqidagi teoremadan foydalanib quyidagini topamiz:

$$\int\limits_{(x,y)}^{(x+\Delta x,y)} P(x,y)dx=P(x+\theta\Delta x,y)\cdot\Delta x \quad (0<\theta<1).$$

Natijada

$$\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}=P(x+\theta\Delta x,y)$$

bo'ladi. Bundan

$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}=\lim_{\Delta x\to 0}P(x+\theta\Delta x,y)=P(x,y)$$

bo'ladi. Demak,

$$\frac{\partial F(x,y)}{\partial x}=P(x,y).$$

Xuddi shunga o'xshash

$$\frac{\partial F(x,y)}{\partial y}=Q(x,y)$$

bo'lishi ko'rsatiladi.

Shunday qilib

$$P(x,y)dx+Q(x,y)dy=\frac{\partial F(x,y)}{\partial x}dx+\frac{\partial F(x,y)}{\partial y}dy=dF(x,y)$$

bo'ladi. ►

4) Agar

$$P(x,y)dx+Q(x,y)dy \qquad (18.34)$$

ifoda (D) sohada berilgan biror funksiyaning to'liq differensiali bo'lsa, u holda

$$\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$$

bo'ladi.

◄Aytaylik, (18.34) ifoda (D) sohada berilgan $F(x,y)$ funksiyaning to'liq differensiali bo'lsin:

$$P(x,y)dx+Q(x,y)dy=dF(x,y).$$

Ravshanki,

$$P(x,y)=\frac{\partial F(x,y)}{\partial x},\quad Q(x,y)=\frac{\partial F(x,y)}{\partial y}.$$

Keyingi tengliklardan ushbuni topamiz:

$$\frac{\partial P(x,y)}{\partial y}=\frac{\partial^2 F(x,y)}{\partial y\partial x},\qquad \frac{\partial Q(x,y)}{\partial x}=\frac{\partial^2 F(x,y)}{\partial x\partial y}.$$

Shartga ko'ra $\frac{\partial P(x,y)}{\partial y}$, $\frac{\partial Q(x,y)}{\partial x}$ lar (D) sohada uzluksiz. Aralash hosilalarning tengligi haqidagi teoremaga binoan (qaralsin, 13-bob, 6-§).

$$\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$$

bo'ladi. ►

Shunday qilib, Grin formulasidan foydalangan holda, yuqoridagi 1)-4) tasdiqlar orasida

$$1)\Rightarrow 2)\Rightarrow 3)\Rightarrow 4)\Rightarrow 1)$$

munosabat borligi ko’rsatildi.

4-§. Birinchi va ikkinchi tur egri chiziqli integrallar orasidagi bog’lanish

Tekislikda sodda silliq $\breve{AB}$ egri chiziq ushbu

$$\begin{matrix} x = x(s), \\ y = y(s) \end{matrix} \qquad (0 \le s \le S)$$

sistema bilan aniqlangan bo’lsin, bunda s - yoy uzunligi (qaralsin, ushbu bobning 1-§) $x(s)$ va $y(s)$ funksiyalar $x'(s)$, $y'(s)$ hosilalarga ega hamda bu hosilalar uzluksiz.

Ravshanki, bu egri chiziq har bir nuqtada urinmaga ega bo’ladi. Agar Ox va Oy o’qlar bilan urinmaning yoy o’sishi tomoniga qarab yo’nalish orasidagi burchak mos ravishda α va β deyilsa, unda

$$x'(s) = \cos\alpha, \quad y'(s) = \cos\beta$$

bo’ladi.

Aytaylik, bu $\breve{AB}$ egri chiziqda $f(x,y)$ funksiya berilgan va uzluksiz bo’lsin. U holda

$$\int_{\breve{AB}} f(x,y)dx$$

integral mavjud bo’ladi va (18.15) formulaga ko’ra

$$\int_{\breve{AB}} f(x,y)dx = \int_0^S f(x(s),y(s))x'(s)ds$$

tenglik o’rinli. Bu tenglikning o’ng tomonidagi integralni quyidagicha

$$\int_0^S f(x(s),y(s))x'(s)ds = \int_0^S f(x(s),y(s))\cos\alpha ds$$

yozish mumkin. Ushbu bobning 1-§ da keltirilgan 1-teoremadan foydalanib quyidagini topamiz:

$$\int_0^S f(x(s),y(s))\cos\alpha ds = \int_{\breve{AB}} f(x,y)\cos\alpha ds .$$

Natijada yuqoridagi tenglikdan

$$\int_{\breve{AB}} f(x,y)dx = \int_{\breve{AB}} f(x,y)\cos\alpha ds$$

bo’lishi kelib chiqadi.

Xuddi shunga o’xshash, tegishli shartda

$$\int_{\breve{AB}} f(x,y)dy = \int_{\breve{AB}} f(x,y)\cos\beta ds$$

va umumiy holda

$$\int_{\breve{AB}} P(x,y)dx + Q(x,y)dy = \int_{\breve{AB}} [P(x,y)\cos\alpha + Q(x,y)\cos\beta]ds$$

bo’ladi.

Mashqlar

18.6. Ushbu

$$J = \int_{\breve{A}B} \left(4\sqrt[3]{x} - 3\sqrt{y}\right) ds$$

integral hisoblansin, bunda $\breve{A}B$ egri chiziq quyidagi

$$x = cos^3 t, \quad y = sin^3 t$$

astroidaning $A(-1;\ 0)$ va $B(0,\ 1)$ nuqtalari orasidagi qismi.

18.7. Ushbu

$$J = \int_{\breve{A}B} x^2 ds$$

integral hisoblansin, bunda $\breve{A}B$ egri chiziq quyidagi

$$x^2 + y^2 = a^2$$

aylananing yuqori qismi.

18.8. $\breve{A}B$ yoyining uzunligi l quyidagi

$$l = \int_{\breve{A}B} ds$$

formula bilan topilishi isbotlansin.

Agar $\breve{A}B$ yoyi

$$x = 2a\,cos\,t - a\,cos\,2t,$$
$$y = 2a\,sin\,t - a\,sin\,2t$$

kardioidadan iborat bo'lsa, uning uzunligi topilsin.

18.9. Tekislikda yopiq C chiziq bilan chegaralangan (D) shaklning yuzi

$$D = \frac{1}{2}\int_C xdy - ydx$$

bo'lishi isbotlansin.

Ushbu

$$x = a\,cos\,t,\ y = в\,sin\,t$$

ellips bilan chegaralangan shaklning yuzi topilsin.

18.10. Agar material egri chiziq $\breve{A}B$ ning zichligi $\rho = \rho(x, y)$ bo'lsa, uning massasi

$$m = \int_{\breve{A}B} \rho(x, y) ds$$

bo'lishi isbotlansin.

Zichligi $\rho(x, y) = \dfrac{y}{x}$ bo'lgan quyidagi

$$y = \frac{x^2}{2}$$

parabolaning $\left(1,\ \dfrac{1}{2}\right)$, $(2,\ 2)$ nuqtalar orasidagi qismning massasi topilsin.

19-BOB

Sirt integrallari

Mazkur kursning 17- bobida $z=f(x,y)$ tenglama aniqlagan silliq (S) sirt bilan tanishgan edik. Bunda $z(x,y)$ funksiya (D) sohada $\left((D)\subset R^2\right)$ berilgan, uzluksiz va $z'_x(x,y)$, $z'_y(x,y)$ xususiy hosilalarga ega hamda bu hosilalar ham (D) da uzluksiz funksiya edi. (S) sirt yuzaga ega bo'lib, uning yuzi

$$S=\iint\limits_{(D)}\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}dxdy \qquad (19.1)$$

ga teng ekanligi ko'rsatildi.

O'sha bobning pirovardida R^3 fazodagi (V) sohada $\left((V)\subset R^3\right)$ berilgan funksiyaning uch karrali integrali bilan tanishib, uni o'rgandik.

Endi R^3 fazodagi (S) sirtda berilgan funksiyaning integrali tushunchasi bilan tanishamiz. Sirt integrali tushunchasini kiritishdan avval, bu erda ham funksiya berilish sohasining bo'laklashishi, bo'laklash bo'laklari, bo'laklashning diametri tushunchalari kiritilishi kerak.

Bu tushunchalar $[a,\ в]$ oraliqni bo'laklashi (qaralsin, 1-qism, 9-bob, 1-§) va tekislikda (D) sohani bo'laklashi (qaralsin, 18-bob, 1-§) kabi kiritiladi va o'xshash xossalarga ega bo'ladi. Shuning uchun bu erda biz bu tushunchalarni kiritilgan hisoblab, bayonimizni bevosita sirt integralining ta'rifidan boshlab ketaveramiz.

1-§. Birinchi tur sirt integrallari

1. Birinchi tur sirt integralining ta'rifi. $f(x,y,z)$ funksiya (S) sirtda $\left((S)\subset R^3\right)$ berilgan bo'lsin. Bu sirtning P bo'laklashni va bu bo'laklashning har bir (S_k) bo'lagida $(k=1,2,...,n)$ ixtiyoriy (ξ_k,η_k,ζ_k) nuqtani olaylik. Berilgan funksiyaning (ξ_k,η_k,ζ_k) nuqtadagi $f(\xi_k,\eta_k,\zeta_k)$ qiymatini (S_k) ning S_k yuziga ko'paytirib, quyidagi yig'indini tuzamiz:

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\cdot S_k$$

1-ta'rif. Ushbu

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\cdot S_k \qquad (19.2)$$

yig'indi $f(x,y,z)$ funksiyaning integral yig'indisi yoki Riman yig'indisi deb ataladi.

(S) sirtning shunday

$$P_1,P_2,...,P_m,... \qquad (19.3)$$

bo'laklashlarini qaraymizki, ularning mos diametrlaridan tashkil topgan

$$\lambda_{P_1},\lambda_{P_2},...,\lambda_{P_m},...$$

ketma-ketlik nolga intilsin: $\lambda_{P_m}\to 0$. Bunda P_m $(m=1,2,...)$ bo'laklashlarga nisbatan $f(x,y,z)$ funksiyaning integral yig'indisini tuzamiz. Natijada (S) sirtning

(19.3) bo'laklashlariga mos integral yig'indilar qiymatlaridan iborat quyidagi ketma-ketlik hosil bo'ladi:

$$\sigma_1, \sigma_2, ..., \sigma_m, ...$$

2-ta'rif. Agar (S) sirtning har qanday (19.3) bo'laklashlari ketma-ketligi $\{P_m\}$ olinganda ham, unga mos integral yig'indi qiymatlaridan iborat $\{\sigma_m\}$ ketma-ketlik, (ξ_k, η_k, ζ_k) nuqtalarni tanlab olinishga bog'liq bo'lmagan holda, hamma vaqt bitta J songa intilsa, bu J yig'indining limiti deb ataladi va u

$$\lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot S_k = J \qquad (19.4)$$

kabi belgilanadi.

Integral yig'indining limitini quyidagicha ham ta'riflash mumkin.

3-ta'rif. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topilsaki, (S) sirtning diametri $\lambda_p < \delta$ bo'lgan har qanday bo'laklashi hamda har bir (S_k) bo'lakdan olingan ixtiyoriy (ξ_k, η_k, ζ_k) lar uchun

$$|\sigma - J| < \varepsilon$$

tengsizlik bajarilsa, J son σ yig'indining limiti deb ataladi va u (19.4) kabi belgilanadi.

4-ta'rif. Agar $\lambda_p \to 0$ da $f(x,y,z)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, $f(x,y,z)$ funksiya (S) sirt bo'yicha integrallanuvchi (Riman ma'nosida integrallanuvchi) funksiya deb ataladi. Bu yig'indining chekli limiti J esa $f(x,y,z)$ funksiyaning birinchi tur sirt integrali deyiladi va u

$$\iint_{(S)} f(x,y,z)\,ds$$

kabi belgilanadi. Demak,

$$\iint_{(S)} f(x,y,z)\,ds = \lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) \cdot S_k$$

2. Uzluksiz funksiya birinchi tur sirt integrali. Endi birinchi tur sirt integrali-ning mavjud bo'lishini ta'minlaydigan shartni topish bilan shug'ullanamiz.

Faraz qilaylik R^3 fazodagi (S) sirt

$$z = z(x,y)$$

tenglama bilan berilgan bo'lsin. Bunda $z = z(x,y)$ funksiya chegaralangan yopiq (D) sohada $((D) \subset R^2)$ uzluksiz va $z'_x(x,y)$, $z'_y(x,y)$ hosilalarga ega hamda bu hosilalar ham (D) da uzluksiz.

1-teorema. Agar $f(x,y,z)$ funksiya (S) sirtda uzluksiz bo'lsa, u holda bu funksiyaning (S) sirt bo'yicha birinchi tur sirt integrali

$$\iint_{(S)} f(x,y,z)\,ds$$

mavjud va

$$\iint_{(S)} f(x,y,z)\,ds = \iint_{(D)} f(x,y,z(x,y))\sqrt{1 + z'^2_x(x,y) + z'^2_y(x,y)}\,dxdy$$

bo'ladi.

◀ (S) sirtning P_S bo'laklashni olaylik. Uning bo'laklari (S_1), (S_2),..., (S_n) bo'lsin. Bu sirt va uning bo'laklarining Oxy tekislikdagi proeksiyasi (D) sohaning P_D bo'laklashni va uning (D_1), (D_2),..., (D_n) bo'laklarini hosil qiladi. P_S bo'laklashiga nisbatan (19.2) yig'indini tuzamiz:

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)\cdot s_k$$

Ma'lumki, $(\xi_k,\eta_k,\zeta_k)\in(S_k)$. Bu nuqta ga akslanuvchi nuqta (ξ_k,η_k) bo'ladi. Demak, $\zeta_k=z(\xi_k,\eta_k)$ (19.1) formulaga binoan

$$S_k=\iint\limits_{(D_k)}\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}\,dxdy$$

bo'ladi.

O'rta qiymat haqidagi teorema (qaralsin, 17-bob, 5-§) dan foydalanib topamiz:

$$S_k=\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}\cdot D_k \quad \left((\xi_k^*,\eta_k^*)\in(D_k)\right).$$

Natijada σ yig'indi quyidagi

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))S_k=$$

$$=\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}\cdot D_k$$

ko'rinishga keladi.

Endi $\lambda_{P_S}\to 0$ da (bu holda $\lambda_{P_D}\to 0$ ham nolga intiladi) yig'indining limitini topish maqsadida uning ifodasini o'zgartirib yozamiz:

$$\sigma=\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\cdot D_k+$$
$$+\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\left[\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\right. \qquad (19.5)$$
$$\left.-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right]\cdot D_k.$$

Bu tenglikning o'ng tomonidagi ikkinchi qo'shiluvchini baholaymiz:

$$\left|\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\left[\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\right.\right.$$
$$\left.\left.-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right]\cdot D_k\right|\le M\sum_{k=1}^{n}\left|\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\right.$$
$$\left.-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right|\cdot D_k,$$

bunda

$$M=max\left|f(x,y,z)\right|$$

Ravshanki,

$$\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}$$

funksiya (D) da uzluksiz, demak, tekis uzluksiz. U holda Kantor teoremasining natijasiga ko'ra $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topiladiki, (D) sohaning diametri $\lambda_{P_D}<\delta$ bo'lgan har qanday P_D bo'laklashi uchun

$$\left|\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right|<\frac{\varepsilon}{MD}$$

bo'ladi. Unda

$$\left|\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\left[\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\right.\right.$$
$$\left.\left.-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right]D_k\right|<M\cdot\frac{\varepsilon}{MD}\sum_{k=1}^{n}D_k=\varepsilon$$

va demak,

$$\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\left[\sqrt{1+z_x'^2(\xi_k^*,\eta_k^*)+z_y'^2(\xi_k^*,\eta_k^*)}-\right.$$
$$\left.-\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\right]D_k=0$$

bo'ladi.

(19.5) tenglikning o'ng tomonidagi birinchi qo'shiluvchi

$$\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\cdot D_k$$

esa

$$f(x,y,z(x,y))\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}$$

funksiyaning integral yig'indisidir. Bu funksiya (D) sohada uzluksiz. Demak, $\lambda_{P_D}\to 0$ da integral yig'indi chekli limitga ega va

$$\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\cdot D_k=$$
$$=\iint\limits_{(D)} f(x,y,z(x,y))\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}\,dxdy$$

bo'ladi. Bu munosabatni e'tiborga olib, (19.5) tenglikda $\lambda_{P_S}\to 0$ da limitga o'tib topamiz:

$$\lim_{\lambda_{P_S}\to 0}\sigma=\lim_{\lambda_{P_S}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))\sqrt{1+z_x'^2(\xi_k,\eta_k)+z_y'^2(\xi_k,\eta_k)}\cdot D_k=$$
$$=\iint\limits_{(D)} f(x,y,z(x,y))\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}\,dxdy.$$

Demak,

$$\iint\limits_{(S)} f(x,y,z)\,ds=\iint\limits_{(D)} f(x,y,z(x,y))\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}\,dxdy\,.\blacktriangleright$$

Bu teorema, bir tomondan, uzluksiz funksiya birinchi tur sirt integralining mavjudligini aniqlab bersa, ikkinchi tomondan, bu integral ikki karrali Riman integrali orqali ifodalanishini ko'rsatadi.

1-eslatma. (S) sirt $x = x(y,z)$ $(y = y(z,x))$ tenglama bilan aniqlangan bo'lib, $x = x(y,z)$ funksiya ($y(z,x)$ funksiya) (D) sohada $((D) \subset R^2)$ uzluksiz va $x'_y(y,z)$, $x'_z(y,z)$ xususiy hosilalarga ($y'_z(z,x)$, $y'_x(z,x)$ xususiy hosilalarga) ega hamda bu hosilalar (D) da uzluksiz bo'lsin.

Agar $f(x,y,z)$ funksiya shu (S) sirtda uzluksiz bo'lsa, u holda bu funksiyaning birinchi tur sirt integrali

$$\iint\limits_{(S)} f(x,y,z)ds$$

mavjud va

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x(y,z),y,z)\sqrt{1+x'^2_y(y,z)+x'^2_z(y,z)}dydz,$$

$$\left(\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x,y(z,x),z)\sqrt{1+y'^2_x(z,x)+y'^2_z(z,x)}dzdx\right)$$

bo'ladi.

2-eslatma. Biz $f(x,y,z)$ funksiya birinchi tur sirt integralining mavjudligi maxsus ko'rinishdagi (S) sirtlar ($z = z(x,y)$, $x = x(y,z)$, $y = y(z,x)$ tenglamalar bilan aniqlangan sirtlar) uchun keltirdik. Aslida funksiya integralining mavjudligi keng sinfdagi sirtlar uchun to'g'ri bo'ladi. Jumladan, agar (S) sirt chekli sondagi yuqorida aytilgan sirtlar yig'indisi sifatida tasvirlangan bo'lsa, unda berilgan va uzluksiz bo'lgan $f(x,y,z)$ funksiyaning sirt integrali mavjud bo'ladi va u mos ikki karrali integrallar yig'indisiga teng bo'ladi.

3. Birinchi tur sirt integrallarining xossalari. Yuqorida keltirilgan teorema uzluksiz funksiyalar birinchi tur sirt integrallarining ikki karrali Riman integrallariga kelishini ko'rsatadi. Binobarin, bu sirt integrallar ham ikki karrali Riman integrallari xossalari kabi xossalarga ega bo'ladi. Ikki karrali Riman integrallarining xossalari 17-bobning 5-§ ida o'rganilgan.

4. Birinchi tur sirt integrallarni hisoblash. Yuqorida keltirilgan teorema birinchi tur sirt integralining mavjudligini tasdiqlabgina qolmasdan, uni hisoblash yo'lini ham ko'rsatadi. Demak, birinchi tur sirt integrallar ikki karrali Riman integrallariga keltirilib hisoblanadi:

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x,y,z(x,y))\sqrt{1+z'^2_x(x,y)+z'^2_y(x,y)}dxdy,$$

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x(y,z),y,z)\sqrt{1+x'^2_y(y,z)+x'^2_z(y,z)}dydz, \qquad (19.6)$$

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x,y(z,x),z)\sqrt{1+y'^2_z(z,x)+y'^2_x(z,x)}dzdx$$

19.1-misol. Ushbu

$$J=\iint\limits_{(S)}(x+y+z)ds$$

integral hisoblansin.

Bunda $(S)-x^2+y^2+z^2=r^2$ sferaning $z=0$ tekislikning yuqorisida joylashgan qismi.

◀Ravshanki, (S) sirt

$$z=\sqrt{r^2-x^2-y^2}$$

tenglama bilan aniqlangan bo'lib, bu sirtda berilgan $f(x,y,z)=x+y+z$ funksiya uzluksizdir. 1-teoremaga ko'ra

$$J=\iint\limits_{(D)}\left(x+y+\sqrt{r^2-x^2-y^2}\right)\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}dxdy$$

bo'ladi, bunda $(D)=\{(x,y)\in R^2:\ x^2+y^2\le r^2\}$.

Endi bu tenglikning o'ng tomonidagi ikki karrali integralni hisoblaymiz:

$$z_x'(x,y)=-\frac{x}{\sqrt{r^2-x^2-y^2}},\quad z_y'(x,y)=-\frac{y}{\sqrt{r^2-x^2-y^2}},$$

$$\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}=\frac{r}{\sqrt{r^2-x^2-y^2}}.$$

Demak,

$$J=\iint\limits_{(D)}\left(x+y+\sqrt{r^2-x^2-y^2}\right)\sqrt{1+z_x'^2(x,y)+z_y'^2(x,y)}dxdy=$$

$$=r\iint\limits_{(D)}\left(\frac{x+y}{\sqrt{r^2-x^2-y^2}}+1\right)dxdy.$$

Keyingi integralda o'zgaruvchilarni almashtiramiz:

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi.$$

Natijada

$$J=r\int_0^{2\pi}\left(\int_0^r\left(\frac{\rho(\cos\varphi+\sin\varphi)}{\sqrt{r^2-\rho^2}}+1\right)\rho d\rho\right)d\varphi=r\int_0^{2\pi}\left(\int_0^r\frac{\rho(\cos\varphi+\sin\varphi)}{\sqrt{r^2-\rho^2}}\rho d\rho\right)d\varphi+$$

$$+r\int_0^{2\pi}\left(\int_0^r\rho d\rho\right)d\varphi=r\int_0^{2\pi}(\cos\varphi+\sin\varphi)d\varphi\int_0^r\frac{\rho^2d\rho}{\sqrt{r^2-\rho^2}}+r\cdot2\pi\cdot\frac{r^2}{2}=\pi r^3.$$

Demak, berilgan integral

$$\iint\limits_{(S)}(x+y+z)dS=\pi r^3$$

bo'ladi.▶

19.2-misol. Ushbu

$$\iint\limits_{(S)}x(y+z)ds$$

integral hisoblansin, bunda $(S) - x = \sqrt{в^2 - y^2}$ silindrik sirtning $z = 0$, $z = c$ $(c > 0)$ tekisliklar orasidagi qismi.

◀Modomiki, bu (S) sirt $x = \sqrt{в^2 - y^2}$ ko'rinishda berilgan ekan, unda integralni hisoblash uchun (19.6) formuladan foydalanish lozimdir.

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} f(x(y,z),y,z)\sqrt{1 + x_y'^2(y,z) + x_z'^2(y,z)}dydz$$

Bunda (D) soha (S) sirtning Oyz tekislikdagi proeksiyasidan iborat:

$$(D) = \left\{(y,z) \in R^2 : x = \sqrt{в^2 - y^2}, z = 0,\ z = c\right\} =$$
$$= \left\{(y,z) \in R^2 : -в \le y \le в,\ 0 \le z \le c\right\}.$$

$x = \sqrt{в^2 - y^2}$ funksiyaning xususiy hosilalari

$$x_y'(y,z) = -\frac{y}{\sqrt{в^2 - y^2}}, \quad x_z'(y,z) = 0$$

bo'ladi. Demak,

$$\iint\limits_{(S)} f(x,y,z)ds = \iint\limits_{(D)} \sqrt{в^2 - y^2}(y+z)\sqrt{1 + \frac{y^2}{в^2 - y^2}}dydz = в\iint\limits_{(D)}(y+z)dydz$$

bo'ladi. Bu tenglikning o'ng tomonidagi ikki karrali integralni hisoblab topamiz:

$$в\iint\limits_{(D)}(y+z)dydz = в\int_{-в}^{в}\left(\int_0^c (y+z)dz\right)dy = в\int_{-в}^{в}\left(yz + \frac{z^2}{2}\right)\Bigg|_{z=0}^{z=c} dy$$

$$= в\int_{-в}^{в}\left(cy + \frac{c^2}{2}\right)dy = \frac{вc}{2}y^2\Big|_{-в}^{в} + \frac{вc^2}{2}y\Big|_{-в}^{в} = в^2c^2$$

Demak,

$$\iint\limits_{(S)} x(y+z)ds = в^2c^2. ▶$$

2-§. Ikkinchi tur sirt integrallari

R^3 fazoda $z = z(x,y)$ tenglama bilan aniqlangan (S) sirtni qaraylik. Bunda $z(x,y)$ funksiya chegarasi bo'lakli-silliq chiziqdan iborat bo'lgan (D) sohada $\left((D) \subset R^2\right)$ berilgan, uzluksiz, $z_x'(x,y)$, $z_y'(x,y)$ xususiy hosilalarga ega hamda bu hosilalar ham uzluksiz. Odatda bunday sirtni silliq sirt deyiladi. Silliq sirt har bir (x_0, y_0, z_0) nuqtasida urinma tekislikka ega bo'ladi.

Endi (S) sirt uning chegarasi bilan kesishmaydigan K yopiq chiziqni olaylik. (x_0, y_0, z_0) nuqta sirtning K yopiq chiziq bilan chegaralangan qismga tegishli bo'lsin. Bu chiziqni Oxy tekisligiga proeksiyalaymiz. Natijada Oxy tekislikda ham $K_П$ yopiq chiziq hosil bo'ladi. Mazkur kursning 18-bob, 2-§ ida tekislikdagi yopiq chiziqning musbat va manfiy yo'nalishlari kiritilgan edi. (S) sirtdagi yopiq chiziqning musbat va manfiy yo'nalishlari ham shu singari kiritiladi. Shuni ham aytish kerakki, yo’nalishning musbat yoki manfiyligini aniqlash xarakatlanayotgan nuqtaga qay tomondan qarashga ham bog’liq.

Sirtning (x_0, y_0, z_0) nuqtasidagi urinma tekislikka shu nuqtada perpendikulyar o’tkazaylik. Bu perpendikulyarning musbat yo’nalishi deb shunday yo’nalish olamizki, uning tomonidan qaralganda ikkala (K hamda K_{Π}) yopiq chiziqlarning yo’nalishlari musbat bo’ladi. Uning manfiy yo’nalishi esa shunday yo’nalishki, u tomondan qaralganda K_{Π} ning musbat yo’nalishiga K ning manfiy yo’nalishi mos keladi. Perpendikulyarning musbat yo’nalishi bo’yicha olingan birlik kesma sirtning (x_0, y_0, z_0) nuqtasidagi normali deyiladi.

Normalning Ox, Oy va Oz o’qlarining musbat yo’nalishlari bilan tashkil qilgan burchaklarini mos ravishda α, β, γ orqali belgilasak,

$$cos\,\alpha = -\frac{z'_x}{\sqrt{1+z'^2_x+z'^2_y}},\quad cos\,\gamma = -\frac{z'_y}{\sqrt{1+z'^2_x+z'^2_y}},\quad cos\,\gamma = \frac{1}{\sqrt{1+z'^2_x+z'^2_y}} \qquad (19.7)$$

bo’ladi va ular normalning yo’naltiruvchi kosinuslari deyiladi.

Isbotlash mumkinki, silliq (S) sirtning barcha nuqtalaridagi perpendikulyarlarning musbat yo’nalishlari (normallari) bir xil bo’ladi. Va, demak, manfiy yo’nalishlari ham. Shunga ko’ra, sirtning ikki tomoni haqida tushuncha kiritiladi.

Sirtning ustki tomoni deb, uning shunday tomoni olinadiki, bu tomondan qaralganda ikkala (K hamda K_{Π}) yopiq chiziqlarning yo’nalishlari musbat bo’ladi.

Sirtning ustki tomoni qaralganda K_{Π} bilan chegaralangan tekis shaklning yuzi musbat ishora bilan, pastki tomoni (ikkinchi tomoni) qaralganda manfiy ishora bilan olinadi.

1. Ikkinchi tur sirt integralining ta’rifi. $f(x,y,z)$ funksiya (S) sirtda beril-gan bo’lsin. Bu sirtning ma’lum bir tomonini olaylik. Sirtning P bo’laklashini va bu bo’laklashning har bir (S_k) bo’lagida $(k=1,2,...,n)$ ixtiyoriy (ξ_k, η_k, ζ_k) nuqta $(k=1,2,...,n)$ olaylik. Berilgan funksiyaning (ξ_k, η_k, ζ_k) nuqtadagi $f(\xi_k, \eta_k, \zeta_k)$ qiymatini (S_k) ning $0xy$ tekislikdagi proeksiyasi (D_k) ning yuziga ko’paytirib quyidagi yig’indini tuzamiz:

$$\sigma = \sum_{k=1}^{n} f(\xi_k, \eta_k, \zeta_k) D_k\,. \qquad (19.8)$$

(S) sirtning shunday

$$P_1,\ P_2,\ ...,\ P_m,\ ... \qquad (19.9)$$

bo’laklashlarini qaraymizki, ularning mos diametridan tashkil topgan

$$\lambda_{P_1}, \lambda_{P_2}, ..., \lambda_{P_m}, ...$$

ketma-ketlik nolga intilsin: $\lambda_{P_m} \to 0$. Bunday P_m $(m=1,2,...)$ bo’laklashlarga nisbatan $f(x,y,z)$ funksiyaning integral yig’indilarini tuzamiz. Natijada (S) sirtning (19.9) bo’laklashlariga mos integrallar yig’indilar qiymatlaridan iborat quyidagi

$$\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_m,\ \ldots$$

ketma-ketlik hosil bo'ladi.

5-ta'rif. Agar (S) sirtning har qanday (19.9) bo'laklashlari ketma-ketligi $\{P_m\}$ olinganda ham, unga mos integrallar yig'indilari qiymatlaridan iborat $\{\sigma_m\}$ ketma-ketlik, (ξ_k,η_k,ζ_k) nuqtalarni tanlab olinishiga bog'liq bo'lmagan holda hamma vaqt bitta J songa intilsa, bu J σ yig'indining limiti deb ataladi va u

$$\lim_{\lambda_P\to 0}\sigma=\lim_{\lambda_P\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k=J \qquad (19.10)$$

kabi belgilanadi.

Integral yig'indining limitini quyidagicha ham ta'riflash mumkin.

6-ta'rif. Agar $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topilsaki, (S) sirtning diametri $\lambda_P<\delta$ bo'lgan har qanday P bo'laklashi hamda har bir (S_k) bo'lakdan olingan ixtiyoriy (ξ_k,η_k,ζ_k) lar uchun

$$|\sigma-J|<\varepsilon$$

tengsizlik bajarilsa, J soni σ yig'indining limiti deb ataladi va u (19.10) kabi belgilanadi.

7-ta'rif. Agar $\lambda_P\to 0$ da $f(x,y,z)$ funksiyaning integral yig'indisi σ chekli limitga ega bo'lsa, $f(x,y,z)$ funksiya (S) sirtning tanlangan tomon bo'yicha integrallanuvchi funksiya deb ataladi. Bu yig'indining chekli limiti J esa, $f(x,y,z)$ funksiyaning (S) sirtning tanlangan tomoni bo'yicha ikkinchi tur sirt integrali deb ataladi va u

$$\iint_{(S)} f(x,y,z)dxdy$$

kabi belgilanadi. Demak,

$$\iint_{(S)} f(x,y,z)dxdy=\lim_{\lambda_P\to 0}\sigma=\lim_{\lambda_P\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k .$$

Funksiya ikkinchi tur sirt integralining quyidagicha

$$\iint_{(S)} f(x,y,z)dxy \qquad (19.11)$$

belgilanishidan, integral (S) sirtning qaysi tomon bo'yicha olinganligi ko'rinmaydi. Binobarin, (19.11) integral to'g'risida gap borganda, har gal integral sirtning qaysi tomon bo'yicha olinayotganligi aytib boriladi.

Ravshanki, $f(x,y,z)$ funksiyaning (S) sirtning bir tomoni bo'yicha olingan ikkinchi tur sirt integrali, funksiyaning shu sirtning ikkinchi tomoni bo'yicha olingan ikkinchi tur integralidan faqat ishorasi bilangina farq qiladi.

Yuqoridagidek

$$\iint_{(S)} f(x,y,z)dydz,\quad \iint_{(S)} f(x,y,z)dzdx$$

ikkinchi tur sirt integrallari ta'riflanadi.

Shunday qilib, sirtda berilgan $f(x,y,z)$ funksiyadan uchta -*Oxy* tekislikdagi proeksiyalar, *Oyz* tekislikdagi proeksiyalar hamda *Ozx* tekislikdagi proeksiyalar vositasida olingan ikkinchi tur sirt integrali tushunchalari kiritiladi.

Umumiy holda, (S) sirtda $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar berilgan bo'lib, ushbu

$$\iint_{(S)} P(x,y,z)dxdy\,, \quad \iint_{(S)} Q(x,y,z)dydz\,, \quad \iint_{(S)} R(x,y,z)dzdx$$

integrallar mavjud bo'lsa, u holda

$$\iint_{(S)} P(x,y,z)dxdy \,\mathrm{Q} \iint_{(S)} Q(x,y,z)dydz \,\mathrm{Q} \iint_{(S)} R(x,y,z)dzdx$$

yig'indi ikkinchi tur sirt integralining umumiy ko'rinishi deb ataladi va u

$$\iint_{(S)} P(x,y,z)dxdy + Q(x,y,z)dydz + R(x,y,z)dzdx$$

kabi belgilanadi. Demak,

$$\iint_{(S)} P(x,y,z)dxdy + \iint_{(S)} Q(x,y,z)dydz + \iint_{(S)} R(x,y,z)dzdx =$$

$$= \iint_{(S)} P(x,y,z)dxdy + Q(x,y,z)dydz + R(x,y,z)dzdx\,.$$

Endi R^3 fazoda biror (V) jism berilgan bo'lsin. Bu qismni o'rab turgan yopiq sirt silliq sirt bo'lib, uni (S) deylik. $f(x,y,z)$ funksiya (V) da berilgan. *Oxy* tekislikka parallel bo'lgan tekislik bilan (V) ni ikki qismga ajratamiz: $(V)=(V_1)+(V_2)$. Natijada uni o'rab turgan (S) sirt ham (S_1) va (S_2) sirtlarga ajraladi. Ushbu

$$\iint_{(S_1)} f(x,y,z)dxdy \,\mathrm{Q} \iint_{(S_2)} f(x,y,z)dxdy \qquad (19.12)$$

integral (agar u mavjud bo'lsa) $f(x,y,z)$ funksiyaning yopiq sirt bo'yicha ikkinchi tur sirt integrali deb ataladi va

$$\oiint_{(S)} f(x,y,z)dxdy$$

kabi belgilanadi. Bunda (19.12) munosabatdagi birinchi integral (S_1) sirtning ustki tomoni, ikkinchi integral esa (S_2) sirtning pastki tomoni bo'yicha olingan. Xuddi shunga o'xshash

$$\oiint_{(S)} f(x,y,z)dydz\,, \quad \oiint_{(S)} f(x,y,z)dzdx$$

hamda, umumiy holda

$$\oiint_{(S)} P(x,y,z)dxdy \,\mathrm{Q}\, Q(x,y,z)dydz \,\mathrm{Q}\, R(x,y,z)dzdx$$

integrallar ta'riflanadi.

2. Uzluksiz funksiya ikkinchi tur sirt integrali. Faraz qilaylik, R^3 fazoda (S) sirt $z=z(x,y)$ tenglama bilan berilgan bo'lsin. Bunda $z=z(x,y)$ funksiya chegaralangan yopiq (D) sohada $\left((D)\subset R^2\right)$ uzluksiz va $z'_x(x,y)$, $z'_y(x,y)$ xususiy hosilalarga ega hamda bu hosilalar ham (D) da uzluksiz.

2-teorema. Agar $f(x,y,z)$ funksiya (S) sirtda uzluksiz bo'lsa, u holda bu funksiyaning (S) sirt bo'yicha olingan ikkinchi tur sirt integrali

$$\iint_{(S)} f(x,y,z)dxdy$$

mavjud va

$$\iint_{(S)} f(x,y,z)dxdy = \iint_{(D)} f(x,y,z(x,y))dxdy$$

bo'ladi.

◄ (S) sirtning P_S bo'laklashini olaylik. Uning bo'laklari (S_1), $(S_2),\ldots,(S_n)$ bo'lsin. Bu sirt va uning bo'laklarining Oxy tekislikdagi proeksiyasi (D) ning P_D bo'laklashini va uning (D_1), $(D_2),\ldots,(D_n)$ bo'laklarini hosil qiladi. P_S bo'laklashga nisbatan ushbu yig'indini tuzamiz:

$$\sigma = \sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k. \qquad (19.8)$$

Agar (S) sirtning ustki tomoni qaralayotgan bo'lsa, u holda barcha D_k lar musbat bo'ladi.

Modomiki, $f(x,y,z)$ funksiya $z=z(x,y)$ sirtda berilgan ekan, u x va y o'zgaruvchilarning quyidagi funksiyasiga aylanadi:

$$f(x,y,z) = f(x,y,z(x,y)).$$

Bundan esa

$$\zeta_k = z(\xi_k,\eta_k) \quad (k=1,2,\ldots n)$$

bo'lishi kelib chiqadi. Natijada (19.8) yig'indi ushbu

$$\sigma = \sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))D_k$$

ko'rinishga keladi. Bu yig'indi $f(x,y,z(x,y))$ funksiyaning integral yig'indisi (ikki karrali integral uchun integral yig'indi) ekanini payqash qiyin emas. Agar $f(x,y,z(x,y))$ funksiyaning (D) da uzluksiz ekanligini e'tiborga olsak, unda $\lambda_{P_D}\to 0$ da

$$\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))D_k$$

yig'indi chekli limitga ega va

$$\lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))D_k = \iint_{(D)} f(x,y,z(x,y))dxdy$$

bo'ladi. Demak,

$$\lim_{\lambda_{P_S}\to 0}\sigma = \lim_{\lambda_{P_S}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,\zeta_k)D_k =$$

$$= \lim_{\lambda_{P_D}\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k,z(\xi_k,\eta_k))D_k = \iint_{(D)} f(x,y,z(x,y))dxdy.$$

Bundan esa

$$\iint_{(S)} f(x,y,z)dxdy = \iint_{(D)} f(x,y,z(x,y))dxdy$$

bo'lishi kelib chiqadi.▶

Agar (S) sirtning pastki tomoni qaralsa, unda D_k lar manfiy bo'lib,

$$\iint_{(S)} f(x,y,z)dxdy = -\iint_{(D)} f(x,y,z(x,y))dxdy$$

bo'ladi.

Xuddi yuqoridagidek, tegishli shartlarda

$$\iint_{(S)} f(x,y,z)dydz\,,\quad \iint_{(S)} f(x,y,z)dzdx$$

integrallar mavjud va

$$\iint_{(S)} f(x,y,z)dydz = \iint_{(D)} f(x(y,z),y,z)dydz\,,$$

$$\iint_{(S)} f(x,y,z)dzdx = \iint_{(D)} f(x,y(x,z),z)dzdx$$

bo'ladi.

1-natija. Yasovchilari Oz o'qiga parallel bo'lgan (S) silindrik sirtni qaraylik. $f(x,y,z)$ funksiya shu sirtda berilgan bo'lsin. U holda

$$\iint_{(S)} f(x,y,z)dxdy$$

mavjud bo'ladi va u nolga teng:

$$\iint_{(S)} f(x,y,z)dxdy = 0\,.$$

Xuddi shunga o'xshash, tegishli shartlarda

$$\iint_{(S)} f(x,y,z)dydz = 0\,,\quad \iint_{(S)} f(x,y,z)dzdx = 0$$

bo'ladi.

Bu tengliklar bevosita ikkinchi tur sirt integrallari ta'rifidan kelib chiqadi.

Yuqorida keltirilgan teoremadan foydalanib, ikkinchi tur sirt integrallari ham ikki karrali Riman integrallari xossalari kabi xossalarga ega bo'lishini ko'rsatish va ularni keltirib chiqarishni o'quvchiga havola etamiz.

3. Ikkinchi tur sirt integrallarini hisoblash. Yuqorida keltirilgan teoremadan foydalanib ikkinchi tur sirt integrallari ikki karrali Riman integrallariga keltirib hisoblanadi:

$$\iint_{(S)} f(x,y,z)dxdy = \iint_{(D)} f(x,y,z(x,y))dxdy\,,$$

$$\iint_{(S)} f(x,y,z)dydz = \iint_{(D)} f(x(y,z),y,z)dydz,$$

$$\iint_{(S)} f(x,y,z)dzdx = \iint_{(D)} f(x,y(z,x),z)dzdx.$$

19.3-misol. Ushbu

$$\iint_{(S)} \left(\frac{x^2}{a^2}+\frac{y^2}{в^2}+kz\right)dxdy$$

integral hisoblansin. Bunda $(S)-\frac{x^2}{a^2}+\frac{y^2}{в^2}+\frac{z^2}{c^2}=1$ ellipsoidning $z=0$ tekislikdan pastda joylashgan qism bo'lib, integral shu sirtning pastki tomon bo'yicha olingan.

◀Ravshanki, bu (S) sirtning tenglamasi quyidagicha bo'lib,

$$z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}$$

uning Oxy tekislikdagi proeksiyasi

$$(D)=\left\{(x,y)\in R^2:\ \frac{x^2}{a^2}+\frac{y^2}{в^2}\le 1\right\}$$

bo'ladi.

(S) sirt ham, bu sirtda berilgan

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{в^2}+kz$$

funksiya ham 2-teoremaning shartlarini qanoatlantiradi. U holda

$$\iint_{(S)} \left(\frac{x^2}{a^2}+\frac{y^2}{в^2}+kz\right)dxdy = -\iint_{(D)} \left(\frac{x^2}{a^2}+\frac{y^2}{в^2}-kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}\right)dxdy$$

bo'ladi. Integral (S) sirtning pastki tomoni bo'yicha olinganligi sababli tenglikning o'ng tomonidagi ikki karrali integral oldiga minus ishorasi qo'yildi.

Endi bu

$$-\iint_{(D)} \left(\frac{x^2}{a^2}+\frac{y^2}{в^2}-kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}\right)dxdy = \iint_{(D)} \left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}-\frac{x^2}{a^2}-\frac{y^2}{в^2}\right)dxdy$$

ikki karrali integralni hisoblaymiz. Ikki karrali integralda o'zgaruvchilarni

$$x=a\rho\cos\varphi,\quad y=в\rho\sin\varphi$$

kabi almashtirib quyidagini topamiz:

$$\iint_{(D)} \left(kc\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{в^2}}-\frac{x^2}{a^2}-\frac{y^2}{в^2}\right)dxdy = \int_0^{2\pi}\left[\int_0^1 \left(kc\sqrt{1-\rho^2}-\rho^2\right)aв\rho d\rho\right]d\varphi =$$

$$= aв\int_0^{2\pi}\left[\int_0^1 \left(kc\rho\sqrt{1-\rho^2}-\rho^2\right)d\rho\right]d\varphi = 2\pi aв\left[-\frac{kc}{2}\frac{\left(1-\rho^2\right)^{3/2}}{3/2}-\frac{\rho^4}{4}\right]_0^1 = 2\pi aв\left(-\frac{1}{4}+\frac{kc}{3}\right).$$

Demak,

$$\iint\limits_{(S)}\left(\frac{x^2}{a^2}+\frac{y^2}{в^2}+kc\right)dxdy=2\pi aв\left(\frac{kc}{3}-\frac{1}{4}\right).\ ▶$$

4. Birinchi va ikkinchi tur sirt integrallari orasida bog'lanish. Biz 18-bob-ning 4-§ da birinchi va ikkinchi tur egri chiziqli integrallar orasidaga bog'lanishni ifodalaydigan formulalarni keltirgan edik.

Shunga o'xshash, birinchi va ikkinchi tur sirt integrallari orasidagi bog'lanishni ifodalovchi formulalar ham mavjud.

(S) sirt va unda berilgan $f(x,y,z)$ va $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiyalar tegishli shartlarni qanoatlantirganda (qaralsin, 2-§ning 1-punkti) ushbu

$$\iint\limits_{(S)} f(x,y,z)dydz=\iint\limits_{(S)} f(x,y,z)\cos\alpha ds,$$

$$\iint\limits_{(S)} f(x,y,z)dzdx=\iint\limits_{(S)} f(x,y,z)\cos\beta ds, \qquad (19.13)$$

$$\iint\limits_{(S)} f(x,y,z)dxdy=\iint\limits_{(S)} f(x,y,z)\cos\gamma ds$$

umumiy holda

$$\iint\limits_{(S)} P(x,y,z)dydz+Q(x,y,z)dzdx+R(x,y,z)dxdy=$$

$$=\iint\limits_{(S)}\left[P(x,y,z)\cos\alpha+Q(x,y,z)\cos\beta+R(x,y,z)\cos\gamma\right]ds$$

formulalarning to'g'riligini isbotlashni o'quvchiga havola etamiz.

3-§. Stoks formulasi

R^3 fazoda $z=z(x,y)$ tenglama bilan aniqlangan silliq (S) sirt berilgan bo'lsin. Bu sirtning chegarasi ∂S bo'lakli-silliq egri chiziq bo'lsin. (S) sirtning Oxy tekislikdagi proeksiyasini (D) deylik. Unda ∂S ning proeksiyasi ∂D dan iborat bo'ladi.

Faraz qilaylik, (S) sirtda $P(x,y,z)$ funksiya berilgan bo'lib, u uzluksiz bo'lsin. Undan tashqari bu funksiya (S) da

$$\frac{\partial P(x,y,z)}{\partial x},\ \frac{\partial P(x,y,z)}{\partial y},\ \frac{\partial P(x,y,z)}{\partial z}$$

xususiy hosilalarga ega va ular uzluksiz bo'lsin.

Ushbu

$$\int\limits_{\partial S} P(x,y,z)dx$$

egri chiziqli integralni qaraylik (uning mavjudligi ravshan). Agar ∂S chiziqning (S) sirtda yotishini e'tiborga olsak, u holda

$$\int\limits_{\partial S} P(x,y,z)dx=\int\limits_{\partial S} P(x,y,z(x,y))dx$$

bo'ladi.

Endi Grin formulasidan foydalanib ushbuni topamiz:

$$\int\limits_{\partial D} P(x,y,z(x,y))dx = -\iint\limits_{(D)} \frac{\partial P(x,y,z(x,y))}{\partial y} dxdy$$

Ravshanki, $P(x,y,z(x,y))$ funksiyaning y o'zgaruvchi bo'yicha xususiy hosilasi

$$\frac{\partial P(x,y,z(x,y))}{\partial y} + \frac{\partial P(x,y,z(x,y))}{\partial z} \cdot z'_y(x,y)$$

bo'ladi.

Ushbu bobning 2-§ idagi (19.7) munosabatlardan

$$z'_y(x,y) = -\frac{\cos\beta}{\cos\gamma}$$

bo'lishini e'tiborga olsak,

$$\iint\limits_{(D)} \left[\frac{\partial P(x,y,z(x,y))}{\partial y} + \frac{\partial P(x,y,z(x,y))}{\partial z} \cdot z'_y(x,y)\right] =$$

$$= \iint\limits_{(D)} \left[\frac{\partial P(x,y,z(x,y))}{\partial y} - \frac{\partial P(x,y,z(x,y))}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] dxdy$$

bo'ladi.

Natijada qaralayotgan integral uchun quyidagi tenglikka ega bo'lamiz:

$$\int\limits_{\partial S} P(x,y,z)dx = -\iint\limits_{(D)} \left[\frac{\partial P(x,y,z(x,y))}{\partial y} - \frac{\partial P(x,y,z(x,y))}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] dxdy \qquad (19.14)$$

2-§ dagi 2-teoremadan foydalanib (19.14) tenglikning o'ng tomonidagi ikki karrali integralni ikkinchi tur sirt integrali orqali ifodalaymiz:

$$\iint\limits_{(D)} \left[\frac{\partial P(x,y,z(x,y))}{\partial y} - \frac{\partial P(x,y,z(x,y))}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] dxdy =$$

$$= \iint\limits_{(S)} \left[\frac{\partial P(x,y,z)}{\partial y} - \frac{\partial P(x,y,z)}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] dxdy .$$

Bu tenglikning o'ng tomonidagi ikkinchi tur sirt integralini, (19.13) formulaga asoslanib, birinchi tur sirt integraliga keltiramiz:

$$\iint\limits_{(S)} \left[\frac{\partial P(x,y,z)}{\partial y} - \frac{\partial P(x,y,z)}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] dxdy =$$

$$= \iint\limits_{(S)} \left[\frac{\partial P(x,y,z)}{\partial y} - \frac{\partial P(x,y,z)}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] \cos\gamma ds = \qquad (19.15)$$

$$= \iint\limits_{(S)} \frac{\partial P(x,y,z)}{\partial y} \cos\gamma ds - \iint\limits_{(S)} \frac{\partial P(x,y,z)}{\partial y} \cos\beta ds .$$

Va nihoyat, yana (19.13) formulalardan foydalanib quyidagini topamiz:

$$\iint_{(S)} \frac{\partial P(x,y,z)}{\partial y} \cos\gamma ds = \iint_{(S)} \frac{\partial P(x,y,z)}{\partial y} dxdy,$$
$$\iint_{(S)} \frac{\partial P(x,y,z)}{\partial z} \cos\beta ds = \iint_{(S)} \frac{\partial P(x,y,z)}{\partial z} dzdx \qquad (19.16)$$

(19.14), (19.15) va (19.16) munosabatlardan

$$\int_{\partial S} P(x,y,z) = \iint_{(S)} \frac{\partial P(x,y,z)}{\partial z} dzdx - \frac{\partial P(x,y,z)}{\partial y} dxdy \qquad (19.17)$$

bo’lishi kelib chiqadi.

Xuddi shunday mulohaza asosida (S) sirt va unda berilgan $Q(x,y,z)$, $R(x,y,z)$ funksiyalar tegishli shartlarni bajarganda ushbu

$$\int_{\partial S} Q(x,y,z)dy = \iint_{(S)} \frac{\partial Q(x,y,z)}{\partial x} dxdy - \frac{\partial Q(x,y,z)}{\partial z} dydz,$$
$$\int_{\partial S} R(x,y,z)dz = \iint_{(S)} \frac{\partial R(x,y,z)}{\partial y} dydz - \frac{\partial R(x,y,z)}{\partial x} dzdx \qquad (19.18)$$

formulalarning o’rinli bo’lishi ko’rsatiladi. (19.17) va (19.18) formulalarni hadlab qo’shib quyidagini topamiz:

$$\int_{\partial S} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$
$$= \iint_{(S)} \left[\frac{\partial Q(x,y,z)}{\partial x} - \frac{\partial P(x,y,z)}{\partial y}\right] dxdy + \left[\frac{\partial R(x,y,z)}{\partial y} = \right. \qquad (19.19)$$
$$\left. - \frac{\partial Q(x,y,z)}{\partial z}\right] dydz + \left[\frac{\partial P(x,y,z)}{\partial z} - \frac{\partial R(x,y,z)}{\partial x}\right] dzdx.$$

Bu Stoks formulasi deb ataladi.

2-natija. Mazkur kursning 18-bob, 3-§ idagi Grin formulasi Stoks formulasining xususiy holidir. Haqiqatdan ham (19.19) Stoks formulasida (S) sirt sifatida Oxy tekislikdagi (D) soha olinsa, unda $z = 0$ bo’lib, (19.19) formuladan

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_{(D)} \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] dxdy$$

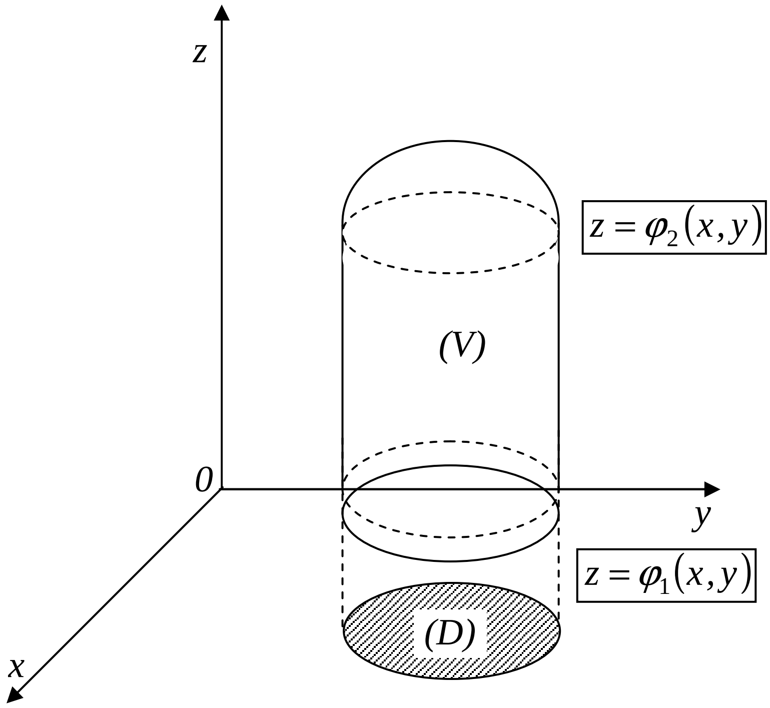
bo’lishi kelib chiqadi. Bu Grin formulasidir.

Shunday qilib, Stoks formulasi (S) sirt bo’yicha olingan II-tur sirt integrali bilan shu sirtning chegarasi bo’yicha olingan egri chiziqli integralni bog’lovchi formuladir.

4-§. Ostrogradskiy formulasi

R^3 fazoda pastdan $z = \varphi_1(x,y)$ tenglama bilan aniqlangan silliq (S_1) sirt bilan, yuqoridan $z = \varphi_2(x,y)$ tenglama yordamida aniqlangan silliq (S_2) sirt bilan, yon tomondan esa yasovchilari Oz o’qiga parallel bo’lgan silindrik (S_3) sirt bilan chegaralangan (V) sohani (jismni) qaraylik. Uning Oxy tekislikdagi proeksiyasi

(D) bo'lib, bu (D) ning chegarasi yuqorida aytilgan silindrik sirtning yo'naltiruvchisi sifatida olinadi. $(\varphi_1(x,y) \le \varphi_2(x,y)\ (x,y) \in (D))$ (64-chizma).



64-chizma

Faraz qilaylik, (V) da $R(x,y,z)$ funksiya berilgan va uzluksiz bo'lsin. Bundan tashqari bu funksiya shu sohada

$$\frac{\partial R(x,y,z)}{\partial z}$$

xususiy hosilaga ega va bu hosila ham uzluksiz.

Ravshanki, bu holda

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z} dxdydz$$

mavjud bo'ladi va 17-bobning 10-§ ida keltirilgan formulaga ko'ra

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z} dxdydz = \iint\limits_{(D)} \left(\int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R(x,y,z)}{\partial z} dz \right) dxdy \qquad (19.20)$$

bo'ladi.

Agar

$$\int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R(x,y,z)}{\partial z} dz = R(x,y,\varphi_2(x,y)) - R(x,y,\varphi_1(x,y))$$

bo'lishini e'tiborga olsak, u holda

$$\iint\limits_{(D)} \left(\int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial R(x,y,z)}{\partial z} dz \right) dxdy = \iint\limits_{(D)} R(x,y,\varphi_2(x,y)) dxdy - \iint\limits_{(D)} R(x,y,\varphi_1(x,y)) dxdy \quad (19.21)$$

bo'ladi. Bu tenglikning o'ng tomonidagi ikki karrali integrallarni 2-§ dagi formulalardan foydalanib, sirt integrallari orqali yozamiz:

$$\iint\limits_{(D)} R(x,y,\varphi_2(x,y))dxdy = \iint\limits_{(S_2)} R(x,y,z)dxdy,$$
$$\iint\limits_{(D)} R(x,y,\varphi_1(x,y))dxdy = \iint\limits_{(S_1)} R(x,y,z)dxdy. \qquad (19.22)$$

Keltirilgan tengliklardagi sirt integrallari sirtning ustki tomoni bo'yicha olingan (19.20), (19.21) va (19.22) munosabatlardan quyidagini topamiz:

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z} dxdydz = \iint\limits_{(S_2)} R(x,y,z)dxdy + \iint\limits_{(S_1)} R(x,y,z)dxdy\,. \qquad (19.23)$$

Bu tenglikning o'ng tomonidagi ikkinchi integral (S_1) sirtning pastki tomoni bo'yicha olingan.

(S_3) sirt yasovchilari Oz o'qiga parallel bo'lgan silindrik sirt bo'lganligidan

$$\iint\limits_{(S_3)} R(x,y,z)dxdy = 0 \qquad (19.24)$$

bo'ladi. (19.23) va (19.24) munosabatlardan

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z} dxdydz = \iint\limits_{(S_1)} R(x,y,z)dxdy + \iint\limits_{(S_2)} R(x,y,z)dxdy +$$
$$+ \iint\limits_{(S_3)} R(x,y,z)dxdy = \oiint\limits_{(S)} R(x,y,z)dxdy$$

bo'lishi kelib chiqadi. Bunda $(S)-(V)$ jismni o'rab turuvchi sirt.
Demak,

$$\iiint\limits_{(V)} \frac{\partial R(x,y,z)}{\partial z} dxdydz = \oiint\limits_{(S)} R(x,y,z)dxdy \qquad (19.25)$$

Xuddi shu yo'l bilan, (V) hamda $P(x,y,z)$, $Q(x,y,z)$ lar tegishli shartlarni qanoatlantirganda quyidagi

$$\iiint\limits_{(V)} \frac{\partial P(x,y,z)}{\partial x} dxdydz = \oiint\limits_{(S)} P(x,y,z)dydz\,, \qquad (19.26)$$

$$\iiint\limits_{(V)} \frac{\partial Q(x,y,z)}{\partial y} dxdydz = \oiint\limits_{(S)} Q(x,y,z)dzdx \qquad (19.27)$$

formulalarning to'g'riligi isbotlanadi.

Yuqoridagi (19.25), (19.26) va (19.27) tengliklarni hadlab qo'shib quyidagilarni topamiz:

$$\iiint\limits_{(V)} \left(\frac{\partial P(x,y,z)}{\partial x} + \frac{\partial Q(x,y,z)}{\partial y} + \frac{\partial R(x,y,z)}{\partial z} \right) dxdydz =$$
$$= \oiint\limits_{(S)} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy.$$

Bu formula Ostrogradskiy formulasi deb ataladi.

Mashqlar

19.4. Ushbu

$$\iint\limits_{(S)} \left(y + z + \sqrt{a^2 + x^2} \right) ds$$

integral hisoblansin, bunda (S) sirt quyidagi

$$x = a^2 - y^2 - z^2$$

paraboloidning Oyz tekisligi bilan kesishishidan hosil bo'lgan sirtning tashqi qismi.

19.6. Sirtning yuzi

$$S = \iint\limits_{(S)} ds$$

formula bilan topilishi isbotlansin.

Ushbu

$$x^2 + y^2 = R \cdot x$$

silindrning

$$x^2 + y^2 + z^2 = R^2$$

sfera ichida joylashgan qismining yuzi topilsin.

20-BOB

Fure qatorlari

Biz yuqorida, kursimiz davomida, murakkab funksiyalarni ulardan soddaroq bo'lgan funksiyalar orqali ifodalash masalalariga bir necha marta duch keldik va ularni o'rgandik. Bu sohadagi klassik masalalardan biri – funksiyalarni darajali qatorlarga yoyishdan iborat bo'lib, u mazkur kursning 14-bobida batafsil o'rganildi.

Agar qaralayotgan funksiyalar davriy funksiyalar bo'lsa, tabiiyki ularni soddaroq davriy funksiyalar bilan ifodalash lozim bo'ladi. Har bir hadi sodda davriy funksiyalar bo'lgan funksional qatorlarni o'rganish murakkab davriy funksiyalarni soddaroq davriy funksiyalar bilan ifodalash masalasini hal etishda muhim rol o'ynaydi.

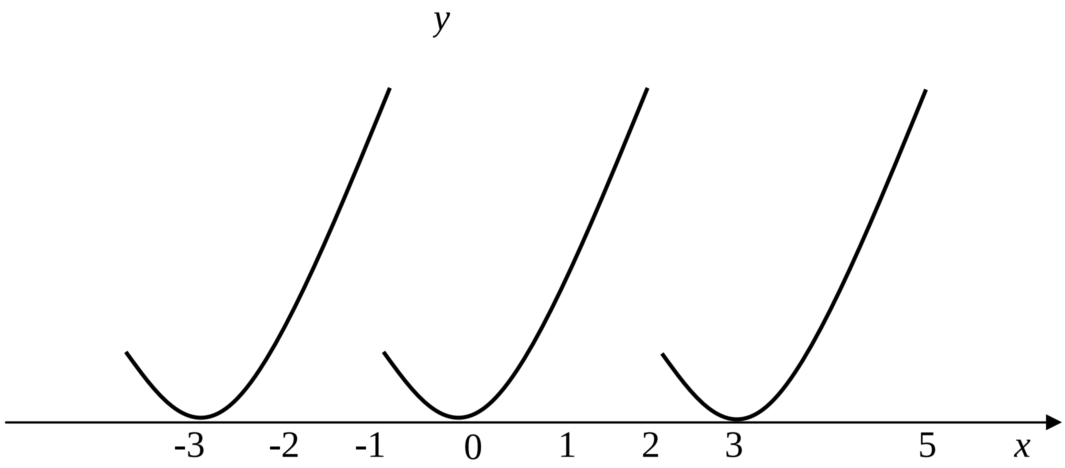
1-§. Ba'zi muhim tushunchalar

1^0. Funksiyalarni davriy davom ettirish. $f(x)$ funksiya $(a, в]$ yarim intervalda berilgan bo'lsin. Bu funksiya yordamida quyidagi

$$f^*(x) = f(x - (в - a)m), \; x \in (a + m(в - a), в + m(в - a)]$$
$$(m = 0, \pm 1, \pm 2, ...) \qquad (20.1)$$

funksiyani tuzamiz. Ravshanki, endi $f^*(x)$ funksiya $(-\infty, +\infty)$ oraliqda berilgan va davriy funksiya bo'ladi. Uning davri $T_0 = в - a$ ga teng. Bajarilgan bu jarayonni funksiyani davriy davom ettirish deyiladi.

Masalan, $(-1, 2]$ oraliqda berilgan $f(x) = x^2$ funksiyani davriy davom ettirishdan hosil bo'lgan funksiyaning grafigi 65-chizmada tasvirlangan.



65-chizma

Agarda berilgan $f(x)$ funksiya $(a,в]$ da uzluksiz funksiya bo'lsa va

$$f(a+0)=\lim_{x\to a+0} f(x)=f(a)\ ,$$

deyilsa, u holda davom ettirilgan $f^*(x)$ funksiya $(-\infty,+\infty)$ da uzluksiz bo'ladi.

$f(x)$ funksiya $[a,в)$ yarim intervalda berilgan bo'lsa uni davriy davom ettirish ham yuqoridagi singari bajariladi:

$$f^*(x)=f(x-(в-a)m),\ x\in[a+m(в-a),в+m(в-a))\ (m=0,\pm1,\pm2,...)$$

Agarda $f(x)$ funksiya $(a,в)$ da berilgan bo'lsa, uni $(-\infty,+\infty)$g' g'$\{a+m(в-a): m=0,\pm1,...\}=X$ to'plamga davriy davom ettirish mumkin:

$$f^*(x)=f(x-(в-a)m),\ x\in(a+m(в-a),в+m(в-a)]\ (m=0,\pm1,\pm2,...)$$

Izoh. $f(x)$ funksiya $[a,в]$ da berilgan bo'lsa, uni $(-\infty,+\infty)$ ga umuman aytganda ikki xil davom ettirish mumkin:

$$f^*(x)=f(x-(в-a)m),\ x\in(a+(в-a)m,в+(в-a)m],$$

$$f^{**}(x)=f(x-(в-a)m),\ x\in[a+(в-a)m,в+(в-a)m)$$

$$(m=0,\pm1,\pm2,...).$$

1-lemma. $f(x)$ funksiya $(a,в]$ oraliqda integrallanuvchi bo'lsin. U holda $f(x)$ ni $(-\infty,+\infty)$ ga davriy davom ettirishdan hosil bo'lgan $f^*(x)$ funksiya ixtiyoriy $(\alpha,\alpha+(в-a)]$ da integrallanuvchi bo'ladi va

$$\int_{\alpha}^{\alpha+(в-a)} f^*(x)dx=\int_a^в f(x)dx \qquad (*)$$

formula o'rinli bo'ladi.

◄ Shartga ko'ra $f(x)$ funksiya $(a,в]$ da integrallanuvchi, $f^*(x)$ funksiyaning tuzilishiga binoan (qaralsin, (20.1)) uning $(\alpha,\alpha+(в-a)]$ $(\forall\alpha\in R)$ da integrallanuvchi bo'lishini topamiz.

Integralning xossasiga ko'ra

$$\int_{\alpha}^{\alpha+(в-a)} f^*(x)dx=\int_{\alpha}^{a} f^*(x)dx+\int_a^в f^*(x)dx+\int_в^{\alpha+(в-a)} f^*(x)dx \qquad (20.2)$$

bo'ladi. Ravshanki, $\forall x \in (a, \text{в}]$ uchun $f^*(x) = f(x)$. Demak,

$$\int_a^{\text{в}} f^*(x)dx = \int_a^{\text{в}} f(x)dx .$$

Endi

$$\int_{\alpha}^{\alpha+(\text{в}-a)} f^*(x)dx$$

integralda $x = y + (\text{в} - a)$ almashtirishni bajaramiz:

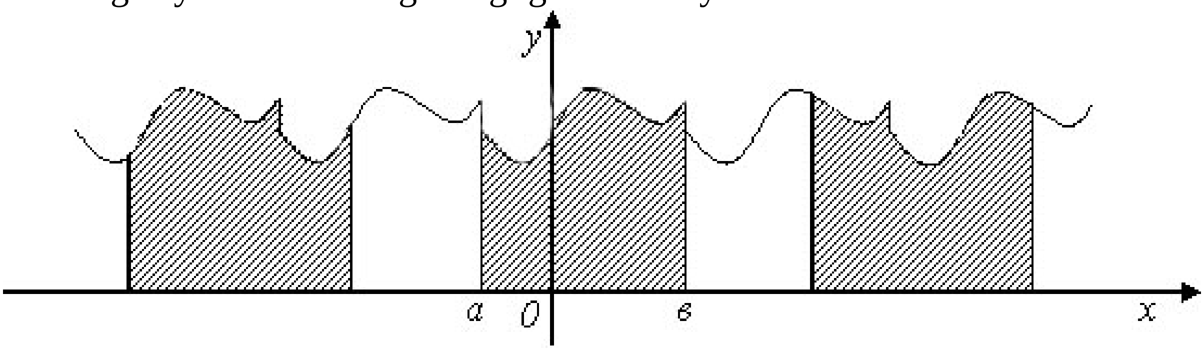
$$\int_{\alpha}^{\alpha+(\text{в}-a)} f^*(x)dx = \int_a^{\alpha} f^*(y + (\text{в} - a))dy = \int_a^{\alpha} f^*(y)dy = -\int_{\alpha}^{a} f^*(y)dy .$$

Natijada (20.2) tenglik ushbu

$$\int_{\alpha}^{\alpha+(\text{в}-a)} f^*(x)dx = \int_a^{\text{в}} f(x)dx$$

ko'rinishga keladi.►

Bu lemmadagi *(*)* formula sodda geometrik ma'noga ega. U 66-chizma shtrixlangan yuzalar bir-biriga tengligini ifodalaydi.



66-chizma

2⁰. Garmonikalar. Ushbu

$$f(x) = A\sin(\alpha x + \beta) \qquad (20.3)$$

funksiyani ko'raylik, bunda A, α, β o'zgarmas sonlar. Bu davriy funksiya bo'lib, uning asosiy davri $T = \dfrac{2\pi}{\alpha}$ ga tengdir. Haqiqatdan ham,

$$f\left(x + \frac{2\pi}{\alpha}\right) = A\sin\left[\alpha\left(x + \frac{2\pi}{\alpha}\right) + \beta\right] = A\sin[(\alpha x + \beta) + 2\pi] = A\sin(\alpha x + \beta) = f(x).$$

Bu

$$f(x) = A\sin(\alpha x + \beta)$$

funksiya garmonika deb ataladi.

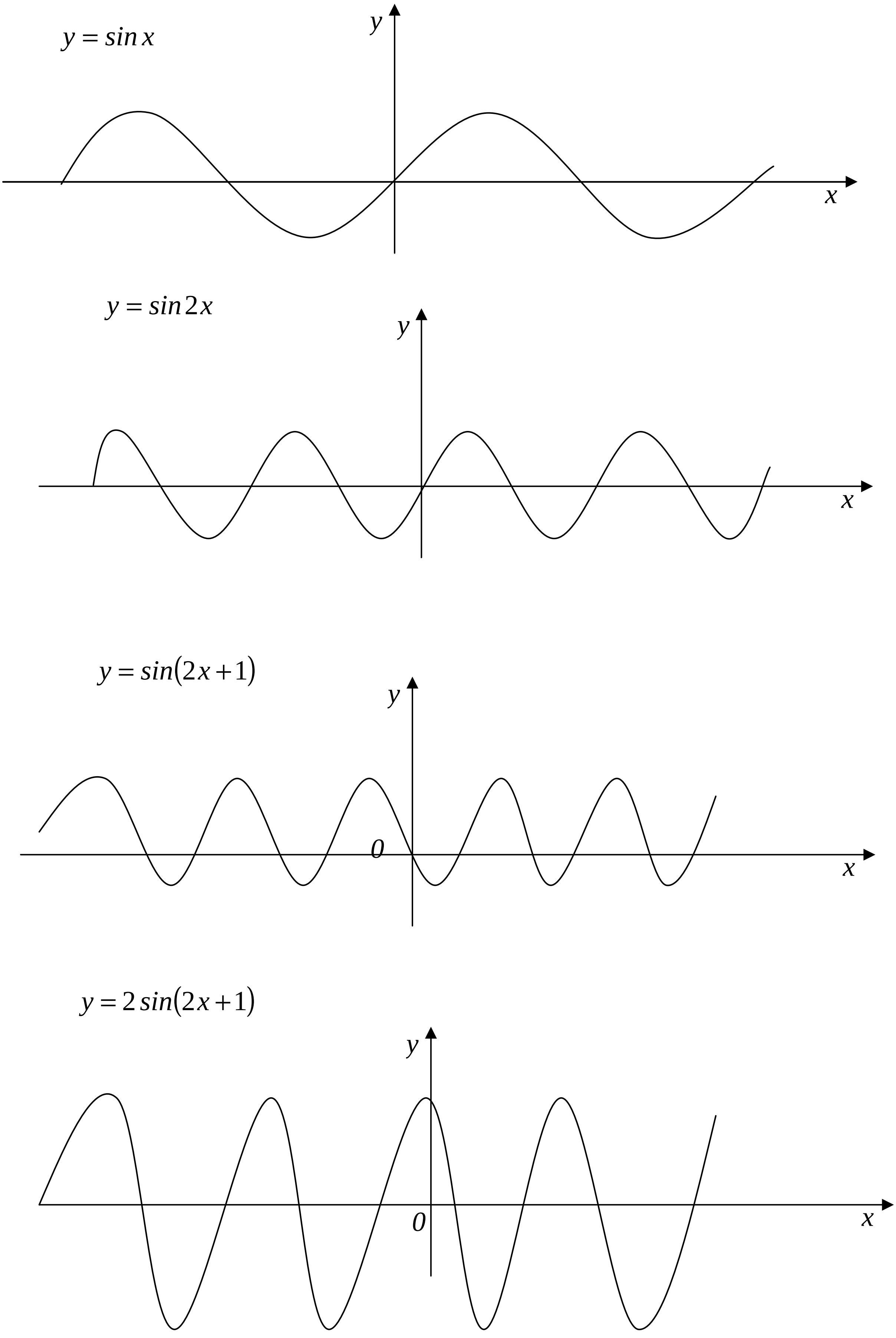
Berilgan

$$f(x) = A\sin(\alpha x + \beta)$$

garmonikaning grafigi, $y = \sin x$ funksiya grafigini Ox va Oy o'qlar bo'yicha siqish (cho'zish) hamda Ox o'qi bo'yicha surish natijasida hosil bo'ladi. Masalan,

$$f(x) = 2\sin(2x + 1)$$

garmonikaning grafigini yasash jarayoni va uning grafigi 67-chizmada tasvirlangan.



67-chizma

Trigonometriyadan ma'lum bo'lgan formuladan foydalanib garmonikani quyidagicha yozish mumkun:

$$f(x) = A sin(\alpha x + \beta) = A(cos\,\alpha x\, sin\,\beta + sin\,\alpha x\, cos\,\beta).$$

Agar

$$A sin\,\beta = a\ ,\ A cos\,\beta = в$$

deb belgilasak, unda garmonika ushbu

$$f(x) = a\, cos\,\alpha x + в\, sin\,\alpha x$$

ko'rinishga keladi.

3^0. Bo'lakli-uzluksizlik va bo'lakli-differensiallanuvchilik. $f(x)$ funksiya $[a, в]$ oraliqda berilgan bo'lsin.

Agar $[a, в]$ oraliqni shunday

$$[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n] \qquad (a_0 = a, a_n = в)$$

bo'laklarga ajratish mumkin bo'lsaki,

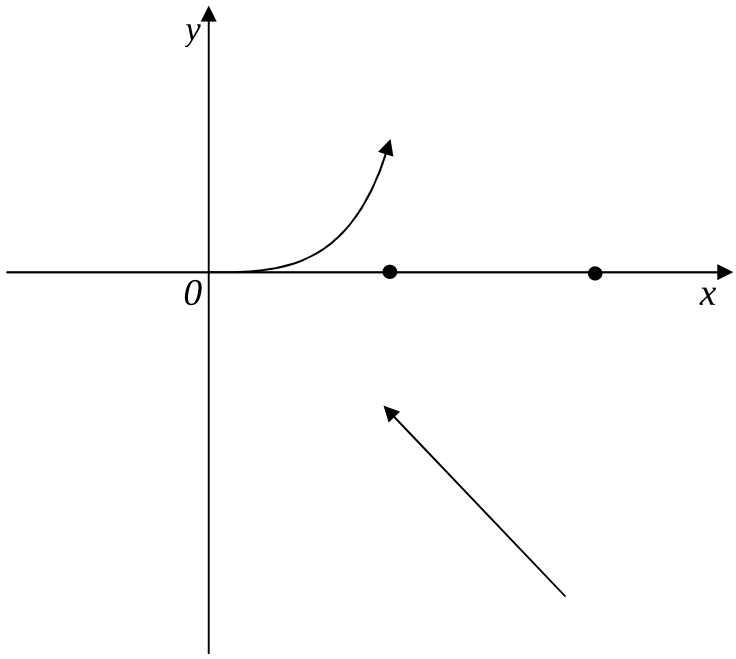
$$([a, в] = [a_0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_{n-1}, a_n])$$

har bir (a_k, a_{k+1}) $(k = 0, 1, \ldots, n-1)$ da $f(x)$ funksiya uzluksiz bo'lsa, hamda $x = a_k$ nuqtalarda chekli o'ng $f(a_k + 0)$ $(k = (0, 1, \ldots, n-1))$ va chap $f(a_k - 0)$ $(k = (0, 1, \ldots, n-1))$ limitlarga ega bo'lsa, u holda $f(x)$ funksiya $[a, в]$ da bo'lakli-uzluksiz deb ataladi.

Masalan, ushbu

$$f(x) = \begin{cases} x^3, & agar\ 0 \le x < 1 \text{ bo'lsa}, \\ 0, & agar\ x = 1 \text{ bo'lsa}, \\ -x, & agar\ 1 < x \le 2 \text{ bo'lsa} \end{cases}$$

funksiya $[0, 2]$ oraliqda bo'lakli-uzluksizdir (68-chizma).

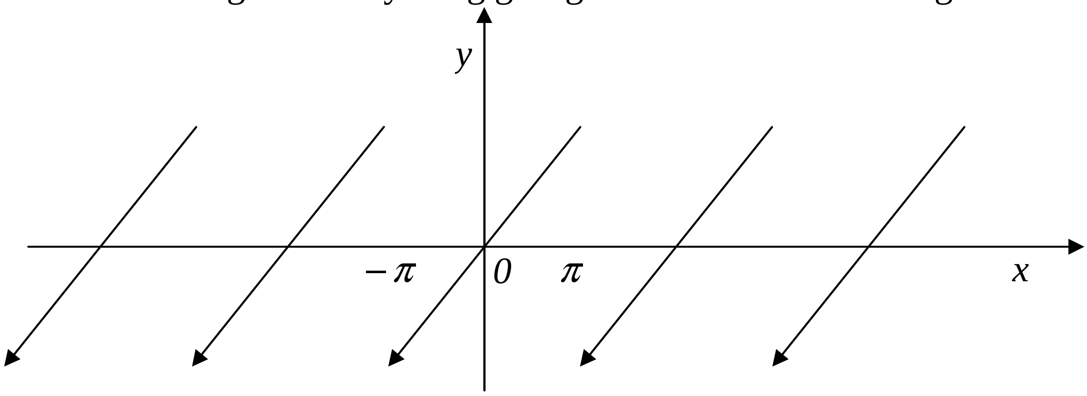


68-chizma

Agar $f(x)$ funksiya $(-\infty,+\infty)$ da berilgan bo'lib, uning istalgan chekli $[\alpha,\beta]$ qismida $([\alpha,\beta]\subset(-\infty,+\infty))$ bo'lakli-uzluksiz bo'lsa, u holda $f(x)$ funksiya $(-\infty,+\infty)$ da bo'lakli-uzluksiz deb ataymiz.

Aytaylik, $f(x)$ funksiya $(a,в]$ da berilgan va bo'lakli-uzluksiz bo'lsin. Bu funksiyani $(-\infty,+\infty)$ ga davriy davom ettirishdan hosil bo'lgan $f^*(x)$ funksiya $(-\infty,+\infty)$ da bo'lakli-uzluksiz bo'ladi.

Masalan $f(x)=x$ $\quad (x\in(-\pi,\pi])$ funksiyani $(-\infty,+\infty)$ ga davriy davom ettirishdan hosil bo'lgan funksiyaning grafigi 69-chizmada tasvirlangan.



69-chizma

Endi bo'lakli-differensiallanuvchanlik tushunchasi bilan tanishamiz.

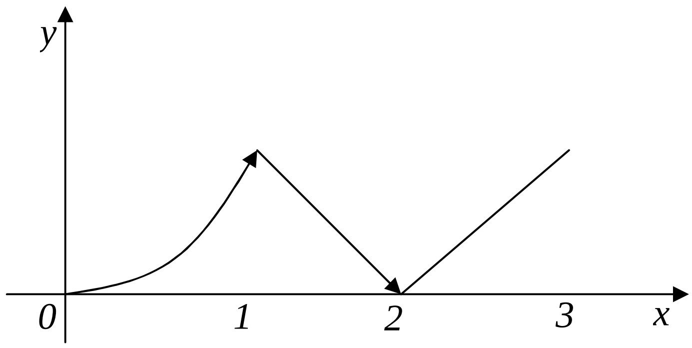
$f(x)$ funksiya $[a,в]$ da berilgan bo'lsin.

Agar $[a,в]$ oraliqni $[a,в]=[a_0,a_1]\cup[a_1,a_2]\cup\ldots\cup[a_{n-1},a_n]$ bo'ladigan shunday $[a_0,a_1],[a_1,a_2],\ldots,[a_{n-1},a_n]$ $(a_0=a,\ a_n=в)$ bo'laklarga ajratish mumkin bo'lsaki, har bir (a_k,a_{k+1}) da $(k=0,1,\ldots,n-1)$ funksiya differensiallanuvchi bo'lsa hamda $x=a_k$ nuqtalarda chekli o'ng $f'(a_k+0)$ $\quad(\mathrm{k}=0,1,\ldots,\mathrm{n}-1)$ va chap $f'(a_k-0)$ $\quad(\mathrm{k}=0,1,\ldots,\mathrm{n}-1)$ hosilalarga ega bo'lsa, u holda $f(x)$ funksiya $[a,в]$ da bo'lakli-differensiallanuvchi deb ataladi.

Masalan, ushbu

$$f(x)=\begin{cases} x^2\ , & \text{agar } 0\le x<1 \ \text{ bo'lsa,}\\ 2-x\,, & \text{agar } 1\le x<2 \ \text{ bo'lsa,}\\ x-2\,, & \text{agar } 2\le x\le 3 \ \text{ bo'lsa}\end{cases}$$

funksiya $[0,3]$ da bo'lakli-differensiallanuvchi bo'ladi. (70-chizma)



70-chizma

Agar $f(x)$ funksiya $(-\infty,+\infty)$ da berilgan bo'lib, uning istalgan chekli $[\alpha,\beta]$ $([\alpha,\beta]\subset(-\infty,+\infty))$ qismida bo'lakli-differensiallanuvchi bo'lsa, u holda $f(x)$ funksiya $(-\infty,+\infty)$ da bo'lakli-differensiallanuvchi deb ataladi.

$f(x)$ funksiya $(a,в]$ da berilgan va bo'lakli-differensiallanuvchi bo'lsa, uni $(-\infty,+\infty)$ ga davriy davom ettirishdan hosil bo'lgan $f^*(x)$ funksiya $(-\infty,+\infty)$ da bo'lakli-differensiallanuvchi bo'ladi.

$f(x)$ funksiya $[a,в]$ da berilgan bo'lsin. Agar $[a,в]$ oraliqni $[a,в]=[a_0,a_1]\cup[a_1,a_2]\cup...\cup[a_{n-1},a_n]$ bo'ladigan shunday $[a_0,a_1],[a_1,a_2],...,[a_{n-1},a_n]$ $(a_0=a,\ a_n=в)$ bo'laklarga ajratish mumkin bo'lsaki, har bir $[a_k,a_{k+1})$ da $(k=0,1,2,...,n-1)$ funksiya $f'(x)$ hosilaga ega va bu hosila uzluksiz bo'lsa hamda $x=a_k$ nuqtalarda chekli o'ng $f'(a_k+0)$ $(k=0,1,2,...,n-1)$ va chap $f'(a_k-0)$ $(k=0,1,2,...,n-1)$ hosilalarga ega bo'lsa, u holda $f(x)$ funksiya $[a,в]$ da bo'lakli-silliq deb ataladi.

2-§. Fure qatorining ta'rifi

Har bir hadi

$$u_n(x)=a_n\cos nx+в_n\sin nx\qquad(\text{n}=0,1,2,...)$$

garmonikadan iborat ushbu

$$a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\qquad(20.4)$$

funksional qatorni qaraylik.

Odatda (20.4) qator trigonometrik qator deb ataladi $a_0,a_1,в_1,a_2,в_2,...$ sonlar esa trigonometrik qatorning koeffitsientlari deyiladi.

Shunday qilib, trigonometrik qator garchand funksional qator bo'lsa ham (uning har bir hadi muayyan funksiyalar bo'lganligi uchun) o'z koeffitsientlari $a_0,a_1,в_1,a_2,в_2,...,a_n,в_n,...$ lar bilan to'la aniqlanadi.

(20.4) trigonometrik qatorning qismiy yig'indisi

$$T_n(x)=a_0+\sum_{k=1}^{n}(a_k\cos kx+в_k\sin kx)$$

trigonometrik ko'phad deb ataladi.

1⁰. Fure qatorining ta'rifi. $f(x)$ funksiya $[-\pi,\pi]$ da berilgan va shu oraliqda integrallanuvchi bo'lsin. U holda

$$f(x)\cos nx,\ f(x)\sin nx\quad(n=1,2,3,...)$$

funksiyalar ham, ikkita integrallanuvchi funksiyalar ko'paytmasi sifatida (qaralsin 1-qism, 9-bob, 7-§) $[-\pi,\pi]$ da integrallanuvchi bo'ladi. Bu funksiyalar integrallarini hisoblab, ularni quyidagicha belgilaylik:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \qquad (n=1,2,3,...), \qquad (20.5)$$

$$в_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx \qquad (n=1,2,3,...)$$

Bu sonlardan foydalanib ushbu

$$T(f;x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+в_n\sin nx) \qquad (20.6)$$

trigonometrik qatorni tuzamiz.

1-ta'rif. $a_0,a_1,b_1,a_2,в_2,...$ koeffitsientlari (20.5) formulalar bilan aniqlangan (20.6) trigonometrik qator $f(x)$ funksiyaning Fure qatori deb ataladi. $a_0,a_1,в_1,a_2,в_2,...$ sonlar esa $f(x)$ funksiyaning Fure koeffitsientlari deyiladi.

Demak, berilgan funksiyaning Fure qatori shunday trigonometrik qatorki, uning koeffitsientlari shu funksiyaga bog'liq bo'lib, (20.5) formulalar bilan aniqlanadi. Shu sababli (20.6) qatorni (uning yaqinlashuvchi yoki uzoqlashuvchi bo'lishidan qat'iy nazar) ushbu "~" belgi bilan quyidagicha yoziladi:

$$f(x)\sim T(f;x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+в_n\sin nx)$$

20.1-misol. Ushbu

$$f(x)=e^{\alpha x} \qquad (-\pi\le x\le\pi\ ,\ \alpha\ne 0)$$

funksiyaning Fure qatori topilsin.

◀(20.5) formuladan foydalanib bu funksiyaning Fure koeffitsientlarini topamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}dx=\frac{1}{\alpha\pi}\left(e^{\alpha\pi}-e^{-\alpha\pi}\right)=\frac{2}{\alpha\pi}sh\alpha\pi,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\cos nxdx=\frac{1}{\pi}\frac{\alpha\cos nx+n\sin nx}{\alpha^2+n^2}e^{\alpha x}\Big|_{-\pi}^{\pi}=(-1)^n\frac{1}{\pi}\frac{2\alpha}{\alpha^2+n^2}sh\alpha\pi,$$

$$в_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{\alpha x}\sin nxdx=\frac{1}{\pi}\frac{\alpha\sin nx+n\cos nx}{\alpha^2+n^2}e^{\alpha x}\Big|_{-\pi}^{\pi}=(-1)^{n-1}\frac{1}{\pi}\frac{2n}{\alpha^2+n^2}sh\alpha\pi,$$

$$(\text{n}=1,2,3,...)$$

(qarang, 1-qism, 8-bob, 2-§).

Demak, berilgan funksiyaning Fure qatori

$$e^{\alpha x}\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+в_n\sin nx)=$$

$$=\frac{2sh\alpha\pi}{\pi}\left[\frac{1}{2\alpha}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2}(\alpha\cos nx-n\sin nx)\right]$$

bo'ladi.▶

2^0. Juft va toq funksiyalarning Fure qatorlari. Juft va toq funksiyalarning Fure qatorlari birmuncha sodda ko'rinishga ega bo'ladi. Biz quyida ularni keltiramiz.

$f(x)$ funksiya $[-\pi,\pi]$ da berilgan juft funksiya bo'lsin. U shu $[-\pi,\pi]$ oraliqda integrallanuvchi bo'lsin. Ravshanki, bu holda $f(x)\cos nx$ juft funksiya, $f(x)\sin nx$ $(n=1,2,3,...)$ esa toq funksiya bo'ladi va ular $[-\pi,\pi]$ da integrallanuvchi bo'ladi.

$f(x)$ funksiyaning Fure koeffitsientlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]=$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx \qquad (n=0,1,2,3,...),$$

$$в_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=0 \qquad (n=1,2,3,...).$$

Demak, juft $f(x)$ funksiyaning Fure koeffitsientlari

$$a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx \qquad (n=0,1,2,...)$$
$$в_n=0 \qquad (n=1,2,3,...) \tag{20.7}$$

bo'lib, Fure qatori esa

$$f(x)\sim T(f;x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

bo'ladi.

Endi $f(x)$ funksiya $[-\pi,\pi]$ da berilgan toq funksiya bo'lsin va u shu $[-\pi,\pi]$ oraliqda integrallanuvchi bo'lsin. Bu holda $f(x)\cos nx$ toq funksiya, $f(x)\sin nx$ $(n=1,2,...)$ esa juft funksiya bo'ladi.

Funksiyaning Fure koeffitsientlarini topamiz:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]=$$

$$=\frac{1}{\pi}\left[-\int_{0}^{\pi}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right]=0 \qquad (n=0,1,2,...),$$

$$в_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\sin nxdx+\int_{0}^{\pi}f(x)\sin nxdx\right]=$$

$$=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nxdx \qquad (n=1,2,...).$$

Demak, toq $f(x)$ funksiyaning Fure koeffitsientlari

$$a_n = 0 \qquad (n = 0,1,2,...)$$
$$в_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \qquad (n = 0,1,2,...) \tag{20.8}$$
bo’lib, Fure qatori esa
$$f(x) \sim T(f;x) = \sum_{n=1}^{\infty} в_n \sin nx$$
bo’ladi.

20.2-misol. $f(x) = x^2 \quad (-\pi \le x \le \pi)$ funksiyaning Fure qatori yozilsin.

◀ (20.7) formulalardan foydalanib berilgan funksiyaning Fure koeffitsientlarini topamiz:
$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2,$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi} x^2 \frac{\sin nx}{n}\bigg|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin nx dx =$$
$$= -\frac{4}{n\pi}\left[\left(-x\frac{\cos nx}{n}\right)\bigg|_0^{\pi} + \int_0^{\pi}\cos nx dx\right] = (-1)^n \frac{4}{n^2} \qquad (n = 1,2,3,...).$$
Demak $f(x) = x^2$ juft funksiyaning Fure qatori ushbu
$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos nx = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ...\right)$$
ko’rinishda bo’ladi. ▶

20.3-misol. Ushbu
$$f(x) = x \quad [-\pi \le x \le \pi]$$
toq funksiyaning Fure qatori yozilsin.

◀ (20.8) formulalardan foydalanib berilgan funksiyaning Fure koeffitsientlarini topamiz:
$$в_n = \frac{\pi}{2}\int_0^{\pi} x \sin nx dx = -\frac{2}{n\pi} x \cos nx\bigg|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos nx dx =$$
$$= -\frac{2}{4}\cos nx = (-1)^{n+1}\frac{2}{n} \qquad (n = 1,2,3,...).$$
Demak, $f(x) = x$ toq funksiyaning Fure qatori quyidagicha bo’ladi:
$$x \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right)$$ ▶

3^0. $[-l,l]$ oraliqda berilgan funksiyaning Fure qatori. $f(x)$ funksiya $[-l,l]$ $(l > 0)$ da berilgan va shu oraliqda integrallanuvchi bo’lsin.

Ravshanki, ushbu
$$t = \frac{\pi}{l}x \tag{20.9}$$
almashtirish $[-l,l]$ oraliqni $[-\pi,\pi]$ oraliqqa o’tkazadi. Agar

$$f(x) = f\left(\frac{l}{\pi}t\right) = \varphi(t)$$

deyilsa, $\varphi(t)$ funksiyani $[-\pi,\pi]$ da berilgan va shu oraliqda integrallanuvchi bo'lishini ko'rish qiyin emas. Bu $\varphi(t)$ funksiyaning Fure qatori quyidagicha bo'ladi:

$$\varphi(t) \sim T(\varphi;t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + в_n \sin nt)$$

bunda,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\cos nt dt \quad (n = 0,1,2,...) \ , \quad в_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\sin nt dt \ \ (n = 1,2,3,...).$$

Yuqoridagi (20.9) tenglikni e'tiborga olsak, unda

$$\varphi\left(\frac{\pi}{l}x\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\frac{\pi}{l}x + в_n \sin n\frac{\pi}{l}x\right)$$

bo'lib, uning koeffitsientlari esa

$$a_n = \frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\cos n\frac{\pi}{l}xdx \quad (n = 0,1,2,...),$$

$$в_n = \frac{1}{l}\int_{-l}^{l}\varphi\left(\frac{\pi}{l}x\right)\sin n\frac{\pi}{l}xdx \quad (n = 1,2,3,...)$$

bo'ladi.

Natijada

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + в_n \sin\frac{n\pi x}{l}\right) \qquad (20.10)$$

ga ega bo'lamiz, bunda

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \quad (n = 0,1,2,...),$$

$$в_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \quad (n = 1,2,3,...) \qquad (20.11)$$

(20.10) ning o'ng tomonidagi trigonometrik qatorni $[-l,l]$ da berilgan $f(x)$ ning Fure qatori deyiladi, (20.11) Fure koeffitsientlari deyiladi.

20.4-misol. Ushbu

$$f(x) = e^x \qquad (-1 \le x \le 1)$$

funksiyaning Fure qatori yozilsin.

◀ (20.11) formulalardan foydalanib, berilgan funksiyaning Fure koeffitsientlarini topamiz:
(bunda $l = 1$)

$$a_0 = \int_{-1}^{1} e^x dx = e - e^{-1},$$

$$a_n = \int_{-1}^{1} e^x \cos n\pi x dx = \frac{n\pi \sin n\pi\, x + \cos n\pi\, x}{1+n^2\pi^2} e^x \Bigg|_{-1}^{1} =$$

$$= \frac{1}{1+n^2\pi^2}\left(e \cos nx - e^{-1} \cos n\pi\right) = (-1)^n \frac{e-e^{-1}}{1+n^2\pi^2} \quad (n = 1,2,...)$$

$$в_n = \int_{-1}^{1} e^x \sin n\pi\pi x d = \frac{\sin n\pi\, x - n\pi \cos n\pi\, x}{1+n^2\pi^2} e^x \Bigg|_{-1}^{1} =$$

$$= \frac{1}{1+n^2\pi^2}\left(en\pi \cos n\pi + e^{-1} n\pi \cos n\pi\right) = \frac{n\pi \cos n\pi}{1+n^2\pi^2}\left(e^{-1} - e\right) =$$

$$= \frac{n\pi(-1)^n}{1+n^2\pi^2}\left(e^{-1} - e\right) = (-1)^{n+1} \frac{e-e^{-1}}{1+n^2\pi^2} n\pi \quad (n = 1,2,...)$$

Demak, $f(x) = e^x$ $(-1 \le x \le 1)$ funksiyaning Fure qatori ushbu

$$e^x \sim \frac{e-e^{-1}}{2} + \left(e - e^{-1}\right)\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1+n^2\pi^2} \cos n\pi x + \frac{(-1)^{n+1}}{1+n^2\pi^2} n\pi \sin n\pi x\right]$$

ko’rinishda bo’ladi.►

Izoh. Ushbu

$$T(f; x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + в_n \sin nx\right)$$

trigonometrik qatorning $(-\infty, +\infty)$ da berilgan 2π davrli funksiya ekanligini ko’rish qiyin emas:

$$T(f; x + 2\pi) = T(f; x).$$

Agar $[-\pi, \pi]$ da berilgan $f(x)$ funksiyani $(-\infty, +\infty)$ ga davriy davom ettirsak (qarang, ushbu bobning 1-§)

$$f^*(x) = f(x - 2\pi m), \quad x \in (-\pi + 2\pi m, \pi + 2\pi m) \quad (m = 0, \pm 1, \pm 2, ...)$$

u holda, ravshanki, $(-\infty, +\infty)$ da

$$f^*(x) \sim T\left(f^*, x\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + в_n \sin nx\right)$$

bo’ladi.

3-§. Lemmalar. Dirixle integrali

Funksiyalarni Fure qatoriga yoyish shartlarini aniqlash, yuqorida aytib o’tganimizdek, Fure qatorlari nazariyasining muhim masalalaridan biri. Uni hal etuvchi teoremani keltirishdan avval ba’zi bir faktlarni o’rganamiz.

1^0. Lemmalar. Quyida keltirilgan lemmalar Fure qatorlari nazariyasida muhim rol o’ynaydi.

2-lemma. $[a, в]$ oraliqda berilgan va integrallanuvchi ixtiyoriy $\varphi(x)$ funksiya uchun

$$\lim_{P\to\infty}\int_a^в \varphi(x)\sin pxdx = 0, \qquad (20.12)$$

$$\lim_{P\to\infty}\int_a^в \varphi(x)\cos pxdx = 0, \qquad (20.13)$$

bo'ladi.

◄ $[a, в]$ oraliqda biror

$$P = \{x_0, x_1, x_2, ..., x_n\} \qquad (a = x_0 < x_1 < x_2 < ... < x_n = в)$$

bo'laklashni olaylik. Integralning xossasiga ko'ra

$$\int_a^в \varphi(x)\sin pxdx = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} \varphi(x)\sin pxdx \qquad (20.14)$$

bo'ladi. $\varphi(x)$ funksiya $[a, в]$ da chegaralangan. Demak,

$$\inf\{\varphi(x): x \in [x_k, x_{k+1}]\} \quad (k = 0,1,2,...,n\text{-}1)$$

mavjud. Uni m_k bilan belgilaymiz;

$$m_k = \inf\{\varphi(x): x \in [x_k, x_{k+1}]\} \quad (k = 0,1,2,...,n\text{-}1)$$

Endi (20.14) integralni

$$\int_a^в \varphi(x)\sin pxdx = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} \varphi(x)\sin pxdx =$$
$$= \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} [\varphi(x) - m_k]\sin pxdx + \sum_{k=0}^{n-1} m_k \int_{x_k}^{x_{k+1}} \sin pxdx = S_1 + S_2 \qquad (20.15)$$

ko'rinishda yozib, so'ngra har bir

$$S_1 = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} [\varphi(x) - m_k]\sin pxdx,$$

$$S_2 = \sum_{k=0}^{n-1} m_k \int_{x_k}^{x_{k+1}} \sin pxdx$$

qo'shiluvchini baholaymiz.

Agar ω_k $\varphi(x)$ funksiyaning $[x_k, x_{k+1}]$ $(k = 0, 1, 2, ..., n-1)$ dagi tebranishi bo'lsa, S_1 uchun ushbu

$$|S_1| \le \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} \omega_k dx \sum_{k=0}^{n-1} \omega_k \Delta x_k \qquad (\Delta x_k = x_{k+1} - x_k) \qquad (20.16)$$

tengsizlikka ega bo'lamiz. Shartga ko'ra $\varphi(x)$ funksiya $[a, в]$ da integrallanuvchi. Unda 1-qism, 9-bob, 5-§ da keltirilgan teoremaga asosan, $\forall \varepsilon > 0$ olinganda ham, shunday $\delta > 0$ topiladiki, $[a, в]$ oraliqning diametri $\lambda_p < \delta$ bo'lgan har qanday P bo'laklashi uchun

$$\sum_{k=0}^{n-1} \omega_k \Delta x_k < \frac{\varepsilon}{2} \qquad (20.17)$$

bo'ladi. (20.16) va (20.17) munosabatlardan

$$|S_1| < \frac{\varepsilon}{2} \qquad (20.18)$$

bo'lishi kelib chiqadi.

Endi $S_2 = \sum_{k=0}^{n-1} m_k \int_{x_k}^{x_{k+1}} sin\, pxdx$ yig'indini baholaymiz. Ravshanki,

$$\left|\int_{x_k}^{x_{k+1}} sin\, pxdx\right| = \left|\frac{cos\, px_k - cos\, px_{k+1}}{p}\right| \le \frac{2}{p}$$

Demak, $|S_2| \le \frac{2}{p}\sum_{k=0}^{n-1}|m_k|$ bo'ladi. p ni etarli katta qilib olish hisobiga

$$\frac{2}{p}\sum_{k=0}^{n-1}|m_k| < \frac{\varepsilon}{2} \qquad (20.19)$$

bo'ladi. Natijada (20.15), (20.18) va (20.19) munosabatlardan etarli katta p lar uchun $\left|\int_a^b \varphi(x) sin\, pxdx\right| < \varepsilon$ bo'lishi kelib chiqadi. Demak,

$$\lim_{p\to\infty}\int_a^b \varphi(x) sin\, pxdx = 0$$

(20.13) munosabatning o'rinli bo'lishi xuddi shunga o'xshash ko'rsatiladi. ►

Xususan, $\varphi(x)$ funksiya $[a, b]$ oraliqda bo'lakli-uzluksiz bo'lsa, uning uchun lemmaning tasdig'i o'rinli bo'ladi.

1-eslatma. Lemmadagi

$$J(p) = \int_a^b \varphi(x) sin\, pxdx, \quad J_1(p) = \int_a^b \varphi(x) cos\, pxdx$$

integrallar, ravshanki, parametrga (p – parametr) bog'liq integrallardir. Mazkur kursning 17-bob, 5-§ ida biz bunday integrallarning limitini integral belgisi ostida limitga o'tib hisoblash haqidagi teoremani isbot qilgan edik. Bu teorema shartlari yuqoridagi integrallar uchun bajarilmaydi ($p \to \infty$ da integral ostidagi funksiyaning limiti mavjud emas) va demak, undan foydalana olmaymiz. Shuning uchun ham 2-lemma yuqorida alohida isbotlandi. Ikkinchi tomondan lemma parametrga bog'liq integrallarning limitini bevosita, integral belgisi ostida limitga o'tmasdan ham, hisoblash mumkin ekanligiga misol bo'ladi.

Yuqoridagi lemma chegaralanmagan funksiyaning hosmas integrali uchun ham umumlashtirilishi mumkin.

$\varphi(x)$ funksiya $[a, b)$ yarim integralda berilgan, b nuqta shu funksiyaning maxsus nuqtasi bo'lsin.

3-lemma. $[a, b)$ da absolyut integrallanuvchi ixtiyoriy $\varphi(x)$ funksiya uchun

$$\lim_{p\to\infty}\int_a^b \varphi(x) sin\, pxdx = 0,$$

$$\lim_{p\to\infty}\int_a^b \varphi(x) cos\, pxdx = 0 \qquad (20.20)$$

bo'ladi.

◄ Ixtiyoriy η $(0 < \eta < b - a)$ olib

$$\int_a^b \varphi(x) sin\, pxdx$$

integralni quyidagicha yozib

$$\int_a^в \varphi(x)\sin pxdx = \int_a^{в-\eta} \varphi(x)\sin pxdx + \int_{в-\eta}^в \varphi(x)\sin pxdx \qquad (20.21)$$

bu tenglikning o'ng tomonidagi har bir qo'shiluvchini baholaymiz.

Qaralayotgan $\varphi(x)$ funksiya $[a, в-\eta)$ da integrallanuvchi bo'lganligi sababli yuqorida keltirilgan 2-lemmaga ko'ra

$$\lim_{p\to\infty} \int_a^{в-\eta} \varphi(x)\sin pxdx = 0$$

bo'ladi. Demak, $\forall\varepsilon > 0$ olganda ham, shunday $p_0 > 0$ topiladiki, barcha $p > p_0$ uchun

$$\left|\int_a^{в-\eta} \varphi(x)\sin pxdx\right| < \frac{\varepsilon}{2} \qquad (20.22)$$

bo'ladi.

Shunga ko'ra $\varphi(x)$ funksiya $[a, в)$ da absolyut integrallanuvchi. Ta'rifga binoan $\forall\varepsilon > 0$ olganda ham shunday $\delta > 0$ topiladiki, $0 < \eta < \delta$ bo'lganda $\int_{в-\eta}^в |\varphi(x)|dx < \frac{\varepsilon}{2}$ bo'ladi. Demak,

$$\left|\int_{в-\eta}^в \sin px\varphi(x)dx\right| \le \int_{в-\eta}^в |\varphi(x)|dx < \frac{\varepsilon}{2} \qquad (20.23)$$

Yuqoridagi (20.21), (20.22) va (20.23) munosabatlardan etarli katta p lar uchun $\left|\int_a^в \varphi(x)\sin pxdx\right| < \varepsilon$ bo'lishi kelib chiqadi. Demak, $\lim_{p\to\infty}\int_a^в \varphi(x)\sin pxdx = 0$

(20.20) munosabatning o'rinli bo'lishi xuddi shunga o'xshash ko'rsatiladi. ►

Isbot etilgan lemmalardan muhim natija kelib chiqadi.

1-natija. $[-\pi, \pi)$ oraliqda bo'lakli-uzluksiz yoki shu oraliqda absolyut integrallanuvchi $f(x)$ funksiyaning Fure koeffitsientlari $n \to \infty$ da nolga intiladi:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = 0,$$

$$\lim_{n\to\infty} в_n = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0,$$

2⁰. Dirixle integrali. Fure qatorining yaqinlashuvchi ekanini o'rganish, bu qator qismiy yig'indilari ketma-ketligining limitini aniqlash demakdir. Shu maqsadda qator qismiy yig'indisini qulay ko'rinishda yozib olamiz.

$f(x)$ funksiya $[-\pi, \pi)$ oraliqda berilgan va absolyut integrallanuvchi (xos yoki xosmas ma'noda) bo'lsin. Bu funksiyaning Fure koeffitsientlarini topib,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt \qquad (k = 0, 1, 2, ...),$$

$$в_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt \qquad (k = 1, 2, ...)$$

so'ngra topilgan koeffitsientlar bo'yicha $f(x)$ funksiyaning Fure qatorini tuzamiz:

$$f(x) \sim T(f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + в_k \sin kx).$$

Endi bu qatorning ushbu

$$F_n(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+в_k\sin kx)$$

qismiy yig'indisini olamiz. Bu yig'indidagi a_k $(k=0,1,2,...)$ va $в_k$ $(k=1,2,...)$ larning o'rniga ularning ifodalarini qo'ysak, u holda

$$F_n(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt+\sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)[\cos kx\cos kt+\sin kt\sin kx]dt.$$

Ma'lumki,

$$\cos kt\cos kx+\sin kt\sin kx=\cos k(t-x).$$

Demak,

$$F_n(f;x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)\right]dt.$$

Integral ostidagi ifoda uchun quyidagi munosabat o'rinli:

$$\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)=\frac{\sin(2n-1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}.$$

Xaqiqatdan ham,

$$2\sin\frac{u}{2}\left[\frac{1}{2}+\sum_{k=1}^{n}\cos ku\right]=\sin\frac{u}{2}+\sum_{k=1}^{n}2\sin\frac{u}{2}\cos ku=$$

$$=\sin\frac{u}{2}+\sum_{k=1}^{n}\left[\sin\left(k+\frac{1}{2}\right)u-\sin\left(k-\frac{1}{2}\right)u\right]=\sin\left(n+\frac{1}{2}\right)u.$$

$$(u=t-x)$$

Bu tenglik yordamida $F_n(f;x)$ yig'indi quyidagicha ifodalanadi:

$$F_n(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{\sin\frac{t-x}{2}}dt \qquad (20.24)$$

(20.24) tenglikning o'ng tomonidagi integral $f(x)$ funksiyaning Dirixle integrali deb ataladi.

Shunday qilib $f(x)$ funksiya Fure qatorining qismiy yig'indisi $F_n(f;x)$ parametrga bog'liq (20.24) ko'rinishidagi integral (Dirixle integrali) dan iborat ekan.

$f^*(x)$ funksiya $f(x)$ funksiyaning $(-\infty,+\infty)$ ga davriy davomi bo'lsin. Binobarin $f^*(x)$ funksiya $(-\infty,+\infty)$ da berilgan, 2π davrli $[-\pi,+\pi)$ da absolyut integrallanuvchi funksiyadir. Qulaylik uchun biz quyida $f(x)$ funksiyaning o'zini $(-\infty,+\infty)$ da berilgan, 2π davrli, $[-\pi,+\pi)$ da absolyut integrallanuvchi funksiya deb hisoblaymiz va $f^*(x)$ o'rniga $f(x)$ yozib ketaveramiz.

Endi,

$$F_n(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{\sin\frac{t-x}{2}}dt$$

integralda $t = x+u$ almashtirish qilamiz. Integral ostidagi funksiya davrli funksiya bo'lganligi sababli, bu almashtirish natijasida integrallash chegarasi o'zgarmasdan qoladi (ushbu bobning 1-§ iga qaralsin).

Natija

$$F_n(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{\sin\frac{u}{2}}du$$

bo'ladi. Bu integralni ushbu

$$F_n(f;x)=\frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{\sin\frac{u}{2}}du+\int_{0}^{\pi} f(x+u)\frac{\sin(2n+1)\frac{u}{2}}{\sin\frac{u}{2}}du\right]$$

ikki qismga ajratib, o'ng tomondagi birinchi integralda u o'zgaruvchini $-u$ ga almashtiramiz. U holda

$$F_n(f;x)=\frac{1}{\pi}\int_{0}^{\pi}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du \qquad (20.25)$$

bo'ladi. Dirixle integrali $F_n(f;x)$ ning bu ko'rinishidan kelgusida foydalaniladi.

Xususan, $f(x)\equiv 1$ bo'lsa, (20.25) munosabatdan

$$1=\frac{2}{\pi}\int_{0}^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du \qquad (n=1,2,3,...) \qquad (20.26)$$

bo'lishi kelib chiqadi. Haqiqatdan ham, bu holda

$$a_0=2,\ a_k=в_k=0 \quad (k=1,2,3,...)$$

bo'lib,

$$F_n(f;x)\equiv 1$$

bo'ladi.

4-§. Fure qatorining yaqinlashuvchiligi

Endi berilgan $f(x)$ funksiya qanday shartlarni bajarganda, uning Fure qatori yaqinlashuvchi bo'lishini topish bilan shug'ullanamiz.

1^0. Lokallashtirish prinsipi. Yuqorida keltirilgan Dirixle integrali

$$F_n(f;x)=\frac{1}{\pi}\int_0^{\pi}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du$$

quyidagi muhim xossaga ega. Ixtiyoriy δ $(0<\delta<\pi)$ sonni olib, (20.25) integralni ikki qismga ajratamiz:

$$F_n(f;x)=\frac{1}{\pi}\int_0^{\delta}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du+$$

$$+\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du=J_1(n,\delta)+J_2(n,\delta).$$

O'ng tomonidagi ikkinchi

$$J_2(n,\delta)=\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du$$

integralning $n\to\infty$ da limiti mavjud va nolga teng. Haqiqatdan ham, berilgan $f(x)$ funksiya $[-\pi,\pi)$ da va demak $[\delta,\pi)$ da absolyut integrallanuvchi bo'lganligidan

$$\varphi(u)=\frac{1}{2\sin\frac{u}{2}}[f(x+u)+f(x-u)]$$

funksiya ham shu oraliqda absolyut integrallanuvchi bo'ladi ($[\delta,\pi)$ da $\sin\frac{u}{2}$ funksiya chegaralangan) va 3-lemmaga asosan

$$\lim_{n\to\infty}J_2(n,\delta)=\lim_{n\to\infty}\int_{\delta}^{\pi}\varphi(u)\sin\left(n+\frac{1}{2}\right)udu=0$$

Natijada quyidagi teoremaga kelamiz.

1-teorema. Ushbu

$$J_1(n,\delta)=\frac{1}{\pi}\int_0^{\delta}[f(x+u)+f(x-u)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du$$

integralning $n\to\infty$ dagi limiti mavjud bo'lgandagina Dirixle integralining $n\to\infty$ dagi limiti mavjud bo'ladi va

$$\lim_{n\to\infty}F_n(f;x)=\lim_{n\to\infty}J_1(n,\delta)$$

Ravshanki, $J_1(n,\delta)$ integralda f funksiyaning $[x-\delta, x+\delta]$ oraliqdagi qiymatlarigina qatnashadi.

Shunday qilib, berilgan $f(x)$ funksiya Fure qatorining x nuqtada yaqinlashuvchi yoki uzoqlashuvchi bo'lishi bu funksiyaning shu nuqta $(x-\delta,\, x+\delta)$ atrofidaga qiymatlarigagina bog'liq bo'lar ekan. Shuning uchun keltirilgan teorema lokallashtirish prinsipi deb yuritiladi.

Uning mohiyatini quyidagicha ham tushintirish mumkin.

Ikkita turli 2π davrli $f(x)$ va $\varphi(x)$ funksiyalarning har biri $[-\pi,+\pi)$ da absolyut integrallanuvchi bo'lsin. Ravshanki, bu funksiyalarning Fure qatorlari ham, umuman aytganda, turlicha bo'ladi. Biror $x_0 \in (-\pi,\pi)$ va δ $(0<\delta<\pi)$ uchun

$$f(x)=\varphi(x), \text{ agar } x\in[x_0-\delta, x_0+\delta],$$
$$f(x)\neq\varphi(x), \text{ agar } x\in[-\pi,\pi)\setminus [x_0-\delta, x_0+\delta]$$

bo'lsa, u holda $n\to\infty$ da bu funksiyalar Fure qatorlari qismiy yig'indilarining x_0 nuqtadagi limitlari yoki bir vaqtda mavjud (bu holda ular bir-biriga teng) bo'ladi, yoki ular bir vaqtda mavjud bo'lmaydi.

Pirovardida, o'quvchilarimiz e'tiborini lokallashtirish prinsipining yana bir muhim tomoniga jalb qilaylik.

Keltirilgan teoremadan $J_1(n,\delta)$ integralning $n\to\infty$ dagi limiti barcha δ $(0<\delta<\pi)$ lar uchun bir vaqtda yoki mavjud bo'lishi, yoki mavjud bo'lmasligi kelib chiqadi.

2^0. Fure qatorining yaqinlashuvchiligi.

2-teorema. 2π davrli $f(x)$ funksiya $[-\pi,\pi)$ oraliqda bo'lakli-differensiallanuvchi funksiya bo'lsa, u holda bu funksiyaning Fure qatori

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+в_k\sin kx)$$

$[-\pi,\pi)$ da yaqinlashuvchi bo'ladi. Uning yig'indisi

$$T(f;x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+в_k\sin kx)=\frac{f(x+0)+f(x-0)}{2}$$

bo'ladi. $\quad (x\in[-\pi,\pi))$

◄ (20.26) tenglikning har ikki tomoni

$$\frac{1}{2}[f(x+0)+f(x-0)]$$

ga ko'paytirib quyidagini topamiz

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}[f(x+0)+f(x-0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du \qquad (20.27)$$

(20.25) va (20.27) munosabatlardan foydalanib ushbu

$$F_n(f,x)-\frac{1}{2}[f(x+0)+f(x-0)]$$

ayirmani quyidagicha yozish mumkin

$$F_n(f;x)-\frac{1}{2}[f(x+0)+f(x-0)]=$$

$$=\frac{1}{\pi}\int_0^{\pi}[f(x+u)+f(x-u)-f(x+0)-f(x-0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du\,.$$

Agar

$$\frac{1}{\pi}\int_0^{\pi}[f(x+u)-f(x+0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du=J_{1n}(f;x),$$

$$\frac{1}{\pi}\int_0^{\pi}[f(x-u)-f(x-0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du=J_{2n}(f;x)$$

deb belgilasak, unda

$$F_n(f;x)-\frac{1}{2}[f(x+0)+f(x-0)]=J_{1n}(f;x)+J_{2n}(f;x)$$

bo'ladi.

Endi $J_{1n}(f;x)$ va $J_{2n}(f;x)$ larni baholaymiz. Ixtiyoriy δ $(0<\delta<\pi)$ sonni olib $J_{1n}(f;x)$ ni ikki qismga ajratib yozamiz:

$$J_{1n}(f;x)=\frac{1}{\pi}\int_0^{\delta}[f(x+u)-f(x+0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du+$$

$$+\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+u)-f(x+0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du\,. \qquad (20.28)$$

Lokallashtirish prinspiga asosan

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+u)-f(x+0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du=0$$

bo'ladi. Demak, $\forall\varepsilon>0$ olinganda ham, shunday $n_0=n_0(\varepsilon,\delta)\in N$ topiladiki, $\forall n>n_0$ uchun

$$\left|\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+u)-f(x+0)]\frac{sin\left(n+\frac{1}{2}\right)u}{2\,sin\frac{u}{2}}du\right|<\frac{\varepsilon}{2} \qquad (20.29)$$

bo'ladi.

Endi (20.28) tenglikning o'ng tomonidagi birinchi integralni baholaylik. Uni δ ni tanlab olish hisobiga etarlicha kichik qila olishimiz mumkinligini ko'rsataylik.

Shartga ko'ra, $f(x)$ funksiya $[-\pi,\pi)$ da bo'lakli-differensiallanuvchi. Binobarin. $\forall x\in[-\pi,\pi)$ nuqtada uning bir tomonli chekli hosilalari, xususan, o'ng hosilasi

$$\lim_{u\to+0}\frac{f(x+u)-f(x+0)}{u}=f^{'}(x+0)$$

mavjud. Demak, shunday $\delta_1>0$ topiladiki $0<u<\delta_1$ bo'lganda

$$\left|\frac{f(x+u)-f(x+0)}{u}\right|\le M_1 \qquad (\mathrm{M}_1=const)$$

bo'ladi.

Shungdek, shunday $\delta_2>0$ topiladiki, $0<u<\delta_2$ bo'lganda

$$\frac{\frac{u}{2}}{sin\frac{u}{2}}\le M_2 \qquad (\mathrm{M}_2=const)$$

bo'ladi.

Agar $\delta=min\left\{\delta_1,\delta_2,\frac{\pi\varepsilon}{2M_1M_2}\right\}$ deyilsa, unda ixtiyoriy $n\in N$ uchun

$$\left|\frac{1}{\pi}\int_{0}^{\delta}\left[\frac{f(x+u)-f(x+0)}{u}\right]\frac{\frac{u}{2}}{sin\frac{u}{2}}sin\left(n+\frac{1}{2}\right)udu\right|\le$$

$$\le\frac{1}{\pi}\int_{0}^{\delta}\left|\frac{f(x+u)-f(x+0)}{u}\frac{\frac{u}{2}}{sin\frac{u}{2}}\right|du\le\frac{1}{\pi}M_1M_2\delta<\frac{\varepsilon}{2} \qquad (20.30)$$

bo'ladi.

Natijada (20.28), (20.29) va (20.30) munosabatlardan $\forall\varepsilon>0$ olinganda ham, shunday $n_0\in N$ topiladiki, barcha $n>n_0$ uchun $|J_{1n}(f;x)|<\varepsilon$ bo'lishi kelib chiqadi.

Ikkinchi integral

$$J_{2n}(f;x)=\frac{1}{\pi}\int_0^{\pi}[f(x-u)-f(x-0)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du$$

ham xuddi shunga o'xshash baholanadi va $|J_{2n}(f;x)|<\varepsilon$ bo'lishi topiladi. Demak, $\forall\varepsilon>0$ olinganda ham, shunday $n_0\in N$ topiladiki, barcha $n>n_0$ uchun

$$\left|F_n(f;x)-\frac{1}{2}[f(x+0)+f(x-0)]\right|<2\varepsilon$$

bo'ladi. Bu esa

$$\lim_{n\to\infty}F_n(f;x)=\frac{1}{2}[f(x+0)+f(x-0)]$$

ekanini bildiradi.

Shunday qilib, $f(x)$ funsiyaning Fure qatori $[-\pi,\pi)$ da yaqinlashuvchi, uning yig'indisi $T(f;x)$ esa $\frac{1}{2}[f(x+0)+f(x-0)]$ ga teng

$$T(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)=\frac{1}{2}[f(x+0)+f(x-0)]\ .$$ ►

Ravshanki, teorema shartlarini qanoatlantiruvchi $f(x)$ funsiyaning uzluksizlik nuqtalarida $T(f;x)=\dfrac{f(x+0)+f(x-0)}{2}=f(x)$ bo'ladi.

$x=\pm\pi$ bo'lganda ushbu bobning 1-§ ida aytilgan ushbu

$$f(\pi+0)=f(-\pi+0)=f(\pi-0)$$

tengliklar e'tiborga olinsa, unda

$$\lim_{n\to\infty}F_n(f;-\pi)=\frac{f(-\pi+0)+f(-\pi-0)}{2}=\frac{f(-\pi+0)+f(\pi-0)}{2}$$

$$\lim_{n\to\infty}F_n(f;\pi)=\frac{f(\pi+0)+f(\pi-0)}{2}=\frac{f(-\pi+0)+f(\pi-0)}{2}$$

bo'ladi. Demak,

$$\lim_{n\to\infty}F_n(f;-\pi)=\lim_{n\to\infty}F_n(f;\pi)=\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$$

ya'ni

$$T(f;-\pi)=T(f;\pi)=\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$$

bo'ladi.

2-natija. Agar 2π davrli $f(x)$ funksiya $[-\pi,\pi]$ da uzluksiz, bo'lakli-differensiallanuvchi va $f(-\pi)=f(\pi)$ bo'lsa, bu funksiyaning Fure qatori $[-\pi,\pi]$ da yaqinlashuvchi, yig'indisi

$$T(f;x)=f(x)\qquad (x\in[-\pi,\pi])$$

bo'ladi.

20.5-misol. Ushbu

$$f(x)=x^2\qquad (x\in[-\pi,\pi])$$

funksiyani Fure qatoriga yoyilsin.

◀ Ma'lumki;

$$x^2 \sim \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k \frac{4}{k^2} \cos kx = \frac{\pi^2}{3} - 4\left(\cos x - \cos\frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^3} - ....\right).$$

Ravshanki, x^2 funksiya $[-\pi,\pi]$ da oraliqda 2-natijaning shartlarini qanoatlantiradi. Shu natijaga ko'ra, $[-\pi,\pi]$ da uning Fure qatori yaqinlashuvchi, yig'indisi esa x^2 ga teng bo'ladi.

$$x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}(-1)^k \frac{4}{k^2} \cos kx = \frac{\pi^2}{3} - 4\left(\cos x - \cos\frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ....\right)$$

$$(x \in [-\pi,\pi])$$ ▶

20.6-misol. Ushbu

$$f(x) = \cos ax \qquad (0 < a < 1)$$

funksiyani Fure qatoriga yoyilsin.

◀ Fure koeffitsientlari

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos axdx = 2\frac{\sin a\pi}{a\pi},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nxdx = \frac{1}{\pi}\int_0^{\pi}(\cos(a+n)x + \cos(a-n)x)dx = (-1)^n \frac{2a}{a^2-n^2}\frac{\sin a\pi}{\pi}$$

$$(n = 1,2,3,...),$$

$$в_n = 0 \quad (n = 1,2,3,...)$$

bo'ladi. Demak, berilgan funksiyaning Fure qatori

$$\cos ax \sim \frac{\sin a\pi}{a\pi} + \frac{2a \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^k}{a^2-k^2}\cos kx$$

bo'ladi. Agar bu $f(x) = \cos ax$ funksiya 2-natijaning shartlarni bajarishini e'tiborga olsak, unda

$$\cos ax = \frac{\sin a\pi}{a\pi} + \frac{2a \sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^k}{a^2-k^2}\cos kx$$

bo'lishini topamiz.

Keyingi tenglikdan $x = 0$ deyilsa

$$1 = \frac{\sin a\pi}{\pi}\left[\frac{1}{a} + 2a\sum_{k=1}^{\infty}\frac{(-1)^k}{a^2-k^2}\right]$$

ya'ni

$$\frac{\pi}{\sin a\pi} = \frac{1}{a} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{a+k} + \frac{1}{a-k}\right)$$

kelib chiqadi. ▶

20.7-misol. Quyidagi

$$f(x) = \begin{cases} -x, & agar \ -\pi \le x \le 0 \ bo'lsa, \\ 0, & agar \ \ 0 < x < \pi \ \ bo'lsa \end{cases}$$

funksiya Fure qatoriga yoyilsin.

◀Bu funksiya yuqoridagi 2-teorema shartini qanoatlantirishini ko'rish qiyin emas.

Berilgan funksiyaning Fure koeffitsientlarini topib, Fure qatorini yozamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = -\frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{0} = \frac{\pi}{2},$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx = -\frac{1}{\pi}\int_{-\pi}^{0} x\cos kxdx = -\frac{1}{k\pi}x\sin kx\Big|_{-\pi}^{o} +$$

$$+\frac{1}{k\pi}\int_{-\pi}^{0}\sin kxdx = \frac{1}{k^2\pi}(\cos k\pi - \cos 0) = \frac{1}{k^2\pi}\left((-1)^k - 1\right)$$

Demak,

$$a_k = \begin{cases} \dfrac{-2}{k^2\pi}, & agar\ k \text{ toq son bo' lsa,} \\ 0\quad, & agar\ k \text{ juft son bo' lsa.} \end{cases}$$

Endi $в_k$ koeffitsientlarini hisoblaymiz:

$$в_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin kxdx = \frac{1}{\pi}x\frac{\cos kx}{k}\Big|_{-\pi}^{0} -$$

$$-\frac{1}{\pi}\int_{-\pi}^{0}\frac{\cos kx}{k}dx = \frac{\cos k\pi}{k} = \frac{(-1)^k}{k}.$$

Shunday qilib, $x \in (-\pi, \pi)$ uchun

$$T(f;x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2} + \sum_{k=1}^{\infty}(-1)^k\frac{\sin kx}{k} = f(x),$$

$x = \pm\pi$ da esa

$$T(f;-\pi) = T(f;\pi) = \frac{0+\pi}{2} = \frac{\pi}{2}$$

bo'ladi.▶

20.8-misol. Ushbu

$$f(x) = \begin{cases} 1\ , \text{ agar } -\pi \le x < 0 \text{ bo'lsa,} \\ -1, \text{ agar } \ 0 \le x < \pi \text{ bo'lsa} \end{cases}$$

funksiya Fure qatoriga yoyilsin.

◀Bu funksiya yuqoridagi teoremaning shartlarini qanoatlantiradi. Berilgan funksiyaning Fure koeffitsientlarini hisoblab, uning Fure qatorini topamiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx - \frac{1}{\pi}\int_{0}^{\pi} dx = 0,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{0}\cos nxdx - \frac{1}{\pi}\int_{0}^{\pi}\cos nxdx = 0 \quad (n = 1,2,3,...),$$

$$в_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{0}\sin nx dx-\frac{1}{\pi}\int_{0}^{\pi}\sin nx dx=$$

$$=-\frac{1}{n\pi}(\cos 0\text{-}\cos n\pi)+\frac{1}{n\pi}(\cos n\pi-\cos 0)=\frac{2}{n\pi}(\cos n\pi-\cos 0)=\frac{2}{n\pi}\left[(\text{-}1)^n-1\right].$$

Demak,

$$в_n=\begin{cases}0 & , \ agar \quad n- \ juft \quad son \ bo'lsa\\ \text{-}\dfrac{4}{n\pi} & , \quad agar \ n \quad toq \ son \ bo'lsa\end{cases}$$

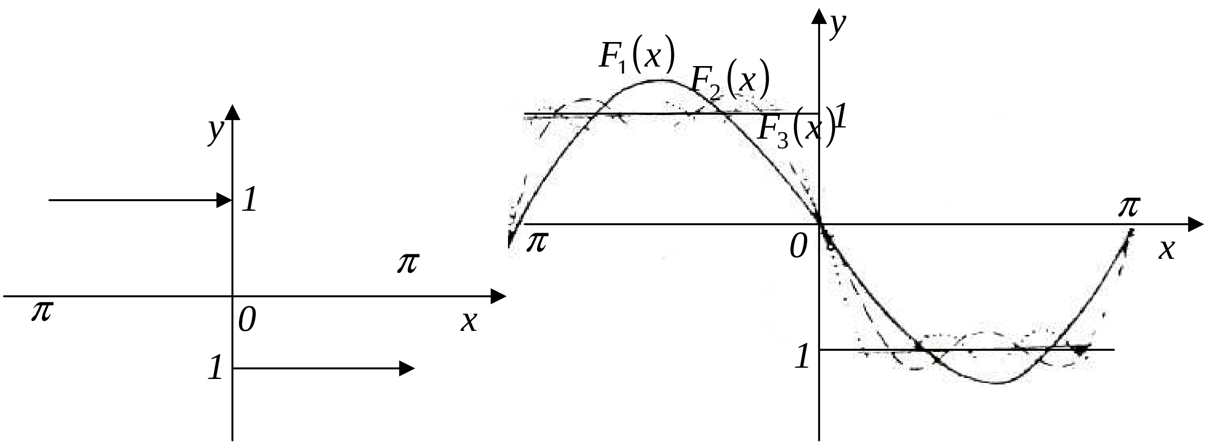
Shunday qilib, berilgan $f(x)$ funksiyaning Fure qatori

$$T(f;x)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=-\frac{4}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+...\right)$$

bo’ladi va uning yig’indisi

$$T(f;x)=\begin{cases} f(x), & agar \quad x\in(-\pi,\pi)\setminus\{0\},\\ \dfrac{f(-0)+f(0)}{2}=\dfrac{1+(-1)}{2}=0 & agar \ x=0\\ \dfrac{f(-\pi-0)+f(-\pi+0)}{2}=0 & agar \ x=-\pi\\ \dfrac{f(\pi-0)+f(\pi+0)}{2}=0 & agar \ x=\pi\end{cases}$$

bo’ladi. 71-chizmada $f(x)$ funksiyaning va uning Fure qatorining va qismiy yig’indilari tasvirlangan.



71-chizma

5-§. Qismiy yig’indilarning bir ekstremal xossasi. Bessel tengsizligi

$f(x)$ funksiya $[a,в]$ oraliqda berilgan. Bu funksiya va uning kvadrati ham shu oraliqda integrallanuvchi bo’lsin. Odatda bunday funksiyalar kvadrati bilan integ-rallanuvchi deb ataladi.

Agar $f(x)$ funksiya $[a,в]$ da kvadrati bilan integrallanuvchi bo’lsa, u shu ora-liqda absolyut integrallanuvchi bo’ladi. Haqiqatdan ham, ushbu

$$|f(x)|\le\frac{1}{2}\left(1+f^2(x)\right)$$

tengsizlikdan foydalanib

$$\int_a^в |f(x)|dx$$

ning mavjud bo'lishini topamiz. Bu esa $f(x)$ funksiyaning $[a,в]$ da absolyut integrallanuvchi ekanini bildiradi.

Ammo $f(x)$ funksiyaning absolyut integrallanuvchi bo'lishidan, uning kvadrati bilan integrallanuvchi bo'lishi har doim kelib chiqavermaydi.

Masalan, ushbu

$$f(x) = \frac{1}{\sqrt{x}}$$

funksiya $(0,1]$ da integrallanuvchi, lekin

$$f^2(x) = \frac{1}{x}$$

funksiya esa $(0,1]$ da integrallanuvchi emas (qaralsin, 16-bob, 5-§).

Demak, kvadrati bilan integrallanuvchi funksiyalar to'plami, absolyut integrallanuvchi funksiyalar to'plamining qismi bo'ladi.

$f(x)$ funksiya $[-\pi,\pi]$ da kvadrati bilan integrallanuvchi funksiya, $T_n(x)$ darajasi n dan katta bo'lmagan trigonometrik ko'phad bo'lsin:

$$T_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx)$$

Ravshanki, bunday ko'phadlar ham $[-\pi,\pi]$ da kvadrati bilan integrallanuvchi bo'ladilar. Koshi-Bunyakovskiy tengsizligidan

$$\int_{-\pi}^{\pi} [f(x) - T_n x]^2 dx \qquad (20.31)$$

integralning ham mavjudligi kelib chiqadi. Bu integral muayyan $f(x)$ da

$$\alpha_0, a_1, \beta_1, \alpha_2, \beta_2, .., \alpha_n, \beta_n, ...$$

larga bog'liq:

$$J = J(\alpha_0, a_1, \beta_1, \alpha_2, \beta_2, .., \alpha_n, \beta_n, ...) = \int_{-\pi}^{\pi} [f(x) - T_n(x)]^2 dx.$$

Endi quyidagi masalani qaraylik. Shu koeffsientlar qanday tanlab olingandan J eng kichik qiymatga ega bo'ladi? Bu masalani hal etish uchun yuqoridagi (20.31) integralni hisoblaylik:

$$\int_{-\pi}^{\pi} [f(x) - T_n(x)]^2 dx = \int_{-\pi}^{\pi} f^2(x)dx - 2\int_{-\pi}^{\pi} f(x)T_n(x)dx + \int_{-\pi}^{\pi} T_n^2(x)dx \qquad (20.32)$$

$f(x)$ funksiya Fure koeffsientlari uchun

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx \qquad (k = 1,2,...)$$

$$в_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx \qquad (k = 1,2,...)$$

formulalardan foydalansak,

$$\int_{-\pi}^{\pi} f(x)T_n(x)dx = \int_{-\pi}^{\pi} f(x)\left[\frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx)\right]dx = \frac{\alpha_0}{2}a_0\pi + \tag{20.33}$$

$$+\sum_{k=1}^{n}(\alpha_k a_k \pi + \beta_k b_k \pi) = \pi\left[\frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n}(\alpha_k a_k + \beta_k b_k)\right]$$

bo'ladi.

Agar

$$\int_{-\pi}^{\pi} \cos kxdx = \int_{-\pi}^{\pi} \sin xdx = 0, \qquad \int_{-\pi}^{\pi} \cos kx \sin kxdx = 0$$

$$\int_{-\pi}^{\pi} \sin^2 kxdx = \int_{-\pi}^{\pi} \cos^2 kxdx = \pi$$

ekanini e'tiborga olsak, u holda

$$\int_{-\pi}^{\pi} T_n^2(x)dx = \int_{-\pi}^{\pi}\left[\frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos x + \beta_k \sin kx)\right]^2 dx = \pi\left[\frac{\alpha_0^2}{2} + \sum_{k=1}^{n}(\alpha_k^2 + \beta_k^2)\right] \tag{20.34}$$

bo'ladi. Yuqoridagi (20.32), (20.33), (20.34) tengliklardan foydalanib quyidagini topamiz:

$$\int_{-\pi}^{\pi}[f(x) - T_n(x)]^2 dx = \int_{-\pi}^{\pi} f^2(x)dx - 2\pi\left[\frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n}\alpha_k a_k + \sum_{k=1}^{n}\beta_k b_k\right] -$$

$$-\pi\left[\frac{\alpha_0^2}{2} + \sum_{k=1}^{n}\alpha_k^2 + \sum_{k=1}^{n}\beta_k^2\right] = \int_{-\pi}^{\pi} f^2(x)dx - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n}a_k^0 + \sum_{k=1}^{b}b_k^2\right] +$$

$$+\pi\left[\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{n}(\alpha_k - a_k)^2 + \sum_{k=1}^{n}(\beta_k - b_k)^2\right].$$

Bu tenglikdan ko'rinadiki,

$$\int_{-\pi}^{\pi}[f(x) - T_n(x)]^2 dx$$

integral

$$\alpha_0 = a_0,$$
$$\alpha_k = a_k, \qquad (k = 1,2,3,...,n)$$
$$\beta_k = в_k$$

bo'lgandagina o'zining eng kichik qiymatiga erishadi va u qiymat

$$\int_{-\pi}^{\pi} f^2(x)dx - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n}a_k^2 + \sum_{k=1}^{n}в_k^2\right]$$

bo'ladi, ya'ni:

$$\min_{\alpha_0,\alpha_1,\beta_1,...,\alpha_n,\beta_n} \int_{-\pi}^{\pi}[f(x) - T_n(x)]^2 dx = \int_{-\pi}^{\pi} f^2(x)dx - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n}a_k^2 + \sum_{k=1}^{n}в_k^2\right].$$

Shunday qilib quyidagi teoremani isbotladik.

3-teorema. $f(x)$ funksiya $[-\pi,\pi]$ da kvadrati bilan integrallanuvchi bo'lsin. Darajasi n dan katta bo'lmagan barcha trigonometrik ko'phadlar $\{T_n(x)\}$ ichida ushbu

$$\int_{-\pi}^{\pi}[f(x)-T_n(x)]^2 dx$$

integralga eng kichik qiymat beruvchi ko'phad $f(x)$ funksiya Fure qatorining $n-$ qismiy yig'indisi bo'ladi:

$$\min_{T_n(x)}\int_{-\pi}^{\pi}[f(x)-T_n(x)]^2 dx=\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2 dx=$$
$$=\int_{-\pi}^{\pi} f^2(x)dx-\pi\left[\frac{a_0^2}{2}+\sum_{k=1}^{n}a_k^2+\sum_{k=1}^{n}в_k^2\right]. \tag{20.35}$$

3-natija. Agar $f(x)$ funksiya $[-\pi,\pi]$ da kvadrati bilan integrallanuvchi bo'lsa, u holda bu funksiyaning Fure koeffsientlari kvadratlaridan tuzilgan:

$$\sum_{k=1}^{\infty}a_k^2, \qquad \sum_{k=1}^{\infty}в_k^2$$

qatorlar yaqinlashuvchi bo'ladi va quyidagi tengsizlik o'rinlidir:

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}a_k^2+\sum_{k=1}^{\infty}в_k^2\leq\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx \tag{20.36}$$

◄ (20.35) munosabatdan

$$\int_{-\pi}^{\pi} f^2(x)dx-\pi\left[\frac{a_0^2}{2}+\sum_{k=1}^{\infty}a_k^2+\sum_{k=1}^{\infty}в_k^2\right]\geq 0$$

ya'ni, $\forall n$ uchun

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}a_k^2+\sum_{k=1}^{n}в_k^2\leq\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

bo'ladi. Bu erda n ni cheksizlikka intiltirib, keltirilgan natijani va tengsizlikni hosil qilamiz.

(20.36) tengsizlik Bessel tengsizligi deb ataladi.

6-§. Yaqinlashuvchi Fure qatori yig'indisining funksional xossalari

Biz mazkur kursning 14-bobida yaqinlashuvchi funsional qatorlar yig'indisining funksional xossalarini batafsil o'rgandik. Ravshanki, berilgan funksiyaning Fure qatori funksional qatorlarning hususiy holidir. Binobarin, tegishli shartlarda Fure qatorlari yig'indilari ham 14-bobda keltirilgan xossalarga ega bo'ladi. Quyida ularni isbotsiz keltiramiz.

$f(x)$ funksiya $[-\pi,\pi]$ da berilgan va uning Fure qatori

$$T(f;x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+в_n\sin nx) \tag{20.37}$$

$[-\pi,\pi]$ da yaqinlashuvchi bo'ladi.

1⁰. Fure qatorlari yig'indisining uzluksizligi. Agar (20.37) qator $[-\pi,\pi]$ da tekis yaqinlashuvchi bo'lsa, u holda bu qatorning $T(f;x)$ yig'indisi $[-\pi,\pi]$ oraliqda uzluksiz funksiya bo'ladi.

2⁰. Fure qatorini hadma-had integrallash. Agar (20.37) qator $[-\pi,\pi]$ da tekis yaqinlashuvchi bo'lsa, u holda (20.37) qator hadlarining integrallaridan tuzilgan.

$$\int_a^в \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left(a_n \int_a^в cos\, nxdx + в_n \int_a^в sin\, nxdx\right) =$$

$$= \frac{a_0}{2}(в - a) + \sum_{n=1}^{\infty}\left(a_n \frac{sin\, nв - sin\, na}{n} + в_n \frac{cos\, na - cos\, nв}{n}\right)$$

qator $(-\pi \le a < в \le \pi)$ ham yaqinlashuvchi bo'ladi va uning yig'indisi

$$\int_a^в T(f;x)dx$$

ga teng bo'ladi, ya'ni

$$\int_a^в T(f;x)dx = \int_a^в \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cos\, nx + в_n\, sin\, nx)\right]dx =$$

$$= \int_a^в \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left[\int_a^в (a_n\, cos\, nx + в_n\, sin\, nx)dx\right]$$

3⁰. Fure qatorini hadma-had differensiallash. Agar (20.37) qator har bir hadining hosilalaridan tuzilgan

$$\sum_{n=1}^{\infty}(-na_n\, sin\, nx + nв_n\, cos\, nx)$$

qator $[-\pi,+\pi]$ da tekis yaqinlashuvchi bo'lsa, u holda berilgan Fure qatorining yig'indisi $T(f;x)$ shu $[-\pi,+\pi]$ da $T'(f;x)$ hosilaga ega va

$$T'(f;x) = \sum_{n=1}^{\infty}(-na_n\, sin\, nx + nв_n\, cos\, nx)$$

bo'ladi.

Shunday qilib, umumiy holdagidek $f(x)$ funksiya Fure qatori yig'indisining funksional xossalarini o'rganishda Fure qatorining tekis yaqinlashuvchi bo'lishi muhim rol o'ynayapti. Binobarin, Fure qatorining tekis yaqinlashuvchi bo'lishini ta'minlaydigan shartlarini aniqlash lozim bo'ladi.

Endi shu haqida teorema keltiramiz

4-teorema. Fure qatorining tekis yaqinlashishi. $f(x)$ funksiya $[-\pi,+\pi]$ oraliqda berilgan, uzluksiz hamda $f(-\pi) = f(\pi)$ bo'lsin. Agar bu funksiya $[-\pi,+\pi]$ oraliqda bo'lakli – silliq bo'lsa, u holda $f(x)$ funksiyaning Fure qatori

$$T(f;x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cos\, nx + в_n\, sin\, nx)$$

$[-\pi,+\pi]$ oraliqda tekis yaqinlashuvchi bo'ladi.

◀Berilgan $f(x)$ funksiya Fure qatorining har bir

$$u_n(x)=a_n\cos nx+в_n\sin nx\qquad (n=1,2,3,...)$$

hadi uchun

$$|u_n(x)|=|a_n\cos nx+в_n\sin nx|\le|a_n|+|в_n|\quad (n=1,2,3,...)$$

bo’ladi.

Endi

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(|a_n|+|в_n|)$$

qatorning yaqinlashuvchi bo’lishini ko’rsatamiz.

Fure koeffitsientlari

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\quad в_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

ni qaraylik.

Bo’laklab integrallash qoidasiga ko’ra

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)d\left(\frac{\sin nx}{n}\right)=\frac{1}{\pi}f(x)\frac{1}{n}\sin nx\Big|_{-\pi}^{\pi}-$$

$$-\frac{1}{n\pi}\int_{-\pi}^{\pi}f'(x)\sin nxdx=-\frac{1}{n}\cdot\frac{1}{\pi}f'(x)\sin nxdx,\qquad (20.38)$$

$$в_n=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)d\left(\frac{\cos nx}{n}\right)=-\frac{1}{\pi}f(x)\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi}+\frac{1}{n\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx=-$$

$$-\frac{1}{n\pi}(-1)^n[f(\pi)-f(-\pi)]+\frac{1}{n\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx$$

Agar $f(-\pi)=f(\pi)$ shartni e’tiborga olsak, u holda

$$в_n=\frac{1}{n}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx\qquad (20.39)$$

bo’ladi.

$f'(x)$ ning Fure koeffitsientlarini a'_n va $в'_n$ desak:

$$a'_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx,\quad в'_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin nxdx,$$

u holda (20.38) va (20.39) munosabatlarga ko’ra

$$a_n=-\frac{1}{n}в'_n,\ в_n=-\frac{1}{n}a'_n\quad (n=1,2,3,...)$$

bo’ladi. Natijada

$$|a_n|+|в_n|=\frac{1}{n}(|a'_n|+|в'_n|)$$

bo’ladi.

Agar

$$\frac{1}{n}(|a'_n|+|в'_n|)=\frac{1}{n}|a'_n|+\frac{1}{n}|в'_n|\le\frac{1}{2}\left(a_n'^2+\frac{1}{n^2}\right)+\frac{1}{2}\left(в_n'^2+\frac{1}{n^2}\right)=\frac{1}{2}\left(a_n'^2+в_n'^2\right)+\frac{1}{n^2}$$

bo’lishini hisobga olsak, unda ushbu

$$|a_n|+|в_n| \le \frac{1}{2}\left(a_n'^2+в_n'^2\right)+\frac{1}{n^2} \qquad (20.40)$$

tengsizlikka ega bo'lamiz.

Shartga ko'ra $f(x)$ funksiya bo'lakli-uzluksizdir. Binobarin, u kvadrati bilan integrallanuvchidir. Shuning uchun bu funksiyaning $a_n', в_n'$ Fure koeffitsientlari Bessel tengsizligini qanoatlantiradi, ya'ni

$$\frac{a_0'^2}{2}+\sum_{n=1}^{\infty}\left(a_n'^2+в_n'^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f'^2(x)dx$$

bo'ladi. Demak,

$$\frac{a_0'^2}{2}+\sum_{n=1}^{\infty}\left(a_n'^2+в_n'^2\right)$$

qator yaqinlashuvchi. Unda yaqinlashuvchi qatorlarning xossalariga ko'ra ushbu

$$\sum_{n=1}^{\infty}\left[\frac{1}{2}\left(a_n'^2+в_n'^2\right)+\frac{1}{n^2}\right] \qquad (20.41)$$

qator ham yaqinlashuvchi bo'ladi.

Yuqorida keltirilgan (20.40) tengsizlikka muvofiq

$$\sum_{n=1}^{\infty}\left(|a_n|+|в_n|\right)$$

qatorning har bir hadi (20.41) qatorning mos hadidan katta emas.

Taqqoslash teoremasiga ko'ra (qaralsin, I-tom, 2-bob, 8-§) (20.39) qator yaqinlashuvchi, demak,

$$\frac{|a_0|}{2}+\sum_{n=1}^{\infty}\left(|a_n|+|в_n|\right)$$

qator yaqinlashuvchi bo'ladi.

Veyershtrass alomatidan (14-bob, 2-§) foydalanib, Fure qatorining $[-\pi,+\pi]$ da tekis yaqinlashuvchi bo'lishini topamiz. ▶

7-§. Funksiyalarni trigonometrik ko'phad bilan yaqinlashtirish

Feyer yig'indisi. $f(x)$ funksiya $[-\pi,+\pi]$ oraliqda berilgan va uzluksiz bo'lsin. Bu funksiya Fure qatorining qismiy yig'indisi

$$F_n(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+в_k\sin kx\right)$$

dan foydalanib, ushbu

$$\sigma_n(f;x)=\frac{1}{n}\left[F_0(f;x)+F_1(f;x)+\ldots+F_{n-1}(f;x)\right],\ \ F_0(f;x)=\frac{a_0}{2} \qquad (20.42)$$

yig'indini tuzamiz. Odatda (20.42) yig'indi funksiyaning Feyer yig'indisi deb ataladi.

$f(x)$ funksiyaning Feyer yig'indisi $\sigma_n(f;x)$ trigonometrik ko'phad bo'ladi. Haqiqatdan ham, Fure qatori qismiy yig'indilarining ifodalari

$$F_0(f;x)=\frac{a_0}{2},$$

$$F_1(f;x)=\frac{a_0}{2}+a_1\cos x+в_1\sin x,$$

$$F_2(f;x)=\frac{a_0}{2}+a_1\cos x+в_1\sin x+a_2\cos 2x+в_2\sin 2x,$$

....,

$$F_{n-1}(f;x)=\frac{a_0}{2}+a_1\cos x+в_1\sin x+...+a_{n-1}\cos(n-1)x+в_{n-1}\sin(n-1)x$$

ga ko'ra

$$\sigma_1(f;x)=\frac{a_0}{2},$$

$$\sigma_2(f;x)=\frac{a_0}{2}+\frac{1}{2}a_1\cos x+\frac{1}{2}в_1\sin x,$$

$$\sigma_3(f;x)=\frac{a_0}{2}+\frac{2}{3}a_1\cos x+\frac{2}{3}в_1\sin x+\frac{1}{3}a_2\cos 2x+\frac{1}{3}в_2\sin 2x,$$

....,

$$\sigma_n(f;x)=\frac{a_0}{2}+\frac{n-1}{n}a_1\cos x+\frac{n-1}{n}в_1\sin x+...+\frac{1}{n}a_{n-1}\cos(n-1)x+$$

$$+\frac{1}{n}в_{n-1}\sin(n-1)x=\frac{a_0}{2}+\sum_{k=1}^{n-1}\left(\frac{n-k}{n}a_k\cos kx+\frac{n-k}{n}a_k\sin kx\right)$$

bo'ladi.

Agar 3-§ da keltirilgan ushbu tenglik

$$F_n(1;x)=1 \quad (n=1,2,3,...)$$

ni e'tiborga olsak, unda (20.42) dan

$$\sigma_n(1;x)=1 \tag{20.43}$$

bo'lishi kelib chiqadi.

(20.42) munosabatdagi $F_k(f;x)$ $(k=0,1,2,...n-1)$ ning o'rniga uning ifodasi

$$F_k(f;x)=\frac{1}{\pi}\int_0^{\pi}[f(x+u)+f(x-u)]\frac{\sin\frac{2k+1}{2}u}{2\sin\frac{u}{2}}du$$

ni qo'yib quyidagini topamiz:

$$\sigma_n(f;x)=\frac{1}{n\pi}\sum_{k=0}^{n-1}\left\{\int_0^{\pi}[f(x+u)+f(x-u)]\frac{\sin\frac{2k+1}{2}u}{2\sin\frac{u}{2}}du\right\}=$$

$$=\frac{1}{2n\pi}\int_0^{\pi}\left[\frac{f(x+u)+f(x-u)}{sin\frac{u}{2}}\sum_{k=0}^{n-1}sin(2k+1)\frac{u}{2}\right]du=$$

$$=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}\left[\frac{f(x+2t)+f(x-2t)}{sin\,t}\sum_{k=0}^{n-1}sin(2k+1)t\right]dt.$$

Integral ostidagi yig'indi uchun

$$\sum_{k=0}^{n-1}sin(2k+1)t=\frac{sin^2 nt}{sin\,t}$$

munosabat o'rinli. Haqiqatdan ham,

$$sin\,t\sum_{k=0}^{n-1}sin(2k+1)t=\sum_{k=0}^{n-1}sin\,t\cdot sin(2k+1)t=$$

$$=\sum_{k=0}^{n-1}\frac{1}{2}[cos\,2kt-cos(2k+2)t]=\frac{1}{2}(1-cos\,2nt)=sin^2 nt.$$

Natijada $f(x)$ funksiyaning Feyer yig'indisi ushbu

$$\sigma_n(f;x)=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}[f(x+2t)+f(x-2t)]\left(\frac{sin\,nt}{sin\,t}\right)^2 dt \qquad (20.44)$$

ko'rinishni oladi. Bu va yuqoridagi (20.43) tenglikdan

$$1=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}2\left(\frac{sin\,nt}{sin\,t}\right)^2 dt \qquad (20.45)$$

bo'lishi kelib chiqadi.

5-teorema (Feyer teoremasi). $f(x)$ funksiya $[-\pi,+\pi]$ oraliqda berilgan, uzluksiz va $f(-\pi)=f(\pi)$ bo'lsin. U holda

$$\lim_{n\to\infty}\sup_{-\pi\le x\le\pi}|\sigma_n(f;x)-f(x)|=0$$

bo'ladi.

◄(20.45) tenglikning har ikki tomonini $f(x)$ ga ko'paytirsak, u holda

$$f(x)=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}2f(x)\left(\frac{sin\,nt}{sin\,t}\right)^2 dt$$

bo'ladi. Bu va (20.44) munosabatdan foydalanib, ushbuni topamiz:

$$\sigma_n(f;x)-f(x)=\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{sin\,nt}{sin\,t}\right)^2 dt. \qquad (20.46)$$

Modomiki, shartga ko'ra $f(x)$ funksiya $[-\pi,+\pi]$ da uzluksiz ekan, u Kantor teoremasiga binoan tekis uzluksiz bo'ladi. Demak, $\forall\varepsilon>0$ olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $|x'-x''|<2\delta$ tengsizlikni qanoatlantiruvchi $\forall x',x''\in[-\pi,+\pi]$ uchun

$$|f(x')-f(x'')|<\frac{\varepsilon}{2} \qquad (20.47)$$

bo'ladi. Shu topilgan δ sonni olib (uni $\delta<\frac{\pi}{2}$ deb hisoblash mumkin), (20.46) integralni ikki qismga ajratamiz:

$$\sigma_n(f;x)-f(x)=J_1(n,\delta)+J_2(n,\delta)$$

bunda

$$J_1(n;\delta)=\frac{1}{n\pi}\int_0^{\delta}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{\sin t}\right)^2dt,$$

$$J_2(n;\delta)=\frac{1}{n\pi}\int_{\delta}^{\frac{\pi}{2}}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{\sin t}\right)^2dt.$$

Endi $J_1(n,\delta)$ va $J_2(n,\delta)$ integrallarni baholaymiz. Yuqoridagi (20.47) munosabatni e'tiborga olib quyidagini topamiz:

$$|J_1(n,\delta)|\le\frac{1}{n\pi}\int_0^{\delta}[|f(x+2t)-f(x)|+|f(x-2t)-f(x)|]\left(\frac{\sin nt}{\sin t}\right)^2dt<$$

$$<\frac{1}{n\pi}\int_0^{\delta}\left(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\right)\left(\frac{\sin nt}{\sin t}\right)^2dt\le\frac{\varepsilon}{n\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin nt}{\sin t}\right)^2dt=\frac{\varepsilon}{2}.$$

Demak, $\forall\varepsilon>0$ olinganda ham, shunday $\delta>0$ topiladiki, barcha $n\in N$ lar uchun $|J_1(n,\delta)|<\frac{\varepsilon}{2}$ bo'ladi.

Endi $J_2(n,\delta)$ integralni baholaymiz.

$$|J_2(n,\delta)|\le\frac{1}{n\pi}\int_0^{\frac{\pi}{2}}|f(x+2t)+f(x+2t)-2f(x)|\left(\frac{\sin nt}{\sin t}\right)^2dt\le\frac{1}{n\pi}\cdot4M\int_{\delta}^{\frac{\pi}{2}}\left(\frac{\sin nt}{\sin t}\right)^2dt$$

bunda $M=\max\limits_{-\pi\le x\le\pi}|f(x)|$. Ravshanki,

$$t\in\left[\delta,\frac{\pi}{2}\right]\ (\delta>0)\text{ da }\left(\frac{\sin nt}{\sin t}\right)^2\le\frac{1}{\sin^2\delta}$$

bo'ladi. Natijada $J_2(n,\delta)$ uchun ushbu $|J_2(n,\delta)|\le\frac{1}{n\pi}\cdot\frac{4M}{\sin^2\delta}\cdot\frac{\pi}{2}=\frac{2M}{n\sin^2\delta}$ bahoga ega bo'lamiz. Agar natural n sonni $n>n_0=\left[\frac{4M}{\varepsilon\sin^2\delta}\right]$ qilib olinsa (bunda $[a]$ - a sonini butun qismi), unda $\frac{2M}{n^2\sin^2\delta}<\frac{\varepsilon}{2}$ va, demak, $|J_2(n,\delta)|<\frac{\varepsilon}{2}$ bo'ladi.

Shunday qilib, $\forall\varepsilon>0$ olinganda ham shunday $\delta=\delta(\varepsilon)>0$ topiladiki, $\forall n\in N$ uchun $|J_1(n,\delta)|<\frac{\varepsilon}{2}$ bo'ladi. Va shu $\varepsilon>0$ va $\delta=\delta(\varepsilon)>0$ larga ko'ra shunday n_0 topiladiki, $\forall n>n_0$ uchun $|J_2(n,\delta)|<\frac{\varepsilon}{2}$ bo'ladi.

Bu tasdiqlarni birlashtirsak, $\forall\varepsilon>0$ uchun shunday $n_0\in N$ topiladiki, $\forall n>n_0$, $\forall x\in[-\pi,\pi]$ uchun $|\sigma_n(f;x)-f(x)|<\varepsilon$ bo'ladi.

Demak, $\lim\limits_{n\to\infty}\sup\limits_{-\pi\le x\le\pi}|\sigma_n(f;x)-f(x)|<\varepsilon$. ►

Natijada, funksiyani trigonometrik ko'phad bilan yaqinlashtirish haqidagi quyidagi teoremaga kelamiz.

6-teorema (Veyershtrass teoremasi). Agar $f(x)$ funksiya $[-\pi,+\pi]$ da berilgan, uzluksiz va $f(-\pi)=f(\pi)$ bo'lsa, u holda shunday $\Im_n(x)$ trigonometrik ko'phad topiladi,

$$\lim_{n\to\infty}\sup_{-\pi\le x\le\pi}|\Im_n(x)-f(x)|=0$$

bo'ladi.

8-§. O'rtacha yaqinlashish. Fure qatorining o'rtacha yaqinlashishi

Funksional ketma-ketlik va qatorlarda tekis yaqinlashish tushunchasi bilan bir qatorda, undan umumiyroq o'rtacha yaqinlashish tushunchasi ham kiritiladi.

1^0. O'rtacha yaqinlashish. $[a,в]$ oraliqda biror $\{f_n(x)\}$:

$$f_1(x),\ f_2(x),\ldots,\ f_n(x),\ldots \qquad (20.48)$$

funksional ketma-ketlik va $f(x)$ funksiya berilgan bo'lib, $f_n(x)$ $(n=1,2,3,\ldots)$ hamda $f(x)$ lar shu oraliqda kvadrati bilan itegrallanuvchi bo'lsin.

2-ta'rif. Agar

$$\lim_{n\to\infty}\int_a^в[f_n(x)-f(x)]^2dx=0$$

bo'lsa, (20.48) funksional ketma-ketlik $f(x)$ funksiyaga $[a,в]$ da o'rtacha yaqinlashadi deb aytiladi.

Masalan, ushbu $\{f_n(x)\}=\{x^n\}$:

$$x,x^2,\ldots,x^n,\ldots\ (x\in[0,1])$$

funksional ketma-ketlik

$$f(x)=\begin{cases}0, & agar,\ x\in[0,1)\ bo'lsa,\\ 1, & agar,\ x=1\quad bo'lsa\end{cases}$$

funksiyaga $[0,1]$ da o'rtacha yaqinlashadi, chunki

$$\int_0^1[f_n(x)-f(x)]^2dx=\int_0^1(x^n-0)^2dx=\int_0^1x^{2n}dx=\frac{1}{2n+1}$$

va demak,

$$\lim_{n\to\infty}\int_0^1 \left(x^n - 0\right)^2 dx = 0$$

7-teorema. Agar (20.48) funksional ketma-ketlik $f(x)$ ga $[a,в]$ da tekis yaqinlashsa, shu (20.48) ketma-ketlik $f(x)$ ga $[a,в]$ da o'rtacha yaqinlashadi.

◀(20.48) ketma-ketlik $f(x)$ tekis yaqinlashsin.

Ta'rifga binoan, $\forall\varepsilon > 0$ olinganda ham shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ va $\forall x \in [a,в]$ uchun bir yo'la

$$|f_n(x) - f(x)| < \sqrt{\frac{\varepsilon}{в - a}}$$

bo'ladi. Demak, $\forall n > n_0$ uchun

$$\left|\int_a^в [f_n(x) - f(x)]^2\right| \le \int_a^в |f_n(x) - f(x)|^2 dx < \int_a^в \frac{\varepsilon}{в - a} dx = \varepsilon$$

bo'ladi. Bu esa

$$\lim_{n\to\infty}\int_a^в [f_n(x) - f(x)]^2 dx = 0$$

ekanini bildiradi. Demak, $\{f_n(x)\}$ ketma-ketlik $f(x)$ funksiyaga $[a,в]$ da o'rtacha yaqinlashadi.▶

2-eslatma. Funksional ketma-ketlikning $[a,в]$ da o'rtacha yaqinlashishidan, uning shu oraliqda tekis yaqinlashishi har doim kelib chiqavermaydi. Masalan, yuqorida ko'rdikki $\{f_n(x)\} = \{x^n\}$ ketma-ketlik

$$f(x) = \begin{cases} 0, \ agar, x \in [0,1) \ bo'lsa, \\ 1, \ agar, \ x = 1 \quad bo'lsa \end{cases}$$

funksiyaga $[0,1)$ da o'rtacha yaqinlashadi. Biroq bu funksional ketma-ketlik shu $f(x)$ funksiyaga $[0,1)$ da tekis yaqinlashmaydi (qaralsin, 14-bob, 2-§).

Yuqorida keltirilgan teorema va eslatma funksional ketma-ketliklarda o'rtacha yaqinlashish tekis yaqinlashish tushunchasiga qaraganda kengroq tushuncha ekanini ko'rsatadi.

Funksional qatorlarda ham o'rtacha yaqinlashish tushunchasi shunga o'xshash kiritiladi.

$[a,в]$ oraliqda

$$\sum_{k=1}^{\infty} u_k(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \qquad (20.49)$$

funksional qator berilgan bo'lsin. Bu qator qismiy yig'indilari

$$S_n(x) = \sum_{k=1}^{\infty} u_k(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

dan iborat $\{S_n(x)\}$ funksional ketma-ketlikni qaraylik.

3-ta'rif. Agar

$$\lim_{n\to\infty}\int_a^в [S_n(x)-S(x)]^2 dx = 0$$

bo'lsa, (20.49) funksional qator $S(x)$ funksiyaga $[a,в]$ da o'rtacha yaqinlashadi deb ataladi.

2^0. Fure qatorining o'rtacha yaqinlashishi. $f(x)$ funksiya $[-\pi,+\pi]$ da berilgan, $T(f;x)$ esa uning Fure qatori

$$T(f;x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos kx + в_k \sin kx)$$

bo'lsin.

8-teorema. Agar $f(x)$ funksiya $[-\pi,+\pi]$ oraliqda uzluksiz va $f(-\pi)=f(\pi)$ bo'lsa, uning Fure qatori $[-\pi,+\pi]$ da $f(x)$ ga o'rtacha yaqinlashadi.

◀Shartga ko'ra $f(x)$ funksiya $[-\pi,+\pi]$ da uzluksiz va $f(-\pi)=f(\pi)$. U holda ushbu bobning 7-§ ida keltirilgan Veyershtrass teoremasmga asosan, $\forall\varepsilon>0$ olinganda ham, shunday trigonometrik ko'phad $\Im_n(x)$ topiladiki, $\forall x\in[-\pi,+\pi]$ uchun

$$|f(x)-\Im_n(x)|<\sqrt{\frac{\varepsilon}{2\pi}}$$

bo'ladi. Bu tengsizlikdan foydalanib,

$$\int_{-\pi}^{\pi}[f(x)-\Im_n(x)]^2 dx < \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}dx=\varepsilon \qquad (20.50)$$

bo'lishini topamiz.

Ma'lumki, $f(x)$ funksiya Fure qatorining qismiy yig'indisi $F_n(f;x)$ uchun

$$\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2 dx = \min_{T_n(x)}\int_{-\pi}^{\pi}[f(x)-T_n(x)]^2 dx \qquad (20.51)$$

bo'ladi (qaralsin, 5-§). Demak, (20.50) va (20.51) munosabatlarga ko'ra

$$\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2 dx \le \int_{-\pi}^{\pi}[f(x)-T_n(x)]^2 dx<\varepsilon \quad (\forall x\in[-\pi,\pi])$$

bo'ladi. Bu esa

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2 dx = 0$$

ya'ni $f(x)$ funksiya Fure qatori $[-\pi,+\pi]$ da o'rtacha yaqinlashishini bildiradi.▶

Biz o'tgan paragrafda $[-\pi,+\pi]$ oraliqda kvadrati bilan integrallanuvchi $f(x)$ funksiya uchun ushbu

$$\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2 dx = \int_{-\pi}^{\pi} f^2(x)dx-\pi\left[\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+в_k^2)\right]$$

tenglikni keltirib chiqargan edik. Bu tenglikdan ko'rinadiki, agar

$$\lim_{n\to\infty}\left\{\int_{-\pi}^{\pi} f^2(x)dx-\pi\left[\frac{a_0^2}{2}+\sum_{k=1}^{n}\left(a_k^2+в_k^2\right)\right]\right\}=0$$

ya’ni

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx=\frac{a_0^2}{2}+\sum_{k=1}^{\infty}\left(a_k^2+в_k^2\right) \qquad (20.52)$$

bo’lsa, u holda

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}[f(x)-F_n(f;x)]^2dx=0$$

bo’ladi va demak, $f(x)$ funksiyaning Fure qatori $[-\pi,+\pi]$ da o’rtacha yaqinlashadi.

Shunday qilib, $f(x)$ funksiyaning Fure qatorining $[-\pi,+\pi]$ da o’rtacha yaqinlashishini ko’rsatishi uchun (20.52) tenglikning o’rinli bo’lishini ko’rsatish zarur va etarli bo’ladi. Odatda (20.52) Parseval tengligi deb ataladi.

9-§. Funksiyalarning ortogonal sistemasi. Umumlashgan Fure qatori

1⁰. Funksiyalarning ortogonal sistemasi. $\varphi(x)$ va $\psi(x)$ funksiyalar $[a,в]$ da berilgan va ular shu oraliqda integrallanuvchi bo’lsin.

4-ta’rif. Agar

$$\int_a^в \varphi(x)\cdot\psi(x)dx=0$$

bo’lsa, $\varphi(x)$ va $\psi(x)$ funksiyalar $[a,в]$ da ortogonal deb ataladi.

Masalan, $\varphi(x)=\sin x$, $\psi(x)=\cos x$ funksiyalar $[-\pi,+\pi]$ da ortogonal bo’ladi, chunki,

$$\int_a^в \varphi(x)\cdot\psi(x)dx=\int_{-\pi}^{\pi}\sin x\cos x dx=0$$

bo’ladi.

$\varphi(x)=x$, $\psi(x)=\frac{3}{2}x^2-1$ funksiyalar $[-1,1]$ da ortogonal bo’ladi, chunki

$$\int_a^в \varphi(x)\cdot\psi(t)dx=\int_{-1}^{1}x\left(\frac{3}{2}x^2-1\right)dx=0.$$

Endi

$$\varphi_0(x),\varphi_1(x),\dots,\varphi_n(x),\dots \qquad (20.52)$$

funksiyalarning har biri $[a,в]$ da berilgan va shu oraliqda integrallanuvchi bo’lsin. Bu (20.52) funksiyalar sistemasini $\{\varphi_n(x)\}$ kabi belgilaymiz.

5-ta’rif. Agar $\{\varphi_n(x)\}$ funksiyalar sistemasining istalgan ikkita $\varphi_k(x)$ va $\varphi_m(x)$ $(k\neq m)$ funksiyalari uchun

$$\int_a^в \varphi_k(x)\,\varphi_m(x)dx=0 \quad (k\neq m)$$

bo'lsa, $\{\varphi_n(x)\}$ funksiyalar sistemasi $[a,в]$ da ortogonal deb ataladi.

Odatda, $k=m$ $(k=0,1,2,...)$ bo'lganda

$$\int_a^в \varphi_k^2(x)dx>0 \quad (k=0,1,2,...)$$

deb qaraladi. Bu integralni λ_k kabi belgilaylik:

$$\lambda_k=\int_a^в \varphi_k^2(x)dx \quad (k=0,1,2,...).$$

Agar (20.52) sistema uchun $\lambda_k=1$ bo'lsa, $\{\varphi_n(x)\}$ normal sistema deyiladi. Agar (20.52) sistema uchun

$$\int_0^в \varphi_k(x)\varphi_m(x)dx=\begin{cases}0, & agar, k\neq m\ bo'lsa,\\ 1, & agar,\ k=m\ bo'lsa\end{cases}$$

bo'lsa, $\{\varphi_n(x)\}$ funksiyalar sistemasi ortonormal deb ataladi.

Masalan, 1) ushbu

$$1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx, ...$$

sistema (trigonometrik sistema) $[-\pi,+\pi]$ da ortogonal bo'ladi, chunki $k\neq m$ bo'lganda

$$\int_{-\pi}^{\pi}\cos kx\cos mxdx=0, \quad \int_{-\pi}^{\pi}\sin kx\sin mxdx=0$$

bo'lib, ixtiyoriy $k,m=0,1,2,...$ bo'lganda $\int_{-\pi}^{\pi}\cos kx\sin mxdx=0$ bo'ladi.

2) Quyidagi

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, ..., \quad \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, ...$$

funksiyalar sistemasi $[-\pi,+\pi]$ da ortonormal bo'ladi. Bu sistemaning $[-\pi,+\pi]$ da ortonormal bo'lishi ravshandir. Uning shu $[-\pi,+\pi]$ da normal bo'lishi esa

$$\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{\pi}}\cos kx\right)^2 dx=\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{\pi}}\sin kx\right)^2 dx=1 \quad (k=0,1,2,...)$$

bo'lishidan kelib chiqadi.

(20.52) sistema berilgan bo'lsin. Uning yordamida tuzilgan ushbu

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)=c_0\varphi_0(x)+c_1\varphi_1(x)+...+c_n\varphi_n(x)+... \qquad (20.53)$$

funksional qator $\{\varphi_n(x)\}$ sistema bo'yicha qator deyiladi, c_0, $c_1,...,$ c_n,... o'zgarmas sonlar esa qatorning koeffitsientlari deyiladi.

Xususan, $\varphi_n(x)=a_n\cos nx+в_n\sin nx$ bo'lganda (20.53) qator trigonometrik qatorga aylanadi.

$f(x)$ funksiya $[a,в]$ oraliqda berilgan va shu oraliqda integrallanuvchi bo'lsin. Ravshanki, $f(x)\cdot\varphi_n(x)$ $(n=0,1,2,3,...)$ funksiya ham $[a,в]$ da integrallanuvchi bo'ladi. Bu funksiyalarning integrallarini hisoblab, ularni quyidagicha belgilaymiz:

$$\alpha_n = \frac{1}{\lambda_n}\int_a^в f(x)\varphi_n(x)dx. \quad (20.54)$$

Bu sonlardan foydalanib ushbu

$$\sum_{n=0}^{\infty}\alpha_n\varphi_n(x) = \alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + ... + \alpha_n\varphi_n(x) + ... \quad (20.55)$$

qatorni tuzamiz.

6-ta'rif. $\alpha_0, \alpha_1, ..., \alpha_n, ...$ - koeffitsientlari (20.54) formula bilan aniqlangan (20.55) qator $f(x)$ funksiyaning $\{\varphi_n(x)\}$ sistema bo'yicha Fure qatori deb ataladi. $\alpha_0, \alpha_1, ..., \alpha_n, ...$-sonlar esa umumlashgan Fure koeffitsientlari deyiladi.

Odatda, $f(x)$ funksiya bilan unga mos umumlashgan Fure qatori "~" belgi orqali quyidagicha yoziladi:

$$f(x) \sim \sum_{n=0}^{\infty}\alpha_n\varphi_n(x) = \alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + ... + \alpha_n\varphi_n(x) + ...$$

Mashqlar

20.9. Ushbu

$$f(x) = e^{-x} \quad (-\pi < x < \pi)$$

funksiyaning Fure qatori topilsin.

20.10.Ushbu

$$f(x) = |cos\, x|$$

funksiyani Fure qatoriga yoyilsin. Yoyilmadan foydalanib quyidagi

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{4n^2-1}, \quad \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

qatorlarning yig'indilari topilsin.

20.11. Ushbu

$$f(x) = x - [x] = \{x\}$$

funksiyani Fure qatoriga yoyilsin.

20.12. $f(x) = x$, $\varphi(x) = x^2$ funksiyalarni $(0, \pi)$ da kosinuslar bo'yicha yoyilmasidan foydalanib, ushbu

$$\sum_{n=1}^{\infty}\frac{cos\, nx}{n^2} = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$$

tenglik isbotlansin.

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